Covariant Linear Perturbation Formalism

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Abstract

Lecture notes on covariant linear perturbation theory and its applications to inflation, dark energy or matter and the cosmic microwave background.

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Contents

1 Apologia

In these informal *Lecture Notes* on formalism, I gather together a few elements of covariant linear perturbation theory and its applications to inflation, dark energy or matter and the cosmic microwave background (CMB). I make no attempt to cite properly the original source material. My usual, lame defense is "it's all in Bardeen's paper [1]" – go read it and ignore these lecture notes!

Seriously though, in addition to [1], I have heavily relied on a few sources in compiling the various sections and refer the reader to references therein. The covariant formalism presented in §2 and gauge transformation material in §3 draws from [2] distilled in [3]. Applications to inflation in §4 draw from [4], to dark energy and matter in §5 from [5], and to the CMB in §6 from [6]. For the last topic, despite great recent progress on the phenomenology, I have limited myself to the formal aspects that relate to covariant perturbation theory. My defense here is "everything that doesn't go out of date as soon as it's written is in Peebles & Yu [7]" – go read it and stop looking for CMB reviews! My other defense is Matias Zaldarriaga was supposed to cover that in this *School* so blame him! As for the equation density and opaqueness of these *Lecture Notes*, I have no excuse – take it as an homage to the Dick Bond lectures of my own student days – at least there are no figures with 100 curves. As Martin White would say, you are physicists; you don't need figures!

2 Formalism

2.1 Covariant Approach

Perturbation theory proceeds by linearizing the Einstein equations

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \,, \tag{1}$$

around a background metric. Here $G_{\mu\nu}$ is the Einstein tensor and $T_{\mu\nu}$ is the stress energy tensor. The Bianchi identity $\nabla_\mu G^{\mu\nu} = 0$ guarantees the covariant conservation of the total stress-energy

$$\nabla_\mu T^{\mu\nu} = 0 \,. \tag{2}$$

The conservation equation is redundant with the Einstein equation but is a particularly useful representation of the equations when there are multiple components to the matter that are separately conserved.

Let us begin by distinguishing between *covariant* equations and *invariant* variables:

Covariant = equations takes same form in all coordinate systems
Invariant = variable takes the same value in all coordinate systems

In an evolving universe, the meaning of a density perturbation necessarily requires the specification of the time slicing in relation to the background and hence there is no such thing as a gauge or coordinate invariant density perturbation. Likewise for elements of the metric. The general covariance of the Einstein and conservation equations, on the other hand, can be preserved. With covariant equations one can, after the fact of their derivation, choose the gauge that best suits a given physical problem, e.g. the evolution of inflationary, dark energy, or CMB fluctuations.

2.2 Metric Representation

Let us take the background metric to be the general homogeneous and isotropic, or Friedmann-Robertson-Walker form. Here the degrees of freedom are the (comoving) spatial curvature K and an overall scale factor for the expansion a such that the line element

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = a^2(-d\eta^2 + \gamma_{ij} dx^i dx^j)\,. \tag{3}$$

The time variable is η the conformal time and we normalize the scale factor to $a = 1$ today, $\eta(a=1) = \eta_0$. The three metric γ_{ij} can be represented in spherical coordinates as

$$\gamma_{ij} dx^i dx^j = dD^2 + D_A^2 d\Omega\,, \tag{4}$$

where D is the comoving distance and $D_A = K^{-1/2} \sin(K^{1/2} D)$ is the angular diameter distance.

A general perturbation to the FRW metric may be represented as

$$\begin{aligned} g^{00} &= -a^{-2}(1-2A)\,, \\ g^{0i} &= -a^{-2} B^i\,, \\ g^{ij} &= a^{-2}(\gamma^{ij} - 2H_L \gamma^{ij} - 2H_T^{ij})\,, \end{aligned} \tag{5}$$

yielding (1) $A \equiv$ a scalar potential; (3) B^i a vector shift; (1) H_L a scalar perturbation to the spatial curvature; (5) H_T^{ij} a trace-free distortion to spatial metric, for a total of (10) degrees of freedom. This gives a complete representation of the symmetric 4×4 metric tensor.

2.3 Matter Representation

Likewise expand the matter stress energy tensor around a homogeneous density ρ and pressure p by introducing 10 degrees of freedom

$$\begin{aligned} T^0_{\ 0} &= -\rho - \delta\rho\,, \\ T^0_{\ i} &= (\rho + p)(v_i - B_i)\,, \\ T_0^{\ i} &= -(\rho + p) v^i\,, \\ T^i_{\ j} &= (p + \delta p)\delta^i_{\ j} + p\Pi^i_{\ j}\,, \end{aligned} \tag{6}$$

yielding (1) $\delta\rho$ a scalar density perturbation; (3) v_i a vector velocity, (1) δp a scalar pressure perturbation; (5) Π_{ij} a tensor anisotropic stress perturbation. So far the treatment of the matter and metric is fully general and applies to any type of matter or coordinate choice including non-linearities in the matter, e.g. cosmological defects.

2.4 Closure

The Einstein equations do not specify a closed system of equations and require the addition of supplemental conditions provided by the microphysics to fix the dynamical degrees of freedom. In the simplest cases this closure is established through equations of state for the background and perturbations.

In counting the dynamical degrees of freedom, one must recall that the conservation equations are redundant with the Bianchi identities and that 4 degrees of freedom are absorbed by the choice of coordinate system

20	Variables (10 metric; 10 matter)
−10	Einstein equations
−4	Conservation equations
+4	Bianchi identities
−4	Gauge (coordinate choice 1 time, 3 space)
6	Degrees of freedom

Without loss of generality these six dynamical degrees of freedom can be taken to be the 6 components of the matter stress tensor.

Since the background is isotropic, the unperturbed matter is described entirely by the pressure $p(a)$ or equivalently $w(a) \equiv p(a)/\rho(a)$, the equation of state parameter. For the perturbations specification of the relationship between δp, Π and the other perturbations, e.g. $\delta\rho$ and v suffices.

2.5 Friedmann Equations

The unperturbed Einstein equation yields the ordinary Friedmann equation which relates the expansion rate to the energy density

$$\left(\frac{\dot{a}}{a}\right)^2 + K = \frac{8\pi G}{3} a^2 \rho \,, \tag{7}$$

where overdots represent conformal time derivatives, and the acceleration Friedmann equation which relates the change in the expansion rate to the equation of state

$$\frac{d}{d\eta}\left(\frac{\dot{a}}{a}\right) = -\frac{4\pi G}{3} a^2 (\rho + 3p) \,, \tag{8}$$

so that $w \equiv p/\rho < -1/3$ implies acceleration of the expansion.

The conservation law $\nabla^\mu T_{\mu\nu} = 0$ implies

$$\frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a} \,, \tag{9}$$

which by virtue of the Bianchi identity can be derived from the Friedmann equation.

The counting exercise for the background becomes

20	Variables (10 metric; 10 matter)
−17	Homogeneity and Isotropy
−2	Einstein equations
−1	Conservation equations
+1	Bianchi identities
1	Degree of freedom

Without loss of generality this degree of freedom can be chosen to be the equation of state $w(a) = p(a)/\rho(a)$.

2.6 Linearization and Eigenmodes

For the inhomogenous universe, the Einstein tensor $G_{\mu\nu}$ is in general constructed out of a nonlinear combination of metric fluctuations. Metric fluctuations typically remain small even in the presence of larger matter fluctuations. Hence we linearize the left hand side of the Einstein equations to obtain a set of partial differential equations that are linear in the variables.

These equations may then be decoupled into a set of ordinary differential equations by employing normal modes under translation and rotation. The scalar, vector and tensor eigenmodes of the Laplacian operator form a complete set

$$\begin{aligned} \nabla^2 Q^{(0)} &= -k^2 Q^{(0)} && \mathrm{S}\,, \\ \nabla^2 Q_i^{(\pm1)} &= -k^2 Q_i^{(\pm1)} && \mathrm{V}\,, \\ \nabla^2 Q_{ij}^{(\pm2)} &= -k^2 Q_{ij}^{(\pm2)} && \mathrm{T}\,. \end{aligned} \tag{10}$$

In a spatially flat ($K = 0$) universe, the eigenmodes are essentially plane waves

$$\begin{aligned} Q^{(0)} &= \exp(i\mathbf{k}\cdot\mathbf{x})\,, \\ Q_i^{(\pm1)} &= \frac{-i}{\sqrt{2}}(\hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2)_i \exp(i\mathbf{k}\cdot\mathbf{x})\,, \\ Q_{ij}^{(\pm2)} &= -\sqrt{\frac{3}{8}}(\hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2)_i(\hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2)_j \exp(i\mathbf{k}\cdot\mathbf{x})\,, \end{aligned} \tag{11}$$

where $\mathbf{e}_1, \mathbf{e}_2$ are unit vectors spanning the plane transverse to $\mathbf{k}$.

Vector modes represent divergence-free vectors (vorticity); tensor modes represent transverse traceless tensors (gravitational waves)

$$\nabla^i Q_i^{(\pm1)} = 0\,, \qquad \nabla^i Q_{ij}^{(\pm2)} = 0\,, \qquad \gamma^{ij} Q_{ij}^{(\pm2)} = 0\,. \tag{12}$$

Curl free vectors and the longitudinal components of tensors are represented with covariant derivatives of the scalar and vector modes

$$\begin{aligned} Q_i^{(0)} &= -k^{-1}\nabla_i Q^{(0)}\,, \\ Q_{ij}^{(0)} &= (k^{-2}\nabla_i\nabla_j + \frac{1}{3}\gamma_{ij})Q^{(0)}\,, \\ Q_{ij}^{(\pm1)} &= -\frac{1}{2k}[\nabla_i Q_j^{(\pm1)} + \nabla_j Q_i^{(\pm1)}]\,, \end{aligned} \tag{13}$$

For the kth eigenmode, the scalar components become

$$\begin{aligned} A(\mathbf{x}) &= A(k)\,Q^{(0)}\,, & H_L(\mathbf{x}) &= H_L(k)\,Q^{(0)}\,, \\ \delta\rho(\mathbf{x}) &= \delta\rho(k)\,Q^{(0)}\,, & \delta p(\mathbf{x}) &= \delta p(k)\,Q^{(0)}\,, \end{aligned} \tag{14}$$

the vectors components become

$$B_i(\mathbf{x}) = \sum_{m=-1}^{1} B^{(m)}(k)\,Q_i^{(m)}\,, \qquad v_i(\mathbf{x}) = \sum_{m=-1}^{1} v^{(m)}(k)\,Q_i^{(m)}\,, \tag{15}$$

and the tensors components become

$$H_{Tij}(\mathbf{x}) = \sum_{m=-2}^{2} H_T^{(m)}(k)\,Q_{ij}^{(m)}\,, \quad \Pi_{ij}(\mathbf{x}) = \sum_{m=-2}^{2} \Pi^{(m)}(k)\,Q_{ij}^{(m)}\,. \tag{16}$$

An arbitrary set of spatial perturbations can be formed through a superposition of the eigenmodes given their completeness.

