

Stability of cosmological solutions in extended quasidilaton massive gravity

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(Dated: August 22, 2014)

We consider the stability of self-accelerating solutions to extended quasidilaton massive gravity in the presence of matter. By making a second or extended fiducial metric dynamical in this model, matter can cause it to evolve from a Lorentzian to Euclidean signature, triggering a ghost instability. We study this possibility with scalar field matter as it can model a wide range of cosmological expansion histories. For the Λ CDM expansion history, stability considerations substantially limit the available parameter space while for a kinetic energy dominated expansion, no choice of quasidilaton parameters is stable. More generally these results show that there is no mechanism intrinsic to the theory to forbid such pathologies from developing from stable initial conditions and that stability can only be guaranteed for particular choices for the matter configuration.

I. INTRODUCTION

de Rham-Gabadadze-Tolley (dRGT) massive gravity [1] is a theory with a massive graviton, which is constructed to remove the Boulware-Deser ghost. In this theory, there are two metrics: the usual spacetime metric and a flat fiducial metric. It possesses a branch of self-accelerated solutions [2–8] where the Universe undergoes de Sitter expansion without a true cosmological constant.

However, because the fiducial metric is non-dynamical, the dRGT model breaks diffeomorphism invariance. In the preferred unitary gauge coordinates where the fiducial metric is Minkowski, the spacetime metric does not take on the Friedmann-Lemaître-Robertson-Walker (FLRW) form. Furthermore on the self-accelerating branch there is no coordinate system where the two metrics are even simultaneously homogeneous and isotropic for spatially flat or closed FLRW solutions [9]. While there exists open FLRW solutions where this is possible [10], they are generally unstable to fluctuations [11, 12]. Though accelerating solutions where one of the two metrics are either inhomogeneous or anisotropic do exist [6, 9, 13–15], this feature of dRGT with a static flat fiducial metric is an obstacle in building a successful cosmology.

Many generalizations of the dRGT model focus on replacing the static flat fiducial metric while retaining the Boulware-Deser ghost free form of the construction. Quasidilaton massive gravity is one of such attempts to make the fiducial metric dynamical. Here the quasidilaton acts as a conformal rescaling of the fiducial metric and so can accommodate the expansion of the Universe in both metrics [16]. Unfortunately, in its original form the model suffers from ghost instabilities [17, 18]. The extended quasidilaton model introduces an extra coupling term between the massive graviton and quasidilaton that cures this instability for vacuum self-accelerating solutions [19]. However making the fiducial metric itself dynamical and dependent on the evolution of the quasidilaton field, opens the possibility that in the presence of matter instabilities develop. It is the purpose of the present paper to investigate the stability of the extended quasidilaton massive gravity in cosmological solutions with matter.

This paper is organized as follows. In §II, we review the model and define notation. In §III, we present homogeneous and isotropic background dynamics of the model. We show that it is still possible to have a self-accelerated solution in the presence of matter components. In §IV, we explore the scalar perturbations around the self-accelerated solution, and check their stability. Starting from summarizing vacuum case, we derive new conditions for stability with matter. We conclude in §V, and we provide techniques used in the main text in Appendix A and B.

Throughout the paper, we will work in natural units where $c = 1$, and the metric signature is $(- + + +)$.

II. EXTENDED QUASIDILATON MASSIVE GRAVITY WITH MATTER

Extended quasidilaton massive gravity is defined by the action [19]

$$S_g = \frac{M_{\text{Pl}}^2}{2} \int d^4x \sqrt{-g} \left[R + 2m_g^2 (\mathcal{L}_2 + \alpha_3 \mathcal{L}_3 + \alpha_4 \mathcal{L}_4) - \frac{\omega}{M_{\text{Pl}}^2} \partial_\mu \sigma \partial^\mu \sigma \right], \quad (1)$$

where m_g is the graviton mass, σ is the quasidilaton scalar field, and ω , α_3 , and α_4 are dimensionless model parameters. The graviton mass term is expressed by

$$\begin{aligned}\mathcal{L}_2 &= \frac{1}{2}([\mathcal{K}]^2 - [\mathcal{K}^2]), \\ \mathcal{L}_3 &= \frac{1}{6}([\mathcal{K}]^3 - 3[\mathcal{K}][\mathcal{K}^2] + 2[\mathcal{K}^3]), \\ \mathcal{L}_4 &= \frac{1}{24}([\mathcal{K}]^4 - 6[\mathcal{K}]^2[\mathcal{K}^2] + 3[\mathcal{K}^2]^2 + 8[\mathcal{K}][\mathcal{K}^3] - 6[\mathcal{K}^4]).\end{aligned}\quad (2)$$