2.7 Covariant Scalar Equations

The Einstein equations for the scalar modes (suppressing 0 superscripts) become

$$
\begin{aligned}
(k^2-3K)[H_L+\frac{1}{3}H_T+\frac{\dot a}{a}(\frac{B}{k}-\frac{\dot H_T}{k^2})] &= 4\pi G a^2\left[\delta\rho+3\frac{\dot a}{a}(\rho+p)\frac{v-B}{k}\right], \\
k^2(A+H_L+\frac{1}{3}H_T)+\left(\frac{d}{d\eta}+2\frac{\dot a}{a}\right)(kB-\dot H_T) &= -8\pi G a^2 p\Pi\,, \\
\frac{\dot a}{a}A-\dot H_L-\frac{1}{3}\dot H_T-\frac{K}{k^2}(kB-\dot H_T) &= 4\pi G a^2(\rho+p)\frac{v-B}{k}\,, \\
\left[2\frac{\ddot a}{a}-2\left(\frac{\dot a}{a}\right)^2+\frac{\dot a}{a}\frac{d}{d\eta}-\frac{k^2}{3}\right]A-\left[\frac{d}{d\eta}+\frac{\dot a}{a}\right](\dot H_L+\frac{kB}{3}) &= 4\pi G a^2(\delta p+\frac{1}{3}\delta\rho)\,. \qquad (17)
\end{aligned}
$$

The conservation equations become the continuity and Navier-Stokes equations

$$
\left[\frac{d}{d\eta}+3\frac{\dot a}{a}\right]\delta\rho+3\frac{\dot a}{a}\delta p=-(\rho+p)(kv+3\dot H_L)\,, \qquad (18)
$$

$$
\left[\frac{d}{d\eta}+4\frac{\dot a}{a}\right](\rho+p)\frac{v-B}{k}=\delta p-\frac{2}{3}(1-3\frac{K}{k^2})p\Pi+(\rho+p)A\,. \qquad (19)
$$

In the absence of anisotropic stress, the Navier-Stokes equation is known as the Euler equation.

These equations are not independent due to the Bianchi identity. For numerical solutions it suffices to retain 2 Einstein equations and the 2 conservation laws. The counting exercise becomes

8	Variables (4 metric; 4 matter)
−4	Einstein equations
−2	Conservation equations
+2	Bianchi identities
−2	Gauge (coordinate choice 1 time, 1 space)
2	Degrees of freedom

Without loss of generality we may choose the dynamical components to be those of the stress tensor δp, Π. Microphysics defines their relationship to the density and velocity perturbations. In fluid dynamics this is the familiar need of a prescription for viscosity and heat conduction (entropy generation) to close the system of equations.

2.8 Covariant Vector Equations

The vector Einstein equations become

$$
\begin{aligned}
(1-\frac{2K}{k^2})(kB^{(\pm1)}-\dot H_T^{(\pm1)}) &= 16\pi G a^2(\rho+p)\frac{v^{(\pm1)}-B^{(\pm1)}}{k}\,, \\
\left[\frac{d}{d\eta}+2\frac{\dot a}{a}\right](kB^{(\pm1)}-\dot H_T^{(\pm1)}) &= -8\pi G a^2 p\Pi^{(\pm1)}\,, \qquad (20)
\end{aligned}
$$

and the conservation equations become

$$
\left[\frac{d}{d\eta}+4\frac{\dot a}{a}\right](\rho+p)\frac{v^{(\pm1)}-B^{(\pm1)}}{k}=-\frac{1}{2}(1-2\frac{K}{k^2})p\Pi^{(\pm1)}\,. \qquad (21)
$$

Since gravity provides no source to vorticity, any initial vector perturbation will simply decay unless vector anisotropic stresses are continuously generated in the matter. In the absence of nonlinearities in the matter, e.g. from cosmological defects, vector perturbations can generally be ignored.

The counting exercise for vectors becomes

8	Variables (4 metric; 4 matter)
−4	Einstein equations
−2	Conservation equations
+2	Bianchi identities
−2	Gauge (coordinate choice 2 space))
2	Degrees of freedom

Without loss of generality, we can choose these to be the vector components of the stress tensor $\Pi^{(\pm 1)}$.

2.9 Covariant Tensor Equations

The Einstein equation for the tensor modes is

$$\left[\frac{d^2}{d\eta^2} + 2\frac{\dot{a}}{a}\frac{d}{d\eta} + (k^2 + 2K)\right] H_T^{(\pm 2)} = 8\pi G a^2 p \Pi^{(\pm 2)} \,, \tag{22}$$

and the counting exercise becomes

4	Variables (2 metric; 2 matter)
−2	Einstein equations
−0	Conservation equations
+0	Bianchi identities
−0	Gauge (coordinate choice 1 time, 1 space)
2	Degrees of freedom

without loss of generality we can choose these to be represented by the tensor components of the stress tensor $\Pi^{(\pm 2)}$.

In the absence of anisotropic stresses and spatial curvature, the tensor equation becomes a simple source-free gravitational wave propagation equation

$$\ddot{H}_T^{(\pm 2)} + 2\frac{\dot{a}}{a}\dot{H}_T^{(\pm 2)} + k^2 H_T^{(\pm 2)} = 0 \,, \tag{23}$$

which has solutions

$$\begin{aligned} H_T^{(\pm 2)}(k\eta) &= C_1 H_1(k\eta) + C_2 H_2(k\eta) \,, \\ H_1(x) &\propto x^{-m} j_m(x) \,, \\ H_2(x) &\propto x^{-m} n_m(x) \,, \end{aligned} \tag{24}$$

where $m = (1-3w)/(1+3w)$. If $w > -1/3$ then the gravitational wave amplitude is constant above horizon $x \ll 1$ and then oscillates and damps. If $w < -1/3$ then gravity wave oscillates and freezes into some value. It will be useful to recall these solutions when considering the Klein Gordon equation for scalar field fluctuations during inflation and dark energy domination.

2.10 Multicomponent universe

With multiple matter components, the Einstein equations of course remain valid but with the associations

$$\begin{aligned} \delta\rho &= \sum_J \delta\rho_J \,, \\ (\rho+p)v^{(m)} &= \sum_J (\rho_J + p_J) v_J^{(m)} \,, \\ p\Pi^{(m)} &= \sum_J p_J \Pi_J^{(m)} \,, \end{aligned} \tag{25}$$

where J indexes the different components, e.g. photons, baryons, neutrinos, dark matter, dark energy, inflaton, cosmological defects. The conservation equations remain valid for each non-interacting subsystem. Interactions can be represented as non-ideality in the specification of the stress degrees of freedom (e.g. viscous and entropic terms) in each subsystem or by explicitly writing down energy and momentum exchange terms in separate conservation equations (see e.g. §6).

3 Gauge

3.1 Semantics

The covariant equations of the last section hold true under any coordinate choice for the relationship between the unperturbed FRW background and the perturbations. Choice of a particular relationship, called a gauge choice can help simplify the equations for the physical conditions at hand. The price to pay is that the perturbation variables are quantities that take on the meaning of say a density perturbation or a spatial curvature perturbation only on a specific gauge – but that is in any case unavoidable.

Since the preferred gauge for simplifying the physics can change as the universe evolves, it is often useful to access the perturbation variables of one gauge from those of another gauge. This operation proceeds by writing down a covariant form for gauge specific variables through the properties of gauge transformation. It is the analogue of deriving covariant equations for the dynamics but for the perturbation variables themselves. Since these gauge specific variables are now thought of in a coordinate independent way, this procedure is called in the literature endowing a variable with a "gauge invariant" meaning. Note neither the numerical values nor the physical interpretation of the variables has changed. "Gauge invariance" here is an operational distinction that indicates a freedom to calculate gauge specific quantities from an arbitrary gauge. Perturbation variables still only take on their given meaning in the given gauge. Under this definition of gauge invariance, the perturbation variables of any fully specified gauge is gauge invariant. We will hereafter avoid using this terminology.

To reduce the opportunity for confusion, we will name the metric perturbations in the various common gauges separately. We will however rely on context to distinguish between matter variables on the various gauges.

3.2 Gauge Transformation

In an evolving inhomogeneous universe, metric and matter fluctuations take on different values in different coordinate systems. Consider a general coordinate transformation

$$\begin{aligned} \eta &= \tilde{\eta} + T \,, \\ x^i &= \tilde{x}^i + L^i \,. \end{aligned} \tag{26}$$

Under this general transformation, the metric and stress energy tensors transform as tensors. The elements of these tensors are the perturbation variables and their numerical values change with the transformation.

The coordinate choice represented by (T, L^i) can be decomposed into scalar and vector modes. With $L^{(0)} = L$, the scalar mode variables transform as

$$\begin{aligned} A &= \tilde{A} - \dot{T} - \frac{\dot{a}}{a} T \,, \\ B &= \tilde{B} + \dot{L} + kT \,, \\ H_L &= \tilde{H}_L - \frac{k}{3} L - \frac{\dot{a}}{a} T \,, \\ H_T &= \tilde{H}_T + kL \,, \end{aligned} \tag{27}$$

for the metric and

$$\begin{aligned} \delta\rho_J &= \delta\tilde{\rho}_J - \dot{\rho}_J T, \\ \delta p_J &= \delta\tilde{p}_J - \dot{p}_J T, \\ v_J &= \tilde{v}_J + \dot{L} \end{aligned} \tag{28}$$

for the matter.

For the vector mode variables

$$\begin{aligned} B^{(\pm 1)} &= \tilde{B}^{(\pm 1)} + \dot{L}^{(\pm 1)} \,, \\ H_T^{(\pm 1)} &= \tilde{H}_T^{(\pm 1)} + kL^{(\pm 1)} \,, \\ v_J^{(\pm 1)} &= \tilde{v}_J^{(\pm 1)} + \dot{L}^{(\pm 1)} \,. \end{aligned} \tag{29}$$

The tensor mode variables are invariant under the gauge transformation as are the components of the anisotropic stress tensor.

A gauge is fully specified if there is an explicit prescription for (T, L) for scalars and $(L^{(\pm 1)})$ for the vectors to get to the desired frame. Gauges are typically defined by conditions on the metric or matter fluctuations. We now consider several common scalar gauge choices and their uses.