Here, square brackets represent the trace of the enclosed matrix. The form of \mathcal{L}_2 , \mathcal{L}_3 , and \mathcal{L}_4 are the same as dRGT massive gravity but the matrix $\mathcal{K}^\mu{}_\nu$ is given by

$$\mathcal{K}^\mu{}_\nu = \delta^\mu{}_\nu - e^{\sigma/M_{\text{Pl}}} \left(\sqrt{g^{-1}\tilde{f}} \right)^\mu{}_\nu, \quad (3)$$

where $(g^{-1}\tilde{f})^\mu{}_\nu = g^{\mu\rho}\tilde{f}_{\rho\nu}$ and $\sqrt{M}^\mu{}_\nu$ is understood as a root of matrix: $\sqrt{M}^\mu{}_\rho\sqrt{M}^\rho{}_\nu = M^\mu{}_\nu$. There are two differences in $\mathcal{K}^\mu{}_\nu$ from dRGT massive gravity: the extended fiducial metric $\tilde{f}_{\mu\nu}$ which is disformally related to the fiducial metric $f_{\mu\nu}$

$$\begin{aligned}\tilde{f}_{\mu\nu} &= f_{\mu\nu} - \frac{\alpha_\sigma}{M_{\text{Pl}}^2 m_g^2} e^{-2\sigma/M_{\text{Pl}}} \partial_\mu \sigma \partial_\nu \sigma, \\ f_{\mu\nu} &= \eta_{ab} \partial_\mu \phi^a \partial_\nu \phi^b,\end{aligned}\quad (4)$$

and its coupling to the quasidilaton. Note that this disformal relation does not guarantee a Lorentzian signature to the extended fiducial metric. More generally a disformal scaling which does not depend on the kinetic term of σ itself does not by construction preserve the signature [20]. Thus we are interested in the question of whether matter can induce an evolution in σ that changes the signature of this metric.

In (4) the 4 Stückelberg fields ϕ^a restore general covariance as they transform as spacetime scalars. In addition, the form of the coupling is chosen so that under a transformation

$$\sigma \rightarrow \sigma + \sigma_0, \quad \phi^a \rightarrow e^{-\sigma_0/M_{\text{Pl}}} \phi^a, \quad (5)$$

where $\sigma_0 = \text{const.}$, the extended fiducial metric transforms as

$$\tilde{f}_{\mu\nu} \rightarrow e^{-2\sigma_0/M_{\text{Pl}}} \tilde{f}_{\mu\nu}, \quad (6)$$

leaving the action invariant as in the original quasidilaton model [16]. The quasidilaton thus allows a rescaling of the extended fiducial metric and in cosmological solutions plays a similar role to the scale factor. The coupling constant α_σ between the massive graviton and the quasidilaton is introduced to stabilize the self-accelerating solution in the absence of matter [19]. Note that the extended fiducial metric is dynamical whereas $f_{\mu\nu}$ is always a coordinatization of the standard Minkowski metric regardless of the dynamics of the Stückelberg fields.

We are interested in how the background and the perturbations for the self-accelerating flat FLRW solution behave if we include matter component. In order to consider a wide range of cosmological background solutions, we take the matter to be a canonical scalar field ξ whose action is given by

$$S_m = \int d^4x \sqrt{-g} \left[-\frac{1}{2} \partial_\mu \xi \partial_\nu \xi - V(\xi) \right]. \quad (7)$$

The total action $S = S_g + S_m$ is thus specified by 5 model parameters $\{m_g, \omega, \alpha_3, \alpha_4, \alpha_\sigma\}$ and a choice of the scalar field potential $V(\xi)$.

III. BACKGROUND

The form of spatially flat cosmological background solutions is defined by homogeneity and isotropy of the spacetime and fiducial metrics, and quasidilaton and matter fields

$$\begin{aligned}ds^2 &= -N(t)^2 dt^2 + a(t)^2 \delta_{ij} dx^i dx^j, \\ \phi^0 &= \phi^0(t), \quad \phi^i = x^i, \\ \sigma &= \bar{\sigma}(t), \quad \xi = \bar{\xi}(t).\end{aligned}\quad (8)$$

Here we have kept a general lapse function $N(t)$ so as to study its implied equation of motion before setting it to unity as in the conventional FLRW metric. The extended fiducial metric is then given by

$$\begin{aligned} -\tilde{f}_{00} &\equiv n(t)^2 = (\dot{\phi}^0)^2 + \frac{\alpha_\sigma}{M_{\text{Pl}}^2 m_g^2} e^{-2\bar{\sigma}/M_{\text{Pl}}} \dot{\bar{\sigma}}^2, \\ \tilde{f}_{ij} &= \delta_{ij}. \end{aligned} \quad (9)$$

It is convenient to introduce the following variables:

$$H \equiv \frac{\dot{a}}{Na}, \quad X \equiv \frac{e^{\bar{\sigma}/M_{\text{Pl}}}}{a}, \quad r \equiv \frac{n}{N}a. \quad (10)$$

The Lagrangian at the background level is then given by

$$\begin{aligned} \mathcal{L} &= \frac{M_{\text{Pl}}^2}{2} a^3 N \left[6 \left(\frac{\dot{a}^2}{N^2 a^2} + \frac{\ddot{a}}{N^2 a} - \frac{\dot{N}\dot{a}}{N^3 a} \right) \right. \\ &\quad + 2m_g^2 (X-1) [3(rX + X - 2) - (X-1)(3rX + X - 4)\alpha_3 + (X-1)^2(rX - 1)\alpha_4] \\ &\quad \left. + \frac{\omega\dot{\sigma}^2}{M_{\text{Pl}}^2 N^2} + \frac{1}{M_{\text{Pl}}^2} \left(\frac{\dot{\xi}^2}{N^2} - 2V \right) \right]. \end{aligned} \quad (11)$$