3.3 Newtonian Gauge

The Newtonian or longitudinal gauge is defined by diagonal metric fluctuations

$$\begin{aligned} B &= H_T = 0\,, \\ \Psi &\equiv A \quad \text{(Newtonian potential)}\,, \\ \Phi &\equiv H_L \quad \text{(Newtonian curvature)}\,, \end{aligned} \tag{30}$$

where Ψ plays the role of the gravitational potential in the Newtonian approximation and Φ is the Newtonian spatial curvature. This condition completely fixes the gauge by giving explicit expressions for the gauge transformation

$$\begin{aligned} L &= -\frac{\tilde{H}_T}{k}\,, \\ T &= -\frac{\tilde{B}}{k} + \frac{1}{k^2}\frac{d}{d\eta}\tilde{H}_T\,. \end{aligned} \tag{31}$$

The Einstein equations become

$$\begin{aligned} (k^2 - 3K)\Phi &= 4\pi G a^2 \left[\delta\rho + 3\frac{\dot{a}}{a}(\rho + p)\frac{v}{k}\right]\,, \\ k^2(\Psi + \Phi) &= -8\pi G a^2 p \Pi\,. \end{aligned} \tag{32}$$

Note that for scales inside the Hubble length $k(\dot{a}/a)^{-1} \gg 1$, the first equation becomes the Poisson equation for the perturbations and if the anisotropic stress of the matter is much less than the energy density perturbation (as in the case of non-relativistic matter), $\Psi \approx -\Phi$.

The conservation laws for the Jth non-interacting subsystem becomes

$$\left[\frac{d}{d\eta} + 3\frac{\dot{a}}{a}\right]\delta\rho_J + 3\frac{\dot{a}}{a}\delta p_J = -(\rho_J + p_J)(k v_J + 3\dot{\Phi})\,, \tag{33}$$

$$\left[\frac{d}{d\eta} + 4\frac{\dot{a}}{a}\right](\rho_J + p_J)\frac{v_J}{k} = \delta p_J - \frac{2}{3}(1 - 3\frac{K}{k^2})p_J\Pi_J + (\rho_J + p_J)\Psi\,. \tag{34}$$

Aside from the $\dot{\Phi}$ term, these are the usual relativistic fluid equations. Since Φ represents a perturbation to the scale factor, $\dot{\Phi}$ represents the perturbation to the redshifting of the density in an expanding universe [see Eqn. (9)].

The Newtonian gauge is useful since it most closely corresponds to Newtonian gravity. A drawback is that as $k(\dot{a}/a)^{-1} \to 0$ there are relativistic corrections to the Poisson equation which make a straightforward implementation of the equations numerically unstable. Likewise in this limit, the metric effects on density though $\dot{\Phi}$ muddle the interpretation of the conservation law.

3.4 Comoving Gauge

The comoving gauge is defined by

$$\begin{aligned} B &= v \quad (T^0_i = 0)\,, \\ H_T &= 0\,, \\ \xi &= A\,, \\ \zeta &= H_L \quad \text{(comoving curvature)}\,, \end{aligned} \tag{35}$$

which completely fixes the gauge through

$$\begin{aligned} T &= (\tilde{v} - \tilde{B})/k\,, \\ L &= -\tilde{H}_T/k\,. \end{aligned} \tag{36}$$

The Einstein equations become

$$\begin{aligned} \dot{\zeta} + Kv/k - \frac{\dot{a}}{a}\xi &= 0\,, \\ \dot{v} + 2\frac{\dot{a}}{a}v + k(\zeta + \xi) &= -8\pi G a^2 p\Pi\,, \end{aligned} \tag{37}$$

and the conservation laws become

$$\left[\frac{d}{d\eta} + 3\frac{\dot{a}}{a}\right]\delta\rho_J + 3\frac{\dot{a}}{a}\delta p_J = -(\rho_J + p_J)(kv_J + 3\dot{\zeta})\,, \tag{38}$$

$$\left[\frac{d}{d\eta} + 4\frac{\dot{a}}{a}\right](\rho_J + p_J)\frac{v_J - v}{k} = \delta p_J - \frac{2}{3}(1 - 3\frac{K}{k^2})p_J\Pi_J + (\rho_J + p_J)\xi\,. \tag{39}$$

In particular the Navier-Stokes equation for the total matter becomes an algebraic relation between total stress fluctuations and the potential

$$(\rho + p)\xi = -\delta p + \frac{2}{3}\left(1 - \frac{3K}{k}\right)p\Pi \tag{40}$$

so that these equations are a complete set.

Eliminating ξ allows us to write a conservation law for the comoving curvature

$$\dot{\zeta} + Kv/k = \frac{\dot{a}}{a}\left[-\frac{\delta p}{\rho + p} + \frac{2}{3}\left(1 - \frac{3K}{k^2}\right)\frac{p}{\rho + p}\Pi\right]. \tag{41}$$

On scales well below the curvature scale, this equation states that the comoving curvature only changes in response to stress gradients which move matter around. This statement corresponds to the non-relativistic intuition that causality should prohibit evolution above the horizon scale. This conservation law is the fundamental virtue of the comoving gauge. An auxiliary consideration is that the comoving gauge variables are numerically stable.

From an alternate gauge choice, one can construct the Newtonian variables via the gauge transformation relation

$$\zeta = \tilde{H}_L + \frac{1}{3}\tilde{H}_T - \frac{\dot{a}}{a}\frac{\tilde{v} - \tilde{B}}{k}\,. \tag{42}$$

For example, since in the Newtonian gauge H_T and v take on the same values

$$\begin{aligned} \Phi &= \zeta + \frac{\dot{a}}{a}\frac{v}{k} \\ &= \zeta + \frac{2}{3}\frac{1}{1+w}\frac{1}{1 + K(\dot{a}/a)^{-2}}[\Psi - \dot{\Phi}/(\dot{a}/a)]\,. \end{aligned} \tag{43}$$

If the curvature is negligible, stress perturbations are negligible, and the equation of state w is constant

$$\Phi = \frac{3 + 3w}{5 + 3w}\zeta\,, \tag{44}$$

so that Φ tends to be of the same order as ζ but changes when the equation of state changes.

The Newtonian curvature can also be obtained through the comoving density perturbations via

$$(k^2 - 3K)\Phi = 4\pi G a^2 \delta\rho\Big|_{\rm comoving}\,, \tag{45}$$

which has the added benefit of taking the form of a simple non-relativistic Poisson equation. This Poisson equation allows us to rewrite the conservation equation for the comoving curvature in the case of negligible background curvature and anisotropic stress as

$$\begin{aligned}\frac{d\ln\zeta}{d\ln a} &= -\frac{\Phi}{\zeta}\frac{2}{3+3w}k^2(\dot a/a)^{-2}\frac{\delta p}{\delta\rho}\Big|_{\rm comoving} \\ &\approx -\frac{2}{5+3w}k^2(\dot a/a)^{-2}\frac{\delta p}{\delta\rho}\Big|_{\rm comoving}\,.\end{aligned} \tag{46}$$

For adiabatic stresses where $\delta p = c_a^2\delta\rho$, the change in the comoving curvature is negligible for $c_a k(\dot a/a)^{-1} \sim c_a k\eta \ll 1$, i.e. on scales below the sound horizon.

3.5 Synchronous Gauge

The synchronous gauge confines the metric perturbations to the spatial degrees of freedom

$$\begin{aligned} A &= B = 0\,, \\ \eta_T &\equiv -\frac{1}{3}H_T - H_L\,, \\ h_L &= 6H_L\,, \\ T &= a^{-1}\int d\eta a\tilde A + c_1 a^{-1}\,, \\ L &= -\int d\eta(\tilde B + kT) + c_2\,. \end{aligned} \tag{47}$$

The metric conditions do not fully specify the gauge and need to be supplemented by a definition of (c_1, c_2). Usually one defines c_1 through the condition that the dark matter has zero velocity in the initial conditions and c_2 through the setting of the initial curvature perturbation. With a completely specified gauge condition, the synchronous gauge is as valid as any other gauge. The variables η_T and h_L form a stable system for numerical solutions and hence the synchronous gauge has been extensively used in numerical solutions.

The Einstein equations give

$$\begin{aligned}\dot\eta_T - \frac{K}{2k^2}(\dot h_L + 6\dot\eta_T) &= 4\pi G a^2(\rho + p)\frac{v}{k}\,, \\ \ddot h_L + \frac{\dot a}{a}\dot h_L &= -8\pi G a^2(\delta\rho + 3\delta p)\,,\end{aligned} \tag{48}$$

while the conservation equations give

$$\left[\frac{d}{d\eta} + 3\frac{\dot a}{a}\right]\delta\rho_J + 3\frac{\dot a}{a}\delta p_J = -(\rho_J + p_J)(kv_J + \frac{1}{2}\dot h_L)\,, \tag{49}$$

$$\left[\frac{d}{d\eta} + 4\frac{\dot a}{a}\right](\rho_J + p_J)\frac{v_J}{k} = \delta p_J - \frac{2}{3}(1 - 3\frac{K}{k^2})p_J\Pi_J\,. \tag{50}$$

Note that the lack of a potential in synchronous gauge implies that there are no gravitational forces in the Navier-Stokes equation. Hence for stress free matter like cold dark matter, zero velocity initially implies zero velocity always.

3.6 Spatially Flat Gauge:

Conversely the spatially flat gauge eliminates spatial metric perturbations

$$\begin{aligned}
H_L &= H_T = 0\,, \\
\alpha_F &\equiv A\,, \\
\beta_F &\equiv B\,, \\
T &= \left(\frac{\dot a}{a}\right)^{-1}\left(\tilde H_L + \frac{1}{3}\tilde H_T\right)\,. \\
L &= -\tilde H_T/k\,,
\end{aligned} \tag{51}$$

The Einstein equations give

$$\begin{aligned}
\dot\beta_F + 2\frac{\dot a}{a}\beta_F + k\alpha_F &= -8\pi G a^2 p\Pi/k\,, \\
\frac{\dot a}{a}\alpha_F - \frac{K}{k}\beta_F &= 4\pi G a^2(\rho+p)\frac{v-\beta_F}{k}\,,
\end{aligned} \tag{52}$$

and the conservation equations give

$$\left[\frac{d}{d\eta} + 3\frac{\dot a}{a}\right]\delta\rho_J + 3\frac{\dot a}{a}\delta p_J = -(\rho_J + p_J)kv_J\,, \tag{53}$$

$$\left[\frac{d}{d\eta} + 4\frac{\dot a}{a}\right](\rho_J + p_J)\frac{v_J - \beta_F}{k} = \delta p_J - \frac{2}{3}(1 - 3\frac{K}{k^2})p_J\Pi_J + (\rho_J + p_J)\alpha_F\,. \tag{54}$$

The spatially flat gauge is useful in that it is the complement of the comoving gauge. In particular the comoving curvature is constructed from Eqn. (42) as

$$\zeta = -\frac{\dot a}{a}\frac{v - \beta_F}{k}\,. \tag{55}$$

This gauge is most often used in inflationary calculations where $v - \beta_F$ is closely related to perturbations in the inflaton field.

3.7 Uniform Density Gauge:

Finally, one can eliminate the density perturbation $\delta\rho = 0$ with the choice

$$\begin{aligned}
H_T &= 0\,, \\
\zeta_\delta &\equiv H_L \\
B_\delta &\equiv B \\
A_\delta &\equiv A \\
T &= \frac{\delta\tilde\rho}{\dot\rho} \\
L &= -\tilde H_T/k
\end{aligned} \tag{56}$$

The Einstein equations become

$$\begin{aligned}
(k^2 - 3K)[\zeta_\delta + \frac{\dot a}{a}\frac{B_\delta}{k}] &= 12\pi G a^2\frac{\dot a}{a}(\rho+p)\frac{v - B_\delta}{k}\,, \\
\frac{\dot a}{a}A_\delta - \dot\zeta_c - \frac{K}{k}B_\delta &= 4\pi G a^2(\rho+p)\frac{v - B_\delta}{k}\,,
\end{aligned} \tag{57}$$

from which A_δ may be eliminated in favor of ζ_δ and B_δ. The conservation equations become

$$\left[\frac{d}{d\eta} + 3\frac{\dot a}{a}\right]\delta\rho_J + 3\frac{\dot a}{a}\delta p_J = -(\rho_J + p_J)(kv_J + 3\dot\zeta_\delta)\,,$$
$$\left[\frac{d}{d\eta} + 4\frac{\dot a}{a}\right](\rho_J + p_J)\frac{v_J - B_\delta}{k} = \delta p_J - \frac{2}{3}(1 - 3\frac{K}{k^2})p_J\Pi_J + (\rho_J + p_J)A_\delta\,. \tag{58}$$

Notice that the continuity equation for the net density perturbation becomes a conservation law

$$\dot\zeta_\delta = -\frac{\dot a}{a}\frac{\delta p}{\rho + p} - \frac{1}{3}kv\,. \tag{59}$$

Furthermore since $\delta\rho = 0$, δp is the non-adiabatic stress [see Eqn. (46)] and the conservation law resembles that of the comoving curvature [see Eqn. (41)]. More specifically, the two curvatures are related by

$$\zeta_\delta = \zeta + \frac{1}{3}\frac{\delta\tilde\rho}{(\rho + p)}\Big|_{\rm comoving}\,. \tag{60}$$

By the same argument as that following Eqn. (46), these two curvatures coincide outside the horizon if the stresses are adiabatic. Hence they are often used interchangeably in the literature. It bears an even simpler relationship to density fluctuations in the spatially flat gauge

$$\zeta_\delta = \frac{1}{3}\frac{\delta\tilde\rho}{(\rho + p)}\Big|_{\rm flat}\,. \tag{61}$$

For a single particle species $\delta\rho/(\rho + p) = \delta n/n$, the number density fluctuation. These simple relationships make ζ_δ and related variables useful for the consideration of relative number density or isocurvature perturbations in a multicomponent system.

4 Inflationary Perturbations

4.1 Horizon, Flatness, Relics Redux

Inflation was originally proposed to solve the horizon, flatness and relic problem: that the cosmic microwave background (CMB) temperature is isotropic across scales larger than Hubble length at recombination; that the spatial curvature scale is at least comparable to the current Hubble length; that the energy density is not dominated by defect relics from phase transitions in the early universe.

Measurements of fluctuations in the CMB imply a stronger version of the problems. They imply spatial curvature fluctuations above the Hubble length and rule out cosmological defects as the primary source of structure in the universe, i.e. they are absent as a significant component of the perturbations as well as the background. The conservation of the comoving curvature above the comoving Hubble length $(\dot a/a)^{-1} \ll 1$ leaves a paradox as to their origin. As long as the Hubble length monotonically increases, there will always be a time $\eta \sim k^{-1}$ before which ζ cannot have been generated dynamically. For sufficiently small k, this time exceeds the recombination epoch at $a \sim 10^{-3}$ where the CMB fluctuations are formed.

The comoving Hubble length decreases if ρ scales more slowly than a^{-2} or $w < -1/3$, i.e. if the expansion accelerates [see Eqn. (7-8)]. Vacuum energy or a cosmological constant can provide for acceleration but not for re-entry into the matter-radiation dominated expansion. On the other hand, a scalar field has both a potential and kinetic energy and provides a mechanism by which acceleration can end. A major success of the inflationary model is that the same mechanism that solves the horizon, flatness and relic problems by making the observed universe today much smaller than the Hubble length at the beginning of inflation *predicts* super-Hubble curvature perturbations from quantum fluctuations in the scalar field. The evolution of scalar field fluctuations can be usefully described in the gauge covariant framework.

4.2 Scalar Fields

The covariant perturbation theory described in the previous sections also applies to the inflationary universe. The stress-energy tensor of a scalar field rolling in the potential $V(\phi)$ is

$$T^{\mu}{}_{\nu} = \nabla^{\mu}\varphi\,\nabla_{\nu}\varphi - \frac{1}{2}(\nabla^{\alpha}\varphi\,\nabla_{\alpha}\varphi + 2V)\delta^{\mu}{}_{\nu}\,. \tag{62}$$

For the backround $\bar\phi \equiv \phi_0$ and

$$\rho_\phi = \frac{1}{2}a^{-2}\dot\phi_0^2 + V\,, \quad p_\phi = \frac{1}{2}a^{-2}\dot\phi_0^2 - V\,. \tag{63}$$

If the scalar field is kinetic energy dominated $w_\phi = p_\phi/\rho_\phi \to 1$, whereas if it is potential energy dominated $w_\phi = p_\phi/\rho_\phi \to -1$.

The equation of motion for the scalar field is simply the energy conservation equation

$$\dot\rho_\phi = -3(\rho_\phi + p_\phi)\frac{\dot a}{a}\,, \tag{64}$$

re-written in terms of ϕ_0 and V

$$\ddot\phi_0 + 2\frac{\dot a}{a}\dot\phi_0 + a^2 V' = 0\,. \tag{65}$$

Likewise for the perturbations $\phi = \phi_0 + \phi_1$

$$\begin{aligned}
\delta\rho_\phi &= a^{-2}(\dot\phi_0\dot\phi_1 - \dot\phi_0^2 A) + V'\phi_1\,,\\
\delta p_\phi &= a^{-2}(\dot\phi_0\dot\phi_1 - \dot\phi_0^2 A) - V'\phi_1\,,\\
(\rho_\phi + p_\phi)(v_\phi - B) &= a^{-2}k\dot\phi_0\phi_1\,,\\
p_\phi\pi_\phi &= 0\,,
\end{aligned} \tag{66}$$

the continuity equation implies

$$\ddot\phi_1 = -2\frac{\dot a}{a}\dot\phi_1 - (k^2 + a^2V'')\phi_1 + (\dot A - 3\dot H_L - kB)\dot\phi_0 - 2Aa^2V'\,, \tag{67}$$

while the Euler equation expresses an identity.

4.3 Gauge Choice

We are interested in inflation as a means of generating comoving curvature perturbations and so naively it would seem that inflationary perturbations should be calculated in the comoving gauge. However for a scalar field dominated universe, the comoving gauge sets $B = v = v_\phi$ and hence $\phi_1 = 0$. In the comoving gauge, the scalar field carries no perturbations by definition! This fact will be useful for treating scalar fields as a dark energy candidate in the next section: in the absence of scalar field fluctuations, energy density and pressure perturbations come purely from the kinetic terms so that $\delta p_\phi = \delta\rho_\phi$ yielding stable perturbations within the Hubble length.

The gauge covariant approach allows us to compute the comoving curvature from variables in another gauge. Since a scalar field transforms as a scalar field

$$\phi_1 = \tilde{\phi}_1 - \dot{\phi}_0 T \tag{68}$$

the comoving gauge is obtained from an arbitrary gauge by the time slicing change $T = \tilde{\phi}_1/\dot{\phi}_0$. The comoving curvature becomes

$$\begin{aligned}\zeta &= \tilde{H}_L - \frac{k}{3}L - \frac{\dot{a}}{a}T\,,\\ &= \tilde{H}_L + \frac{\tilde{H}_T}{3} - \frac{\dot{a}}{a}\frac{\tilde{\phi}_1}{\dot{\phi}_0}\,,\end{aligned} \tag{69}$$

and hence the simplest gauge from which to calculate the comoving curvature is one in which $H_L = H_T = 0$, i.e. the spatially flat gauge. In this case

$$\zeta = -\frac{\dot{a}}{a}\frac{\tilde{\phi}_1}{\dot{\phi}_0} \tag{70}$$

and so a calculation of $\tilde{\phi}_1$ trivially gives ζ. Notice that the proportionality constant involves $\dot{\phi}_0$. The slower the background field is rolling, the larger the curvature fluctuation implied by a given field fluctuation. The reason is that the time surfaces must be warped by a correspondingly larger amount to compensate $\tilde{\phi}_1$. We will hereafter drop the tildes and assume that the scalar field fluctuation applies to the spatially flat gauge.

4.4 Perturbation Evolution

In the spatially flat gauge the scalar field equation of motion can be written in a surprisingly compact form. Beginning with the spatially flat gauge equation

$$\ddot{\phi}_1 = -2\frac{\dot{a}}{a}\dot{\phi}_1 - (k^2 + a^2 V'')\phi_1 + (\dot{\alpha}_F - k\beta_F)\dot{\phi}_0 - 2\alpha_F a^2 V'\,.$$

the metric terms may be eliminated through Einstein equations

$$\begin{aligned}\alpha_F &= 4\pi G a^2 \left(\frac{\dot{a}}{a}\right)^{-1} (\rho_\phi + p_\phi)(v_\phi - \beta_F)/k\\ &= 4\pi G \left(\frac{\dot{a}}{a}\right)^{-1} \dot{\phi}_0 \phi_1\,,\end{aligned}$$

and ($k^2 \gg |K|$)

$$\begin{aligned}k\beta_F &= 4\pi G a^2 \left(\frac{\dot{a}}{a}\right)^{-1} \left[\delta\rho_\phi + 3\frac{\dot{a}}{a}(\rho_\phi + p_\phi)(v_\phi - \beta_F)/k\right]\\ &= 4\pi G \left[\left(\frac{\dot{a}}{a}\right)^{-1} (\dot{\phi}_0\dot{\phi}_1 + a^2 V'\phi_1) - \left(\frac{\dot{a}}{a}\right)^{-2} (4\pi G\dot{\phi}_0)^2 \dot{\phi}_0\phi_1 + 3\dot{\phi}_0\phi_1\right]\end{aligned} \tag{71}$$

so that α_F, $\dot{\alpha}_F - k\beta_F \propto \phi_1$ with proportionality that depends only on the background evolution, i.e. the Einstein and scalar field equations reduce to a single second order differential equation with the form of an expansion damped oscillator

$$\ddot{\phi}_1 + 2\frac{\dot{a}}{a}\dot{\phi}_1 + [k^2 + f(\eta)]\phi_1 \, . \tag{72}$$