From this Lagrangian we can derive the equations of motion. Variation of the action with respect to ϕ^0 gives

$$\frac{d}{dt} \left[\frac{\dot{\phi}^0}{n} a^4 X (X-1) J \right] = 0, \quad (12)$$

where

$$J \equiv 3 + 3(1-X)\alpha_3 + (1-X)^2\alpha_4. \quad (13)$$

Since the Stückelberg field do not couple to the matter field, this equation is the same in the presence or absence of matter and we follow Ref. [19] in studying its solutions. From (12), we obtain $X(1-X)J\dot{\phi}^0/n \propto a^{-4}$ which asymptotically vanishes as the Universe expands. We focus on the branch with $J = 0$, and hereafter X shall denote the root of $J = 0$. In this branch of cosmological solutions, $X \equiv e^{\bar{\sigma}/M_{\text{Pl}}}/a = \text{const}$ and the quasidilaton in the background plays the same role as the scale factor allowing the extended fiducial metric to scale with the expansion. This solution implies $\dot{\bar{\sigma}} = M_{\text{Pl}}NH$ and

$$\left(\frac{\dot{\phi}^0}{n} \right)^2 = 1 - \frac{\alpha_\sigma e^{-2\bar{\sigma}/M_{\text{Pl}}} \dot{\bar{\sigma}}^2}{M_{\text{Pl}}^2 m_g^2 n^2} = 1 - \frac{\alpha_\sigma H^2}{m_g^2 X^2 r^2}. \quad (14)$$

If we insist that the fiducial and extended fiducial metrics have a Lorentzian signature then $\dot{\phi}_0/n$ is real and

$$\alpha_\sigma < \frac{m_g^2 X^2 r^2}{H^2}. \quad (15)$$

Variation with respect to N and a give the Friedmann equations

$$3 \left(1 - \frac{\omega}{6} \right) M_{\text{Pl}}^2 H^2 = M_{\text{Pl}}^2 \Lambda_X + \frac{\dot{\xi}^2}{2} + V, \quad (16)$$

$$-2 \left(1 - \frac{\omega}{6} \right) M_{\text{Pl}}^2 \dot{H} = \dot{\xi}^2. \quad (17)$$

After deriving the equation of motion for N , we set $N = 1$ for the following. Here, we define

$$\Lambda_X \equiv m_g^2 (X-1)^2 [(X-1)\alpha_3 - 3]. \quad (18)$$

Therefore, the total energy consists of the matter component and an effective cosmological constant induced by the graviton mass term, which leads to a self-accelerated expansion of the Universe. To make $\Lambda_X \sim m_g^2$ responsible for the late-time acceleration, one needs $m_g \sim H_0$ and its positivity requires

$$(X-1)\alpha_3 - 3 > 0. \quad (19)$$

In addition, we note that the effective gravitational constant for background is given by a rescaling of the Planck mass

$$\tilde{M}_{\text{Pl}}^2 \equiv M_{\text{Pl}}^2 \left(1 - \frac{\omega}{6}\right) \quad (20)$$

and $\tilde{M}_{\text{Pl}}^2 > 0$ requires

$$\omega < 6. \quad (21)$$

By defining the effective critical density $\tilde{\rho}_{cr} \equiv 3\tilde{M}_{\text{Pl}}^2 H_0^2$, the Friedmann equations (16), (17) take their usual form. In particular for the Λ CDM expansion history with $\Omega_i \equiv \rho_i/\tilde{\rho}_{cr}$, $H^2/H_0^2 = \Omega_\Lambda + \Omega_m a^{-3}$. Setting Ω_Λ to satisfy observational constraints determines m_g/H_0 as

$$\frac{m_g^2}{H_0^2} = \frac{(6 - \omega)\Omega_\Lambda}{2(X - 1)^2[(X - 1)\alpha_3 - 3]}. \quad (22)$$

From the equation of motion for the quasidilaton $\bar{\sigma}$, we obtain

$$r = 1 + \frac{\omega(3H^2 + \dot{H})}{3m_g^2 X^2 [(X - 1)\alpha_3 - 2]}. \quad (23)$$

Therefore, r is not constant in general, a crucial distinction from the case without a matter field. It is only constant if H itself is constant, or if $3H^2 + \dot{H} = 0$. In particular, the latter case implies that the Universe is dominated by the stiff matter, whose equation of state parameter is $w = 1$. This phase could take place if the expansion is dominated by the kinetic energy of the scalar field. In this case, $r = 1$, which we shall see has interesting consequences for stability.

Finally, the matter field $\bar{\xi}$ obeys the usual equation for a minimally coupled scalar field

$$\ddot{\bar{\xi}} + 3H\dot{\bar{\xi}} + \frac{dV}{d\xi} = 0. \quad (24)$$

Since the equation of state parameter for the scalar field is $w \geq -1$, it typically dominates the energy density and the expansion rate in the past. We shall use the flexibility in choosing the potential to mimic the various stages of the standard Λ CDM model. In particular, we can reproduce any power law expansion $a \propto t^p$ by using the potential for power-law inflation. Furthermore, it is possible to reproduce an expansion which is equivalent to that with nonrelativistic matter and a cosmological constant. This case is studied in the Appendix A.