The expression for $f(\eta)$ can be given explicitly in terms of the parameters

$$\epsilon \equiv \frac{3}{2}(1 + w_\phi) = \frac{\frac{3}{2}\dot{\phi}_0^2/a^2V}{1 + \frac{1}{2}\dot{\phi}_0^2/a^2V} \, , \tag{73}$$

which represents the deviation from a de Sitter expansion and

$$\delta \equiv \frac{\ddot{\phi}_0}{\dot{\phi}_0}\left(\frac{\dot{a}}{a}\right)^{-1} - 1 \, , \tag{74}$$

which represents the deviation from the overdamped limit of $d^2\phi_0/dt^2 = 0$, where $dt = ad\eta$. When small, these quantities are known as the slow-roll parameters. The Friedmann equations become

$$\begin{aligned} \left(\frac{\dot{a}}{a}\right)^2 &= 4\pi G\dot{\phi}_0^2\epsilon^{-1} \, , \\ \frac{d}{d\eta}\left(\frac{\dot{a}}{a}\right) &= \left(\frac{\dot{a}}{a}\right)^2 (1 - \epsilon) \, , \end{aligned} \tag{75}$$

and the background field equation becomes

$$\dot{\phi}_0 \frac{\dot{a}}{a}(3 + \delta) = -a^2 V' \, , \tag{76}$$

Together they yield the equation of motion for ϵ

$$\dot{\epsilon} = 2\epsilon(\delta + \epsilon)\frac{\dot{a}}{a} \, . \tag{77}$$

It is straightforward now to show that defining $u \equiv a\phi$ the perturbation equation becomes

$$\ddot{u} + [k^2 + g(\eta)]u = 0 \, , \tag{78}$$

where

$$\begin{aligned} g(\eta) \equiv f(\eta) + \epsilon - 2 &= -\left(\frac{\dot{a}}{a}\right)^2 [2 + 3\delta + 2\epsilon + (\delta + \epsilon)(\delta + 2\epsilon)] - \frac{\dot{a}}{a}\dot{\delta} \\ &= -\frac{\ddot{z}}{z} \end{aligned} \tag{79}$$

and

$$z \equiv a\left(\frac{\dot{a}}{a}\right)^{-1}\dot{\phi}_0 \, . \tag{80}$$

For any background field, Eqn. (78) can be solved to yield the evolution of the scalar field fluctuation.

4.5 Slow Roll Limit

In the slow roll limit $\epsilon, \delta \ll 1$ and the calculation simplifies dramatically. The slow roll parameters are usually written in terms of the derivatives of the potential

$$\begin{aligned} \epsilon &= \frac{\frac{3}{2}\dot\phi_0^2/a^2V}{1+\frac{1}{2}\dot\phi_0^2/a^2V} \approx \frac{1}{16\pi G}\left(\frac{V'}{V}\right)^2 , \\ \delta &= \frac{\ddot\phi_0}{\dot\phi_0}\left(\frac{\dot a}{a}\right)^{-1} - 1 \approx \epsilon - \frac{1}{8\pi G}\frac{V''}{V} . \end{aligned} \tag{81}$$

The slow roll limit requires a very flat potential.

In the slow roll limit, the perturbation equation is simply

$$\ddot u + [k^2 - 2\left(\frac{\dot a}{a}\right)^2]u = 0 . \tag{82}$$

With the conformal time measured from the end of inflation

$$\begin{aligned} \tilde\eta &= \eta - \eta_{\rm end} , \\ \tilde\eta &= \int_{a_{\rm end}}^{a} \frac{da}{Ha^2} \approx -\frac{1}{aH} , \end{aligned} \tag{83}$$

where $aH = \dot a/a$, the equation becomes

$$\ddot u + [k^2 - \frac{2}{\tilde\eta^2}]u = 0 . \tag{84}$$

This equation has the exact solution

$$u = A(k + \frac{i}{\tilde\eta})e^{-ik\tilde\eta} , \tag{85}$$

where A is a constant. For $|k\tilde\eta| \gg 1$ (early times, inside Hubble length) the field behaves as free oscillator

$$\lim_{|k\tilde\eta|\to\infty} u = Ake^{-ik\tilde\eta} . \tag{86}$$

For $|k\tilde\eta| \ll 1$ (late times, $\gg$ Hubble length), the fluctuation freezes in

$$\begin{aligned} \lim_{|k\tilde\eta|\to 0} u &= \frac{i}{\tilde\eta}A = iHaA , \\ \phi_1 &= iHA , \\ \zeta &= -iHA\left(\frac{\dot a}{a}\right)\frac{1}{\dot\phi_0} , \end{aligned}$$

which in the slow-roll approximation can be simplified through

$$\left(\frac{\dot a}{a}\right)^2\frac{1}{\dot\phi_0^2} = \frac{8\pi G a^2 V}{3}\frac{3}{2a^2V\epsilon} = \frac{4\pi G}{\epsilon} = \frac{4\pi}{m_{\rm pl}^2\epsilon} , \tag{87}$$

yielding the comoving curvature power spectrum

$$\Delta_\zeta^2 \equiv \frac{k^3|\zeta|^2}{2\pi^2} = \frac{2k^3}{\pi}\frac{H^2}{\epsilon m_{\rm pl}^2}A^2 . \tag{88}$$

All that remains is to set the normalization constant A through quantum fluctuations of the free oscillator.

4.6 Quantum Fluctuations

Inside the Hubble length, the classical equation of motion for u is the simple harmonic oscillator equation

$$\ddot{u} + k^2 u = 0\,, \tag{89}$$

which can be quantized as

$$\hat{u} = u(k,\eta)\hat{a} + u^*(k,\eta)\hat{a}^\dagger\,, \tag{90}$$

and normalized to zero point fluctuations in the Minkowski vacuum $[\hat{u}, d\hat{u}/d\eta] = i$,

$$u(k,\eta) = \frac{1}{\sqrt{2k}} e^{-ik\tilde{\eta}}\,. \tag{91}$$

Thus $A = (2k^3)^{-1/2}$ and curvature power spectrum becomes

$$\Delta_\zeta^2 \equiv \frac{1}{\pi}\frac{H^2}{\epsilon m_{\rm pl}^2}\,. \tag{92}$$

The curvature power spectrum is scale invariant to the extent that H is constant during inflation. Evolution in H produces a tilt in the spectrum

$$\begin{aligned} \frac{d\ln\Delta_\zeta^2}{d\ln k} &\equiv n_S - 1 \\ &= 2\frac{d\ln H}{d\ln k} - \frac{d\ln\epsilon}{d\ln k}\,, \end{aligned} \tag{93}$$

evaluated at Hubble crossing when the fluctuation freezes

$$\frac{d\ln H}{d\ln k}\Big|_{-k\tilde{\eta}=1} = \frac{k}{H}\frac{dH}{d\tilde{\eta}}\Big|_{-k\tilde{\eta}=1}\frac{d\tilde{\eta}}{dk}\Big|_{-k\tilde{\eta}=1} \tag{94}$$

$$= \frac{k}{H}(-aH^2\epsilon)\big|_{-k\tilde{\eta}=1}\frac{1}{k^2} = -\epsilon\,, \tag{95}$$

where $aH = -1/\tilde{\eta} = k$. Finally with

$$\frac{d\ln\epsilon}{d\ln k} = -\frac{d\ln\epsilon}{d\ln\tilde{\eta}} = -2(\delta+\epsilon)\frac{\dot{a}}{a}\tilde{\eta} = 2(\delta+\epsilon)\,, \tag{96}$$

the tilt becomes

$$n_S = 1 - 4\epsilon - 2\delta \tag{97}$$

in the slow-roll approximation.

4.7 Gravitational Waves

Any nearly massless degree of freedom will acquire quantum fluctuations during inflation. The inflaton is only special in that it carries the energy density of the universe. Other degrees of freedom result in isocurvature perturbations. In particular consider the gravitational wave degrees of freedom. Their classical equation of motion resembles the scalar field equation

$$\ddot{H}_T^{(\pm2)} + 2\frac{\dot{a}}{a}\dot{H}_T^{(\pm2)} + k^2 H_T^{(\pm2)} = 0\,, \tag{98}$$

and hence acquires fluctuations in same manner as ϕ. Setting the normalization with

$$\phi_1 \to H_T^{(\pm 2)} \sqrt{\frac{3}{16\pi G}} \,, \tag{99}$$

the gravitational wave power spectrum in each component (polarization components $H_T^{(\pm 2)} = (h_+ \pm i h_\times)/\sqrt{6}$) is

$$\Delta_H^2 = \frac{16\pi G}{3 \cdot 2\pi^2} \frac{H^2}{2} = \frac{4}{3\pi} \frac{H^2}{m_{\rm pl}^2} \,. \tag{100}$$

The gravitational wave fluctuations are scale invariant with power $\propto H^2 \propto V \propto E_i^4$ where E_i is the energy scale of inflation. A detection of gravitational waves from inflation would yield a measurement of the energy scale of inflation.

Finally the tensor tilt

$$\frac{d \ln \Delta_H^2}{d \ln k} \equiv n_T = 2 \frac{d \ln H}{d \ln k} = -2\epsilon \,, \tag{101}$$

which yields a consistency relation between tensor-scalar ratio and tensor tilt

$$\frac{\Delta_H^2}{\Delta_\zeta^2} = \frac{4}{3}\epsilon = -\frac{2}{3} n_T \,. \tag{102}$$

5 Dark Matter and Energy

5.1 Degrees of Freedom

A dark component interacts with ordinary matter only through gravity and hence its observable properties are completely specified by the degrees of freedom in its stress-energy tensor. We have seen that without loss of generality these can be taken as the elements of the symmetric 3×3 stress tensor. Two of these elements represent the scalar degrees of freedom that influence the formation of structure. In the background only one of these remain due to isotropy leaving only the pressure p_D of the dark component or equivalently the equation of state $w_D = p_D/\rho_D$. Let us generalize the concept of equations of state to the fluctuations in the stress tensor.