To summarize, we choose the model parameters, namely, $\{m_g, \omega, \alpha_3, \alpha_4, \alpha_\sigma\}$ in order to satisfy requirements on the background evolution. Since

$$3 + 3(1 - X)\alpha_3 + (1 - X)^2\alpha_4 = 0 \quad (25)$$

on the self accelerating branch and we also need to satisfy a condition on X (19) for positivity of the effective cosmological constant, it is useful to choose first α_3 and X and determine α_4 by (25). A specific example of a set of parameters which satisfy (25) and (19) is

$$\alpha_3 = 4, \quad \alpha_4 = 9, \quad X = 2. \quad (26)$$

For ω , we only need to satisfy (21) in order to guarantee the positivity of the gravitational constant. For α_σ , (15) is necessary if all metrics have Lorentzian signatures. We shall see in the next section that this condition can be alternately viewed as a requirement for the stability of fluctuations around the background solution which generalizes the vacuum results of Ref. [19]. Then we set m_g (22) using the observational data for Ω_Λ . For instance, for parameter set (26), $\omega = 4$, and $\Omega_\Lambda = 0.7$, we obtain $(m_g/H_0)^2 = 0.7$. After specifying all the parameters, the evolution of $H(t)$ and $\bar{\xi}(t)$ are given by (17) and (24), and $r(t)$ is given by (23). Importantly, this makes the bound on α_σ time dependent beyond the vacuum solutions.

IV. SCALAR PERTURBATIONS

We will work in the unitary gauge, where the perturbation for the Stückelberg field vanishes. This gauge condition completely fixes the gauge degree of freedom and requires the most general parameterization of scalar metric

fluctuations

$$\begin{aligned}\delta g_{00} &= -2\Phi, \\ \delta g_{0i} &= a\partial_i B, \\ \delta g_{ij} &= a^2 \left[2\delta_{ij}\Psi + \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\partial_\ell\partial^\ell \right) E \right],\end{aligned}\tag{27}$$

and two dimensionless perturbation for quasidilaton and matter field

$$\begin{aligned}\sigma &= \bar{\sigma} + M_{\text{Pl}}\delta\sigma, \\ \xi &= \bar{\xi} + M_{\text{Pl}}\delta\xi,\end{aligned}\tag{28}$$

and we will work in a Fourier space.

Since the quadratic action does not have kinetic term for B and Φ as expected, we can eliminate them by using their equations of motion. We are then left with four variables, Ψ , E , $\delta\sigma$, and $\delta\xi$.

A. Vacuum case

Let us first review the case without matter component. To analyze this case, we only need to switch off ξ and V , and use $\dot{H} = 0$. Then we have three variable Ψ , E , and $\delta\sigma$. However, the kinetic terms for Ψ and $\delta\sigma$ can be combined in the form of $(\dot{\Psi} - \dot{\delta\sigma})^2$. Therefore, one nondynamical degree of freedom still remains in the quadratic Lagrangian. We define a new notation as

$$\begin{aligned}\phi_1 &\equiv \Psi - \delta\sigma : \text{dynamical}, \\ \phi_2 &\equiv E : \text{dynamical}, \\ \phi_3 &\equiv \Psi + \delta\sigma : \text{nondynamical}.\end{aligned}\tag{29}$$

After integrating out ϕ_3 , the kinetic terms are $K_{ij}\dot{\phi}_i\dot{\phi}_j$ for $i, j = 1, 2$. The no-ghost condition is given by the positivity of all the eigenvalues of the kinematic matrix K_{ij} , which is equivalent to imposing

$$\det K = \frac{M_{\text{Pl}}^4\omega^2 a^2 H^2 k^6}{r^2(r-1)^2} \frac{2A(r-1)^2(k/aH)^2 + 3(\omega-6)(A-r^2)}{4(A-1)(k/aH)^2 + \omega(6-\omega)} > 0,\tag{30}$$

$$K_{22} = \frac{k^4 M_{\text{Pl}}^2 \omega [2(A-1)(k/aH)^2 + 3(6-\omega)]}{18 [4(A-1)(k/aH)^2 + \omega(6-\omega)]} > 0,\tag{31}$$

where

$$A \equiv \frac{\alpha_\sigma H^2}{m_g^2 X^2}.\tag{32}$$

Note that H is given by (16) without the matter component, r is given by (23) with $\dot{H} = 0$, and both H and r are constant.

We would like to derive a condition for model parameters to make both of (30) and (31) positive for all wavenumber k . We start from deriving necessary conditions from taking high- k and low- k limit. For $k/aH \gg 1$,

$$\frac{A}{A-1} > 0, \quad \omega > 0.\tag{33}$$

For $k/aH \ll 1$, the K_{22} condition is automatically satisfied and

$$\frac{r^2 - A}{\omega} > 0,\tag{34}$$

Therefore, in addition to $\omega < 6$ from the positivity of the effective gravitational constant, the necessary condition for the stability is

$$\omega > 0, \text{ and } A < r^2, \text{ and } [A > 1 \text{ or } A < 0].\tag{35}$$

Now let us check the sufficiency of the conditions. We note that $A < 0$ is not sufficient. For instance, we can choose wavenumber

$$\left(\frac{k}{aH}\right)^2 = \frac{3(1-\epsilon)(6-\omega)}{2(1-A)}, \quad (36)$$

which is positive by the virtue of $A < 0$. Here, we choose some small positive ϵ , which satisfies $0 < \epsilon < \text{Min}\{1, 6-\omega\}$. For this wavenumber, K_{22} is a positive number times

$$\frac{\epsilon}{\omega - 6 + \epsilon}, \quad (37)$$

which is negative. On the other hand, $A > 1$ is sufficient, because all the terms appeared in the expressions of $\det K$ and K_{22} are positive for $A > 1$, combined with $0 < \omega < 6$ and $A < r^2$.