5.2 Generalized Equations of State

It is convenient to separate out the non-adiabatic stress or entropy contribution

$$p_D \Gamma_D = \delta p_D - c_{Da}^2 \delta \rho_D \,, \tag{103}$$

where the adiabatic sound speed is

$$c_{Da}^2 \equiv \frac{\delta p}{\delta \rho}\Big|_{\Gamma_D = 0} = \frac{\dot p_D}{\dot \rho_D} = w_D - \frac{1}{3} \frac{\dot w_D}{1 + w_D} \left(\frac{\dot a}{a} \right)^{-1} \,. \tag{104}$$

Therefore, $p_D = w_D \rho_D$ does *not* imply $\delta p_D = w_D \delta \rho_D$ and furthermore if $\Gamma_D \neq 0$, the function $w_D(a)$ does not completely specify the pressure fluctuation. For the dark energy, $w_D < 0$ and is slowly-varying compared with the expansion rate $(\dot a/a)$ such that $c_{Da}^2 < 0$ The adiabatic pressure fluctuation produces accelerated collapse rather than support for the density perturbation. Therefore a dynamical dark energy component must have substantial non-adiabatic stress fluctuations $\Gamma_D \neq 0$ to be phenomenologically viable.

One cannot simply parameterize the pressure fluctuation by a non-adiabatic sound speed $\delta p_D/\delta\rho_D$ since this is not a gauge invariant quantity and the non-adiabatic stress fluctuation is. The gauge covariant approach allows us to define the equation of state covariantly.

As we have seen the comoving gauge gives the closest general relativistic analogue to non-relativistic physics. The generalization of the comoving gauge to an individual dark component has the conditions

$$B = v_D\,, \qquad H_T = 0\,, \tag{105}$$

which we will call the rest gauge since $T^0_i = 0$. Taking

$$\left.\frac{\delta p}{\delta\rho}\right|_{\rm rest} = c_D^2\,, \tag{106}$$

we obtain via gauge transformation

$$\begin{aligned} \delta p_D &= c_D^2\delta\rho_D - (c_D^2\dot\rho_D - \dot p_D)\frac{v_D - B}{k} \\ &= c_D^2\delta\rho_D + 3(1+w_D)\frac{\dot a}{a}\rho_D(c_D^2 - c_{Da}^2)\frac{v_D - B}{k}\,, \end{aligned} \tag{107}$$

yielding a manifestly gauge invariant non-adiabatic stress

$$p_D\Gamma_D = (c_D^2 - c_{Da}^2)\left[\delta\rho_D + 3(1+w_D)\frac{\dot a}{a}\rho_D\frac{v_D - B}{k}\right]\,. \tag{108}$$

The anisotropic stress can also affect the density perturbations. A familiar example is that of a fluid, where it represents viscosity and damps density perturbations. More generally, the anisotropic stress component is the amplitude of a 3-tensor that is linear in the perturbation. A natural choice for its source is kv_D, the amplitude of the velocity shear tensor $\partial^i v_D^j$. However it must also be gauge invariant and generated by the corresponding shear term in the metric fluctuation H_T. Gauge transforming from a gauge with $H_T = 0$ yields an invariant source of $kv_D - \dot H_T$. These requirements are satisfied by

$$\left(\frac{d}{d\eta} + 3\frac{\dot a}{a}\right) w_D\pi_D = 4c_{Dv}^2(kv_D - \dot H_T)\,. \tag{109}$$

5.3 Examples

Cold dark matter provides a trivial example of a dark component. Here stress fluctuations are negligible compared with the rest energy density and so $w_{CDM} = c_D^2 = c_{Dv}^2 = 0$. Scalar field dark energy provides another simple example. Here the competition between kinetic and potential energy can drive $w_\phi < 0$. As we have already seen a slowly rolling scalar field has $c_\phi^2 = 1$ by virtue of the absence of scalar field fluctuations in the rest gauge. In this case the fluctuations bear only the kinetic contributions and so the relationship is exact.

The non-adiabatic sound speed c_D^2 is also useful in characterizing k-essence, a scalar field with a non-canonical kinetic term. Since in the rest gauge the potential fluctuation vanishes, the sound speed directly reflects the modification to the kinetic term. A special case occurs when the kinetic

Type	w_D	c_D^2	c_{Dv}^2
CDM	0	0	0
Λ	-1	–	–
Massless ν	1/3	1/3	1/3
Tight coupled γ	1/3	1/3	0
Hot/Warm DM	$1/3 \to 0$	same	same
Quintessence	variable	1	0
k-essence	variable	variable	0
Phantom energy	< -1	1	0
Decaying ν	$1/3 \to 0 \to 1/3$	same	same
Axions	0	small	0
Fuzzy Dark Matter	0	scale dependent	0

term has the wrong sign. Here $w_\phi < -1$ and the energy density increases as the universe expands. This type of matter has been dubbed phantom dark energy.

The viscosity term is relevant for dark radiation components. Massless neutrinos can be approximated by $w_\nu = c_\nu^2 = c_{\nu v}^2 = 1/3$. A massive neutrino has an equation of state that goes from $w_{m\nu} = 1/3$ to 0 as the neutrinos become non-relativistic. Fitting to the numerical integration of the distribution gives

$$w_{m\nu} = \frac{1}{3}\left[1 + (a/a_{\rm nr})^{2p}\right]^{-1/p}, \tag{110}$$

with $p = 0.872$ and $a_{\rm nr} = 6.32 \times 10^{-6}/\Omega_\nu h^2$. We can model its behavior as $c_{m\nu}^2 = c_{m\nu v}^2 = w_D$.

A summary of the phenomenological parameterization of various particle candidates are given in the table.

5.4 Initial Conditions

With the equations of motion for the dark components defined, all that remains is to specify the initial conditions. Conservation of the comoving curvature outside the Hubble length allows us to ignore the microphysics of the intermediate reheating phase between inflation and the radiation dominated universe. One can then simply take the initial conditions for structure formation to be in the radiation dominated universe. Accounting for the neutrino anisotropic stress as described in the previous section, the initial conditions for the total perturbations in the comoving gauge become

$$\begin{aligned}
\delta &= A_\delta (k\eta)^2 \zeta\,,\\
v &= A_v (k\eta)\zeta\,,\\
\Pi &= A_{\Pi\nu} f_\nu (k\eta)^2 \zeta\,,
\end{aligned} \tag{111}$$

where $\delta = \delta\rho/\rho$, and η is now understood as the conformal time elapsed after the inflationary epoch. The constants are

$$\begin{aligned} A_\delta &= \frac{4}{9}\frac{1+2f_\nu/5}{1+4f_\nu/15}(1-3K/k^2)\,, \\ A_v &= -\frac{1}{3}\frac{1}{1+4f_\nu/15}\,, \\ A_{\Pi\nu} &= -\frac{4}{15}\frac{1}{1+4f_\nu/15}\,, \end{aligned} \tag{112}$$

and

$$f_\nu = \frac{\rho_\nu}{\rho_\gamma+\rho_\nu}\,. \tag{113}$$

The dark component initial conditions can then be determined by detailed balance

$$\begin{aligned} \delta_J &= A_{\delta J}(k\eta)^2\zeta \\ v_J - v &= A_{\Delta v J}(k\eta)^3\zeta\,, \\ \Pi_J &= A_{\Pi J}(k\eta)^2\zeta \end{aligned} \tag{114}$$

with

$$\begin{aligned} A_{\Pi J} &= \frac{c_{Jv}^2}{w_J}A_{\Pi\nu}\,, \\ A_{\Delta v J} &= \frac{\frac{2}{15}[2-3(w_J-c_J^2)](f_\nu - \frac{4}{1+w_J}c_{Jv}^2) - [2(c_J^2-\frac{1}{3}) + (w_J-c_J^2)](1+\frac{2}{5}f_\nu)}{(4-3c_J^2)[2-3(w_J-c_J^2)] + 9(c_J^2-c_{Ja}^2)c_J^2}(1-3\frac{K}{k^2})A_v\,, \\ A_{\delta J} &= \frac{3}{4}(1+w_J)\frac{A_\delta - 6(c_J^2-c_{Ja}^2)A_{\Delta v J}}{1-3(w_J-c_J^2)/2}\,. \end{aligned} \tag{115}$$

These initial conditions apply to all particle species J, including the photons and neutrinos, save that for the baryons rapid Thomson scattering with the photons sets $v_b = v_\gamma$. In fact for most numerical purposes one can simply set $v_J - v = 0$ in the initial conditions and let the velocity differences arise dynamically in the integration. The initial conditions in an arbitrary gauge can be established from these relations and the gauge transformation properties of the perturbations.

6 Cosmic Microwave Background

6.1 Boltzmann Equation

The gauge covariant formalism is useful for CMB studies as well. The interpretation of CMB anisotropy formation is simplest in the Newtonian gauge where the manifestations of gravitational redshift and infall correspond to Newtonian intuition. The numerical solution of the perturbation equations is best handled under the comoving or synchronous gauge where the fundamental perturbation variables are stable. The link to inflationary initial conditions is best seen in the comoving gauge. The gauge covariant approach allows one to calculate in one gauge and interpret in or relate to another.

Most of the physics of CMB perturbation evolution is contained in the general discussion of §2. However the cosmic microwave background differs from the dark components in that it undergoes interactions with the baryons that can exchange energy and momentum. Thus the conservation law for its stress energy tensor must be supplemented with an interaction term. CMB observable properties are also not the energy density and velocity perturbations but the higher order angular distribution of its temperature and polarization.

Generally the CMB is described by the phase space distribution function photons in each of the two polarization states $f_a(\mathbf{x}, \mathbf{q}, \eta)$, where $\mathbf{x}$ is the comoving position and $\mathbf{q}$ is the photon momentum. The evolution of the distribution function under gravity and collisions is governed by the Boltzmann equation

$$\frac{d}{d\eta} f_{a,b} = C[f_a, f_b]\,, \tag{116}$$

where C denotes the collision term.

In absence of collisions, the Boltzmann equation becomes the Liouville equation. Rewriting the variables in terms of the photon propagation direction $\hat{\mathbf{n}}$,

$$\frac{d}{d\eta} f_a(\mathbf{x}, \hat{\mathbf{n}}, q, \eta) = \dot{f}_a + n^i \nabla_i f_a + \dot{q} \frac{\partial}{\partial q} f_a = 0\,. \tag{117}$$

The last term represents the gravitational redshifting (or Sachs-Wolfe effect) of the photons under the metric and is given by the geodesic equation as

$$\frac{\dot{q}}{q} = -\frac{\dot{a}}{a} - \frac{1}{2} n^i n^j \dot{H}_{Tij} - \dot{H}_L + n^i \dot{B}_i - n^i \nabla_i A\,. \tag{118}$$

We have in fact already derived these gravitational effects in the general covariant perturbation formalism. To establish this fact note that the stress energy tensor of the photons is the integral of the photon distribution function over momentum states

$$T^{\mu\nu} = \int \frac{d^3 q}{(2\pi)^3} \frac{q^\mu q^\nu}{E} (f_a + f_b)\,. \tag{119}$$

The Liouville equation then expresses the conservation of the stress-energy tensor. Given that the CMB distribution is observed to be close to blackbody, it is suffices to calculate the evolution of these integrated quantities. In particular let us define the temperature perturbation

$$\Theta(\mathbf{x}, \hat{\mathbf{n}}, \eta) = \frac{1}{4} \delta_\gamma = \frac{1}{4\rho_\gamma} \int \frac{q^3 dq}{2\pi^2} (f_a + f_b) - 1 \tag{120}$$

and likewise for the linear polarization states Q and U as the temperature differences between the polarization states. The redshift terms then become the metric terms in the continuity and Navier-Stokes equations.