Therefore, the no-ghost condition in the absence of matter component is given by

$$0 < \omega < 6, \quad 1 < \frac{\alpha_\sigma H^2}{m_g^2 X^2} < r^2. \quad (38)$$

Note that we need $r > 1$, namely, $(X-1)\alpha_3 - 2 > 0$, which is satisfied if we impose (19). This condition is necessary to establish stability in asymptotic future of cosmological solutions of the self accelerating branch. Furthermore note that since H and r are constant here, the stability condition for α_σ depends only on the choices for the other parameters of the quasidilaton model. We shall next consider how these conditions generalize in the presence of matter.

B. Matter with $-1 \leq w < 1$

Now we turn our attention to examine the no-ghost condition in the presence of matter field, but we omit the case with $w = 1$ for reasons which shall be made clear in §IV D. In addition to the perturbation for the metric and the quasidilaton, we introduce matter perturbation $\delta\xi$. As in the absence of matter, the quadratic Lagrangian with matter does not have kinetic terms for Φ and B . Thus, we can derive two constraint equations and make use of them to eliminate Φ and B . After eliminating Φ and B , we are left with four perturbative variables, namely, $(E, \Psi, \delta\sigma, \delta\xi)$. Without matter, we have two dynamical degrees of freedom. Therefore, we anticipate that with matter we should have three dynamical degrees of freedom, and there is one nondynamical degrees of freedom which should be expressed by certain linear combination of $(E, \Psi, \delta\sigma, \delta\xi)$. Indeed, the determinant of the kinematic matrix for $(E, \Psi, \delta\sigma, \delta\xi)$ vanishes, which implies the existence of nondynamical field. By examining the sub-kinematic matrices, we find that the kinematic matrix for $(\Psi, \delta\sigma, \delta\xi)$ is the minimal one whose determinant vanishes. Let us denote the eigenvalues by $\lambda_1 = 0$, λ_2 , λ_3 , and their eigenvectors by $\vec{v}_1^T \equiv (1/\Xi, 1/\Xi, 1)$, $\vec{v}_2^T \equiv (v_{21}, v_{22}, 1)$, $\vec{v}_3^T \equiv (v_{31}, v_{32}, 1)$. Here,

$$\Xi \equiv \frac{\dot{\xi}}{M_{\text{Pl}} H} \quad (39)$$

and explicit forms for λ_2 , λ_3 , \vec{v}_2 , and \vec{v}_3 are given in Appendix B. Now, we define a new basis thorough $(\Psi, \delta\sigma, \delta\xi) = (\vec{v}_1, \vec{v}_2, \vec{v}_3)(\psi_1, \psi_2, \psi_3)$, and diagonalize the sub-kinematic matrix. After rewriting the quadratic Lagrangian in terms of new basis, we obtain the kinematic matrix for ψ_1, ψ_2, ψ_3 , and $\psi_4 \equiv E$ as

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & * \\ 0 & 0 & \lambda_3 & * \\ 0 & * & * & * \end{pmatrix}, \quad (40)$$

where a star denotes nonzero components. Therefore, we can eliminate the nondynamical variable ψ_1 by using its constraint equation, and end up with the quadratic Lagrangian for three dynamical degrees of freedom (ψ_2, ψ_3, ψ_4) . We would like to examine the necessary conditions for the kinematic matrix K_{ij} for $i, j = 2, 3, 4$ to be positive definite. As in the matterless case, we investigate the sign of the determinants of the matrix and its subblocks, $\det K$, $K_{33}K_{44} - K_{34}^2$ and K_{44} in the high- k and low- k limit.

First, let us focus on the high- k limit. The leading order terms for $k/aH \gg 1$ are given by

$$K_{44} = \frac{k^4}{72} M_{\text{Pl}}^2 a^3 (\omega + \Xi^2) + \dots, \quad (41)$$

$$\begin{vmatrix} K_{33} & K_{34} \\ K_{34} & K_{44} \end{vmatrix} = \frac{\omega k^4}{144} M_{\text{Pl}}^4 a^6 (\Xi v_{32} - 1)^2 + \dots, \quad (42)$$

$$\det K = \frac{\omega^2 A k^2}{96 r^2 \Xi^2 (A - 1)} M_{\text{Pl}}^6 a^{11} (3H^2 + \dot{H})(2 + \Xi^2)^2 [(1 - 2\omega)^2 + 2\Xi^2 + \Xi^4] + \dots. \quad (43)$$

Here, v_{32} is understood as the leading order term at the high- k limit. Let us determine the constraints on model parameters that are necessary for these quantities to be positive. First, (41) is always positive. From (42) being positive, we have $\omega > 0$. Since $\omega < 6$ from the positivity of the effective gravitational constant, we obtain $0 < \omega < 6$. Last, (43) provides $A(A - 1) > 0$, namely, ($A > 1$ or $A < 0$). To also satisfy sufficient conditions for the matterless case, we choose $A > 1$.

Next, we focus on the low- k limit. The leading order terms for $k/aH \ll 1$ are

$$K_{44} = \frac{k^4}{12} M_{\text{Pl}}^2 a^3 + \dots, \quad (44)$$

$$\begin{vmatrix} K_{33} & K_{34} \\ K_{34} & K_{44} \end{vmatrix} = \frac{\omega(r^2 - A)k^2}{8r^2(r - 1)^2} M_{\text{Pl}}^4 a^8 (3H^2 + \dot{H})(v_{31} - v_{32})^2 + \dots, \quad (45)$$

$$\det K = \frac{3\omega(r^2 - A)k^2}{16r^2(r^2 - 1)} M_{\text{Pl}}^6 a^{11} H^2 [(v_{31} - v_{32})(\Xi v_{21} - 1) - (v_{21} - v_{22})(\Xi v_{31} - 1)]^2 + \dots. \quad (46)$$

Here, v_{21} , v_{22} , v_{31} , v_{32} are understood as the leading order terms at the low- k limit. From (45) and (46), we obtain $A < r^2$.