6.2 Eigenmodes

As in the case of the purely spatial perturbations, we decompose the temperature and polarization in normal modes of the spatial and angular distributions. For the k-th spatial eigenmode

$$\begin{aligned} \Theta(\mathbf{x}, \hat{\mathbf{n}}, \eta) &= \sum_{\ell m} \Theta_\ell^{(m)} G_\ell^m(\mathbf{k}, \mathbf{x}, \hat{\mathbf{n}})\,, \\ [Q \pm iU](\mathbf{x}, \hat{\mathbf{n}}, \eta) &= \sum_{\ell m} [E_\ell^{(m)} \pm i B_\ell^{(m)}]_{\pm 2} G_\ell^m(\mathbf{k}, \mathbf{x}, \hat{\mathbf{n}})\,. \end{aligned} \tag{121}$$

They are generated by recursion (${}_0G = G$)

$$n^i({}_sG_\ell^m)_{|i} = \frac{q}{2\ell+1}\left[{}_s\kappa_\ell^m({}_sG_{\ell-1}^m) - {}_s\kappa_{\ell+1}^m({}_sG_{\ell+1}^m)\right] - i\frac{qms}{\ell(\ell+1)}\,{}_sG_\ell^m\,, \tag{122}$$

where

$$\begin{aligned} q^2 &= k^2 + (|m|+1)K\,, \\ {}_s\kappa_\ell^m &= \sqrt{\left[\frac{(\ell^2-m^2)(\ell^2-s^2)}{\ell^2}\right]\left[1-\frac{\ell^2}{q^2}K\right]}. \end{aligned} \tag{123}$$

The lowest order modes begin the recursion and are related to the spatial harmonics as

$$\begin{aligned} {}_0G_j^m &= n^{i_1}\ldots n^{i_{|m|}}Q_{i_1\ldots i_{|m|}}^{(m)}\,, \\ {}_{\pm2}G_2^m &\propto (\hat{m}_1 \pm i\hat{m}_2)^{i_1}(\hat{m}_1 \pm i\hat{m}_2)^{i_2}Q_{i_1 i_2}^{(m)}\,, \end{aligned} \tag{124}$$

where $\hat{\mathbf{m}}_1$, $\hat{\mathbf{m}}_2$ span the plane perpendicular to $\hat{\mathbf{n}}$. The normalization is set so that

$${}_sG_\ell^m(0,0,\hat{\mathbf{n}}) = (-i)^\ell\sqrt{\frac{4\pi}{2\ell+1}}\,{}_sY_\ell^m(\hat{\mathbf{n}})\,. \tag{125}$$

Here the spin spherical harmonics ${}_sY_\ell^m$ are the eigenfunctions of the 2D Laplace operator on a rank s tensor. They are given explicitly by rotation matrices as

$${}_sY_\ell^m(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}}\mathcal{D}_{-ms}^\ell(\phi,\theta,0)\,. \tag{126}$$

The meaning of these modes becomes clear in a spatially flat cosmology. Here the modes are simply the direct product of plane waves and spin-spherical harmonics

$$\begin{aligned} G_\ell^m(\mathbf{k},\mathbf{x},\hat{\mathbf{n}}) &\equiv (-i)^\ell\sqrt{\frac{4\pi}{2\ell+1}}Y_\ell^m(\hat{\mathbf{n}})\exp(i\mathbf{k}\cdot\mathbf{x})\,, \\ {}_{\pm2}G_\ell^m(\mathbf{k},\mathbf{x},\hat{\mathbf{n}}) &\equiv (-i)^\ell\sqrt{\frac{4\pi}{2\ell+1}}{}_{\pm2}Y_\ell^m(\hat{\mathbf{n}})\exp(i\mathbf{k}\cdot\mathbf{x})\,. \end{aligned} \tag{127}$$

The main content of the Liouville equation is purely geometrical and describes the projection of inhomogeneities into anisotropies. Photon propagation takes gradients in the spatial distribution and converts them to anisotropy as

$$\hat{\mathbf{n}}\cdot\nabla e^{i\mathbf{k}\cdot\mathbf{x}} = i\hat{\mathbf{n}}\cdot\mathbf{k}e^{i\mathbf{k}\cdot\mathbf{x}} = i\sqrt{\frac{4\pi}{3}}kY_1^0(\hat{\mathbf{n}})e^{i\mathbf{k}\cdot\mathbf{x}}\,. \tag{128}$$

This dipole term adds to angular dependence through the addition of angular momentum

$$\sqrt{\frac{4\pi}{3}}Y_1^0Y_\ell^m = \frac{\kappa_\ell^m}{\sqrt{(2\ell+1)(2\ell-1)}}Y_{\ell-1}^m + \frac{\kappa_{\ell+1}^m}{\sqrt{(2\ell+1)(2\ell+3)}}Y_{\ell+1}^m\,, \tag{129}$$

where $\kappa_\ell^m = {}_s\kappa_\ell^m = \sqrt{\ell^2-m^2}$ is related to the Clebsch-Gordon coefficients.

6.3 Collision Term

The dominant collision process for CMB photons is Thomson scattering off of free electrons which has the differential cross section

$$\frac{d\sigma}{d\Omega} = \frac{3}{8\pi}|\hat{\mathbf{E}}' \cdot \hat{\mathbf{E}}|^2 \sigma_T \,, \tag{130}$$

where $\hat{\mathbf{E}}'$ and $\hat{\mathbf{E}}$ denote the incoming and outgoing directions of the electric field or polarization vector.

To evaluate the collision term we begin in the electron rest frame and in a coordinate system fixed by the scattering plane, spanned by incoming and outgoing directional vectors $-\hat{\mathbf{n}}' \cdot \hat{\mathbf{n}} = \cos\beta$, where β is the scattering angle. Denoting $\Theta_{\parallel}$ as the in-plane polarization temperature fluctuation and $\Theta_{\perp}$ as the perpendicular polarization state, we obtain the geometrical content of the transfer equation

$$\Theta_{\parallel} \propto \cos^2\beta\, \Theta'_{\parallel}, \qquad \Theta_{\perp} \propto \Theta'_{\perp} \,, \tag{131}$$

where the proportionality reflects the scattering rate

$$\dot{\tau} = n_e \sigma_T a \,. \tag{132}$$

To calculate the Stokes parameters in this basis, we also need to calculate the polarization states with axes rotated by 45°

$$\hat{\mathbf{E}}_1 = \frac{1}{\sqrt{2}}(\hat{\mathbf{E}}_{\parallel} + \hat{\mathbf{E}}_{\perp}) \,, \qquad \hat{\mathbf{E}}_2 = \frac{1}{\sqrt{2}}(\hat{\mathbf{E}}_{\parallel} - \hat{\mathbf{E}}_{\perp}) \,, \tag{133}$$

yielding the transfer

$$\begin{aligned} \Theta_1 &\propto |\hat{\mathbf{E}}_1 \cdot \hat{\mathbf{E}}_1|^2 \Theta'_1 + |\hat{\mathbf{E}}_1 \cdot \hat{\mathbf{E}}_2|^2 \Theta'_2 \\ &\propto \frac{1}{4}(\cos\beta + 1)^2 \Theta'_1 + \frac{1}{4}(\cos\beta - 1)^2 \Theta'_2 \\ \Theta_2 &\propto |\hat{\mathbf{E}}_2 \cdot \hat{\mathbf{E}}_2|^2 \Theta'_2 + |\hat{\mathbf{E}}_2 \cdot \hat{\mathbf{E}}_1|^2 \Theta'_1 \\ &\propto \frac{1}{4}(\cos\beta + 1)^2 \Theta'_2 + \frac{1}{4}(\cos\beta - 1)^2 \Theta'_1 \,. \end{aligned} \tag{134}$$

Now the transfer properties of the Stokes parameters

$$\Theta \equiv \frac{1}{2}(\Theta_{\parallel} + \Theta_{\perp}), \quad Q \equiv \frac{1}{2}(\Theta_{\parallel} - \Theta_{\perp}), \quad U \equiv \frac{1}{2}(\Theta_1 - \Theta_2) \tag{135}$$

arranged in a vector $\mathbf{T} \equiv (\Theta,\, Q + iU,\, Q - iU)$ becomes

$$\mathbf{T} \propto \mathbf{S}(\beta)\mathbf{T}' \,, \tag{136}$$

$$\mathbf{S}(\beta) = \frac{3}{4} \begin{pmatrix} \cos^2\beta + 1 & -\frac{1}{2}\sin^2\beta & -\frac{1}{2}\sin^2\beta \\ -\frac{1}{2}\sin^2\beta & \frac{1}{2}(\cos\beta + 1)^2 & \frac{1}{2}(\cos\beta - 1)^2 \\ -\frac{1}{2}\sin^2\beta & \frac{1}{2}(\cos\beta - 1)^2 & \frac{1}{2}(\cos\beta + 1)^2 \end{pmatrix} ,$$

where the normalization factor is set by photon conservation in the scattering.

Finally convert the the polarization quantities referenced to the scattering basis to a fixed basis on the sky by noting that under a rotation $\mathbf{T}' = \mathbf{R}(\psi)\mathbf{T}$ where

$$\mathbf{R}(\psi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-2i\psi} & 0 \\ 0 & 0 & e^{2i\psi} \end{pmatrix}, \tag{137}$$

giving the scattering matrix

$$\mathbf{R}(-\gamma)\mathbf{S}(\beta)\mathbf{R}(\alpha) = \frac{1}{2}\sqrt{\frac{4\pi}{5}} \begin{pmatrix} Y_2^0(\beta,\alpha) + 2\sqrt{5}Y_0^0(\beta,\alpha) & -\sqrt{\frac{3}{2}}Y_2^{-2}(\beta,\alpha) & -\sqrt{\frac{3}{2}}Y_2^{2}(\beta,\alpha) \\ -\sqrt{6}\,{}_2Y_2^0(\beta,\alpha)e^{2i\gamma} & 3\,{}_2Y_2^{-2}(\beta,\alpha)e^{2i\gamma} & 3\,{}_2Y_2^{2}(\beta,\alpha)e^{2i\gamma} \\ -\sqrt{6}\,{}_{-2}Y_2^0(\beta,\alpha)e^{-2i\gamma} & 3\,{}_{-2}Y_2^{-2}(\beta,\alpha)e^{-2i\gamma} & 3\,{}_{-2}Y_2^{2}(\beta,\alpha)e^{-2i\gamma} \end{pmatrix}, \tag{138}$$

where α, γ are the angles required to rotate into and out of the scattering frame.

Finally, by employing the addition theorem for spin spherical harmonics

$$\sum_m {}_{s_1}Y_\ell^{m*}(\hat{\mathbf{n}}')\ {}_{s_2}Y_\ell^{m}(\hat{\mathbf{n}}) = (-1)^{s_1-s_2}\sqrt{\frac{2\ell+1}{4\pi}}\ {}_{s_2}Y_\ell^{-s_1}(\beta,\alpha)e^{is_2\gamma} \tag{139}$$

the scattering in the electron rest frame into the Stokes states becomes

$$\begin{aligned} C_{\rm in}[\mathbf{T}] &= \dot\tau \int \frac{d\hat{\mathbf{n}}'}{4\pi}\mathbf{R}(-\gamma)\mathbf{S}(\beta)\mathbf{R}(\alpha)\mathbf{T}(\hat{\mathbf{n}}') \\ &= \dot\tau \int \frac{d\hat{\mathbf{n}}'}{4\pi}(\Theta', 0, 0) + \frac{1}{10}\dot\tau \int d\hat{\mathbf{n}}' \sum_{m=-2}^{2} \mathbf{P}^{(m)}(\hat{\mathbf{n}}, \hat{\mathbf{n}}')\mathbf{T}(\hat{\mathbf{n}}'), \end{aligned} \tag{140}$$

where the quadrupole coupling term is

$$\mathbf{P}^{(m)}(\hat{\mathbf{n}}, \hat{\mathbf{n}}') = \begin{pmatrix} Y_2^{m*}(\hat{\mathbf{n}}')\,Y_2^m(\hat{\mathbf{n}}) & -\sqrt{\frac{3}{2}}\,{}_2Y_2^{m*}(\hat{\mathbf{n}}')\,Y_2^m(\hat{\mathbf{n}}) & -\sqrt{\frac{3}{2}}\,{}_{-2}Y_2^{m*}(\hat{\mathbf{n}}')\,Y_2^m(\hat{\mathbf{n}}) \\ -\sqrt{6}Y_2^{m*}(\hat{\mathbf{n}}')\,{}_2Y_2^m(\hat{\mathbf{n}}) & 3\,{}_2Y_2^{m*}(\hat{\mathbf{n}}')\,{}_2Y_2^m(\hat{\mathbf{n}}) & 3\,{}_{-2}Y_2^{m*}(\hat{\mathbf{n}}')\,{}_2Y_2^m(\hat{\mathbf{n}}) \\ -\sqrt{6}Y_2^{m*}(\hat{\mathbf{n}}')\,{}_{-2}Y_2^m(\hat{\mathbf{n}}) & 3\,{}_2Y_2^{m*}(\hat{\mathbf{n}}')\,{}_{-2}Y_2^m(\hat{\mathbf{n}}) & 3\,{}_{-2}Y_2^{m*}(\hat{\mathbf{n}}')\,{}_{-2}Y_2^m(\hat{\mathbf{n}}) \end{pmatrix}. \tag{141}$$

The full scattering matrix involves difference of scattering into and out of state

$$C[\mathbf{T}] = C_{\rm in}[\mathbf{T}] - C_{\rm out}[\mathbf{T}]. \tag{142}$$

In the electron rest frame

$$C[\mathbf{T}] = \dot\tau \int \frac{d\hat{\mathbf{n}}'}{4\pi}(\Theta', 0, 0) - \dot\tau\mathbf{T} + C_P[\mathbf{T}] \tag{143}$$

which describes isotropization in the rest frame. All moments have $e^{-\tau}$ suppression except for isotropic temperature Θ_0. Here C_P is the P or $\ell = 2$ term of Eqn. (140).

Transformation into the background frame simply induces a dipole term

$$C[\mathbf{T}] = \dot\tau\left(\hat{\mathbf{n}}\cdot\mathbf{v}_b + \int \frac{d\hat{\mathbf{n}}'}{4\pi}\Theta', 0, 0\right) - \dot\tau\mathbf{T} + C_P[\mathbf{T}], \tag{144}$$

yielding the final form of the collision term.

6.4 Temperature-Polarization Hierarchy

The Boltzmann equation in normal modes then becomes

$$\begin{aligned}
\dot\Theta_\ell^{(m)} &= q\left[\frac{\kappa_\ell^m}{2\ell+1}\Theta_{\ell-1}^{(m)} - \frac{\kappa_{\ell+1}^m}{2\ell+3}\Theta_{\ell+1}^{(m)}\right] - \dot\tau\Theta_\ell^{(m)} + S_\ell^{(m)}\,,\\
\dot E_\ell^{(m)} &= k\left[\frac{{}_2\kappa_\ell^m}{2\ell-1}E_{\ell-1}^{(m)} - \frac{2m}{\ell(\ell+1)}B_\ell^{(m)} - \frac{{}_2\kappa_{\ell+1}^m}{2\ell+3}\right] - \dot\tau E_\ell^{(m)} + \mathcal{E}_\ell^{(m)}\,,\\
\dot B_\ell^{(m)} &= k\left[\frac{{}_2\kappa_\ell^m}{2\ell-1}B_{\ell-1}^{(m)} + \frac{2m}{\ell(\ell+1)}B_\ell^{(m)} - \frac{{}_2\kappa_{\ell+1}^m}{2\ell+3}\right] - \dot\tau E_\ell^{(m)} + \mathcal{B}_\ell^{(m)}\,,
\end{aligned} \tag{145}$$

where the gravitational and scattering sources are

$$\begin{aligned}
S_\ell^{(m)} &= \begin{pmatrix} \dot\tau\Theta_0^{(0)} - \dot H_L^{(0)} & \dot\tau v_b^{(0)} + kA^{(0)} + \dot B^{(0)} & \dot\tau P^{(0)} - \frac{2}{3}\sqrt{1-3K/k^2}\dot H_T^{(0)} \\ 0 & \dot\tau v_b^{(\pm1)} + \dot B^{(\pm1)} & \dot\tau P^{(\pm1)} - \frac{\sqrt{3}}{3}\sqrt{1-2K/k^2}\dot H_T^{(\pm1)} \\ 0 & 0 & \dot\tau P^{(\pm2)} - \dot H_T^{(\pm2)} \end{pmatrix}\,,\\
\mathcal{E}_\ell^{(m)} &= -\dot\tau\sqrt{6}P^{(m)}\delta_{\ell,2}\,,\\
\mathcal{B}_\ell^{(m)} &= 0\,,
\end{aligned} \tag{146}$$

with

$$P^{(m)} \equiv \frac{1}{10}(\Theta_2^{(m)} - \sqrt{6}E_2^{(m)})\,. \tag{147}$$

The physical content of the coupling hierarchy is that an inhomogeneity in the temperature or polarization distribution will eventually become a high multipole order anisotropy by “free streaming” or simple projection.

6.5 Integral Solution

Since the hierarchy equations simply represents geometric projection, their implicit solution can be written as the projection of the gravitational and scattering sources at a distance. This operation proceeds by writing the normal modes themselves in spherical coordinates,

$$\begin{aligned}
G_{\ell_s}^m &= \sum_\ell (-i)^\ell\sqrt{4\pi(2\ell+1)}\alpha_{\ell_s\ell}^{(m)}(k,D)Y_\ell^m(\hat{\mathbf{n}})\,,\\
{}_{\pm2}G_{\ell_s}^m &= \sum_\ell (-i)^\ell\sqrt{4\pi(2\ell+1)}[\epsilon_{\ell_s\ell}^{(m)} \pm \beta_{\ell_s\ell}^{(m)}](k,D)\,{}_{\pm2}Y_\ell^m(\hat{\mathbf{n}})\,,
\end{aligned} \tag{148}$$

where $D = \eta_0 - \eta$. Summing over the sources

$$\begin{aligned}
\frac{\Theta_\ell^{(m)}(k,\eta_0)}{2\ell+1} &= \int_0^{\eta_0} d\eta e^{-\tau}\sum_{\ell_s} S_{\ell_s}^{(m)}\alpha_{\ell_s\ell}^{(m)}(k,D)\,,\\
\frac{E_\ell^{(m)}(k,\eta_0)}{2\ell+1} &= \int_0^{\eta_0} d\eta e^{-\tau}\sum_{\ell_s} \mathcal{E}_{\ell_s}^{(m)}\epsilon_{\ell_s\ell}^{(m)}(k,D)\,,\\
\frac{B_\ell^{(m)}(k,\eta_0)}{2\ell+1} &= \int_0^{\eta_0} d\eta e^{-\tau}\sum_{\ell_s} \mathcal{E}_{\ell_s}^{(m)}\beta_{\ell_s\ell}^{(m)}(k,D)\,.
\end{aligned} \tag{149}$$

Note that the polarization has only an $\ell_s = 2$ source.

In a flat cosmology, the radial projection kernels are related to spherical Bessel functions

$$e^{i\mathbf{k}\cdot\mathbf{x}} = \sum_\ell (-i)^\ell\sqrt{4\pi(2\ell+1)}j_\ell(kD)Y_\ell^0(\hat{\mathbf{n}}) \tag{150}$$

by the recoupling of the "spin" angular dependence of the source ℓ_s to the "orbital" angular dependence of the plane waves. For example

$$\begin{aligned}
\alpha^{(0)}_{0\ell}(k,D) &\equiv j_\ell(kD)\,,\\
\alpha^{(0)}_{1\ell}(k,D) &\equiv j'_\ell(kD)\,,\\
\epsilon^{(0)}_{2\ell}(k,D) &= \sqrt{\frac{3}{8}\frac{(\ell+2)!}{(\ell-2)!}}\frac{j_\ell(kD)}{(kD)^2}\,,\\
\beta^{(0)}_{2\ell}(k,D) &= 0\,.
\end{aligned} \tag{151}$$

In a curved geometry, the radial projection kernels are related to the ultraspherical Bessel functions by the same coupling of angular momenta.

6.6 Power Spectra

The two point statistics of the temperature and polarization fields are described by their power spectra

$$C_\ell^{XX'} = \frac{2}{\pi}\int \frac{dk}{k}\sum_m \frac{k^3\langle X_\ell^{(m)*} X_\ell^{'(m)}\rangle}{(2\ell+1)^2}\,, \tag{152}$$

where $X, X' \in \Theta, E, B$.

7 Epilogue

How to conclude *Lecture notes* in which no conclusions have been drawn? I leave you instead with a thought:

> What goes on being hateful about analysis is that it implies that the analyzed is a completed set. The reason why completion goes on being hateful is that it implies everything can be a completed set.
>
> –Chuang-tzu, 23

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