With the high- k and low- k results combined, the necessary conditions for stability are

$$0 < \omega < 6, \quad \frac{m_g^2 X^2}{H^2(t)} < \alpha_\sigma < \frac{m_g^2 X^2}{H^2(t)} r^2(t), \quad (47)$$

which is identical to (38) for the case without matter. However, the crucial difference is that $H = H(t)$ and $r = r(t)$ are time dependent and the condition must be satisfied for all time with a single value of the constant α_σ . This means that it is possible to choose parameters for which the system is initially stable but evolve into an instability. We shall show in the next section explicit examples that do so. Physically, this means that these backgrounds have their fiducial metrics evolve from a Lorentzian to a Euclidean signature, thus triggering the instability. By making the second metric $\tilde{f}_{\mu\nu}$ dynamical in the extended quasidilaton scenario, stability depends not just on the intrinsic model parameters but also on the matter content and evolution of the Universe.

Indeed, by introducing a scalar field for the matter with an arbitrary potential, we have allowed for the possibility of any expansion history for matter whose equation of state parameter varies from $-1 \leq w < 1$. For instance, we can describe any expansion evolving as $w = \text{const.}$ or $a \propto t^{2/3(1+w)}$ by using the same potential that describes power law inflation. We shall next derive a more explicit condition from (47) with the help of the Λ CDM expansion.

C. Λ CDM expansion history

Given the observational success of the Λ CDM expansion history, it is worthwhile to explore the explicit constraints on parameters for this form. In Appendix A, we show that it is possible to construct two different scalar field potentials that reproduce the Λ CDM expansion history. The first case is the usual axion model where the field oscillates in a quadratic potential with $m \gg H$. The second, more novel case is a rolling field where the kinetic and potential energy driven to be equal through an attractor. While these models have the same background expansion history, the axion model is equivalent to CDM in that it is gravitationally unstable whereas the rolling field is not. The two models indicate that our condition (47) is not dependent on whether matter is gravitationally unstable in the linear regime.

Using the definition of Λ_X and the expansion history $H^2 = H_0^2(\Omega_\Lambda + \Omega_m a^{-3})$, we obtain

$$\frac{m_g^2 X^2}{H^2} = \frac{X^2}{2(X - 1)^2} \frac{6 - \omega}{(X - 1)\alpha_3 - 3} \frac{\Omega_\Lambda}{\Omega_\Lambda + \Omega_m a^{-3}}, \quad (48)$$

$$r = 1 + \frac{\omega}{6 - \omega} \frac{(X - 1)^2}{X^2} \frac{(X - 1)\alpha_3 - 3}{(X - 1)\alpha_3 - 2} \left(2 + \frac{\Omega_m}{\Omega_\Lambda} a^{-3} \right). \quad (49)$$

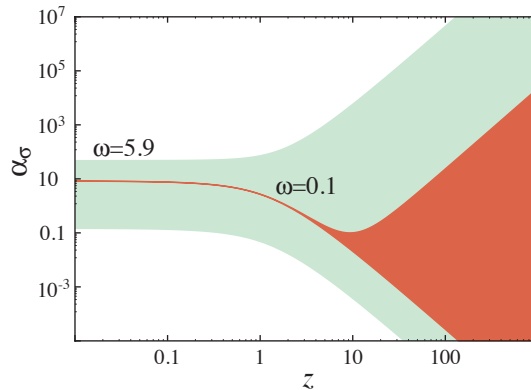


Figure 1. Evolution in redshift of the region between the upper and lower bounds upper bound for α_σ (47) for the Λ CDM expansion history. A constant α_σ that satisfies the bounds exists for $\omega = 5.9$ (light green shaded region) but does not for $\omega = 0.1$ (dark red shaded region) yielding an additional constraint on ω . In the latter case, all choices of $\alpha_\sigma > 0$ are initially stable but evolve to an instability. Here $\alpha_3 = 4$, $\alpha_4 = 9$, and $X = 2$.

From (48) and (49), the allowed range for α_σ (47) can be expressed in terms of X , α_3 , ω , and Ω_Λ . Note that as a decreases, the lower bound on α_σ monotonically weakens since $H \geq H_0$ while the upper bound asymptotically weakens since $\lim_{a \rightarrow 0} r^2/H^2 \propto a^{-3}$.

On the other hand the appearance of the factor $\omega/(6-\omega)$ in (49) for r has important consequences for the evolution of the upper bound near $a = 1$ or redshift $z = 0$. For $\omega \rightarrow 0$, the growing part of r is suppressed and near $z = 0$ can drop below the growth of H^2 thereby tightening the upper bound. Figure 1 illustrates these properties by showing the region between the upper and lower bounds as a function of redshift z for parameter set (26) with $\omega = 5.9$ (light green shaded) compared with 0.1 (dark red shaded). For $\omega = 5.9$, it is possible to choose a constant α_σ that satisfies the bound for all z whereas it is impossible for $\omega = 0.1$. Hence, small ω is not allowed and we gain an additional constraint beyond $0 < \omega < 6$ from requiring that the Λ CDM expansion history be stable.

The existence of a constant α_σ which satisfy the bound (47) requires

$$\frac{m_g^2 X^2}{H_0^2} < \text{Min} \left[\frac{m_g^2 X^2}{H(t)^2} r(t)^2 \right] \quad (50)$$

should be satisfied. By using (48) and (49), we can rewrite this condition as

$$\frac{\omega}{6-\omega} > B, \quad (51)$$

where

$$B = \frac{X^2}{2(X-1)^2} \frac{(X-1)\alpha_3 - 2}{(X-1)\alpha_3 - 3} (\sqrt{1 + \Omega_\Lambda} - 1), \quad (52)$$

or

$$\frac{6B}{1+B} < \omega < 6. \quad (53)$$

For instance, for parameter set (26) and $\Omega_\Lambda = 0.7$, we obtain $3.29 < \omega < 6$. This means that requiring Λ CDM stability eliminates half the parameter space that was available in the matterless case. We emphasize that these parameters would otherwise appear to grant stability both at the initial and current epochs. They represent models whose extended fiducial metric evolve from Lorentzian to Euclidean and back to Lorentzian.

The condition for α_σ (47) is then

$$\frac{X^2}{2(X-1)^2} \frac{(6-\omega)\Omega_\Lambda}{(X-1)\alpha_3 - 3} < \alpha_\sigma < \frac{2\omega}{(X-1)\alpha_3 - 2} \left[1 + \frac{2\omega}{6-\omega} \frac{(X-1)^2}{X^2} \frac{(X-1)\alpha_3 - 3}{(X-1)\alpha_3 - 2} \right]. \quad (54)$$

Here we used (22). For instance, for parameter set (26) and $\Omega_\Lambda = 0.7$, $\omega = 4$, we obtain $(m_g/H_0)^2 = 0.7$ and $1.43 < \alpha_\sigma < 5$.

Thus, in addition to the conditions which we mentioned in the end of Sec. III, we need to choose ω to satisfy (53) and α_σ to satisfy (54) given a value for Ω_Λ that satisfies observational constraints on the expansion history.

D. Matter with $w = 1$

Finally, for the completeness of our analysis, there is a special case that occurs for $w = 1$ or a kinetic energy dominated scalar field. Since $3H^2 + \dot{H} = 0$, $r - 1 = 0$ and (47) would imply that no constant α_σ can satisfy the stability bounds. However since the derivation involves many expressions that assume these quantities are finite, we study this case separately.

As in the $w \neq 1$ case, we study the conditions that would make the kinematic matrix K_{ij} for $i, j = 2, 3, 4$ be positive definite. However, $\det K$ is negative definite:

$$\det K = - \frac{3m_g^4 M_{\text{Pl}}^6 X^4 a^{13} [(X-1)\alpha_3 - 2]^2 (\omega - 8)^2}{64 [8(k/C)^2 + \omega + 6]^2} \times [64(5\omega^2 - 18\omega + 49)(k/C)^4 + 16(\omega^3 + 18\omega^2 - 61\omega + 294)(k/C)^2 + (\omega^2 - \omega + 42)^2], \quad (55)$$

where

$$C \equiv m_g a X \sqrt{(X-1)\alpha_3 - 2}. \quad (56)$$

Note that the second line of (55) is positive definite for any wavenumber under $0 < \omega < 6$. Thus, there is no choice of parameters that makes the extended quasidilaton model stable for matter with a kinetic dominated equation of state. This is compatible with the naive interpretation of the bound (47).

Furthermore note that a pure $w = 1$ expansion history is not strictly necessary for the bound (47) to have no solution. For a multicomponent matter system, so long as the kinetic term of the scalar field dominates the expansion, any additional subdominant matter component with equation of state parameter $w_j < 0$ will also cause a failure of solutions. Suppose that total energy is dominated by the kinetic term of the scalar field but has other components: $H^2 = H_0^2 (\Omega_s a^{-6} + \sum_j \Omega_j a^{-3(1+w_j)})$. Then,

$$\lim_{a \rightarrow 0} \frac{r-1}{H} = \frac{\omega}{6-\omega} \frac{(X-1)^2}{X^2} \frac{(X-1)\alpha_3 - 3 \sum_j (1-w_j)\Omega_j a^{-3w_j}}{(X-1)\alpha_3 - 2 \Omega_\Lambda \Omega_s^{1/2} H_0}. \quad (57)$$

The right hand side asymptotically vanishes as $a \rightarrow 0$ for $w_j < 0$, which implies that r/H approaches to $1/H$ in the past and no constant α_σ can satisfy the stability bounds (47). This includes the case where the additional component is from the self-accelerating background. For $w_j = 0$, solutions would only exist for special choices of parameters, e.g. $\omega \rightarrow 6$, so that $r/H \gg 1/H$ and it allows a constant α_σ to satisfy the bound (47).

V. CONCLUSIONS

We considered cosmological self-accelerated solutions of the extended quasidilaton theory in the presence of matter components. By treating the matter as a scalar field with a canonical kinetic term but an arbitrary potential, we have allowed for a wide range of background expansion histories that may occur in a cosmological setting. Examining the quadratic Lagrangian for the scalar perturbations around these background solutions, we obtained necessary conditions for stability (47). While these appear identical in form to the case without matter, they provide time-dependent constraints on the fundamental parameters of the theory. By demanding the Λ CDM expansion history be stable, we obtained the constraints (53) and (54) for model parameters ω and α_σ , for given value of Ω_Λ which are considerably stronger than the case without matter. We also showed that the self-accelerated solution is unstable for any choice of model parameters if the expansion is governed by matter with $w = 1$ or a kinetic energy dominated scalar field.

More generally, these results arise because in this model the extended fiducial metric is dynamical. In particular there is nothing intrinsic to its dynamics that forbids an evolution of this metric from a Lorentzian to a Euclidean signature. Backgrounds that evolve through such a transition develop a ghost instability. Thus the presence of certain types of matter can induce evolution to an instability that is not present in the initial conditions or apparent from just the parameters of the extended quasidilaton model itself.

ACKNOWLEDGMENTS

We thank A. Joyce and L.T. Wang for useful discussions. HM was supported in part by JSPS Postdoctoral Fellowships for Research Abroad. WH was supported by U.S. Dept. of Energy contract DE-FG02-13ER41958 and the Kavli Institute for Cosmological Physics at the University of Chicago through grants NSF PHY-0114422 and NSF PHY-0551142.

Appendix A: Λ CDM expansion history with scalar fields

In the main text, we use a scalar field to model the Λ CDM expansion history from the matter dominated to the acceleration epoch. It is well known that an axionic model where the field oscillates in a quadratic potential with $m \gg H$ satisfies these conditions averaged over oscillations. Here we provide a novel explicit construction of an alternate case where the field is rolling rather than oscillating and the Λ CDM expansion history arises from attractor behavior. We begin with the expansion history itself which can be written as

$$a(t) = \left(\frac{\Omega_m}{\Omega_\Lambda} \right)^{1/3} \sinh^{2/3} \left(\frac{3}{2} \sqrt{\Omega_\Lambda} H_0 t \right), \quad (\text{A1})$$

or

$$H(t) = \sqrt{\Omega_\Lambda} H_0 \coth \left(\frac{3}{2} \sqrt{\Omega_\Lambda} H_0 t \right). \quad (\text{A2})$$

Here we consider the Ω_Λ contribution to be from the quasidilaton or more generally, a contribution that is external to the scalar field system. Therefore, for the scalar system to combine with Ω_Λ to form the Λ CDM expansion history, we require the energy in the scalar field to scale as $\rho \propto a^{-3}$ and the pressure $p = 0$. This condition then requires $\dot{\xi}^2/2 = V(\bar{\xi}) = \rho/2$ or

$$\bar{\xi} = \pm \frac{2\tilde{M}_{\text{Pl}}}{\sqrt{3}} \log \left[\frac{\tanh \left(\frac{3}{4} \sqrt{\Omega_\Lambda} H_0 t \right)}{\frac{3}{4} \sqrt{\Omega_\Lambda}} \right], \quad (\text{A3})$$

and finally

$$V = \frac{3\Omega_\Lambda H_0^2 \tilde{M}_{\text{Pl}}^2}{8} \left[\frac{3}{4} \sqrt{\Omega_\Lambda} e^{\frac{\sqrt{3}\bar{\xi}}{2\tilde{M}_{\text{Pl}}}} - \left(\frac{3}{4} \sqrt{\Omega_\Lambda} e^{\frac{\sqrt{3}\bar{\xi}}{2\tilde{M}_{\text{Pl}}}} \right)^{-1} \right]^2. \quad (\text{A4})$$

This solution (A3) is an attractor of this potential. It also holds for a true cosmological constant rather than the quasidilaton effective cosmological constant with the replacement $\tilde{M}_{\text{Pl}} \rightarrow M_{\text{Pl}}$.

Note that this system differs from axionic scalar field solutions that also satisfy the Λ CDM expansion history. In our solution the kinetic and potential energies are set to be equal instantaneously whereas for an axion they only average to the same values over many oscillations. This difference also appears in the dynamics of perturbations. Here they are gravitationally stable due to the field fluctuations having sound speed unity in a slowly varying background. In the axion case, the rapid oscillation of the background allows for growing modes in the energy density perturbations that behave like CDM. Unlike the axionic case, this model is an example of a system that is indistinguishable from Λ CDM from the expansion history but easily distinguishable in the growth of structure.

Appendix B: Eigensystem for sub-kinematic matrix

Here we give explicit forms for the eigenvalues and eigenvectors of the sub-kinematic matrix for $(\Psi, \delta\sigma, \delta\xi)$ used in Sec. IV B. Diagonalizing the matrix yields the eigenvalues

$$\lambda_1 \equiv 0, \quad \lambda_2 \equiv \frac{1}{4}(p_0 - \sqrt{q}), \quad \lambda_3 \equiv \frac{1}{4}(p_0 + \sqrt{q}), \quad (\text{B1})$$

and the corresponding eigenvectors

$$\vec{v}_1^T \equiv (1/\Xi, 1/\Xi, 1), \quad \vec{v}_2^T \equiv (v_{21}, v_{22}, 1), \quad \vec{v}_3^T \equiv (v_{31}, v_{32}, 1), \quad (\text{B2})$$

where recall $\Xi = \dot{\xi}/M_{\text{Pl}}H$ and

$$\begin{aligned} v_{21} &\equiv \frac{p_1 - (\omega + \Xi^2)\sqrt{q}}{d + \Xi\sqrt{q}}, & v_{22} &\equiv \frac{p_2 + \omega\sqrt{q}}{d + \Xi\sqrt{q}}, \\ v_{31} &\equiv \frac{p_1 + (\omega + \Xi^2)\sqrt{q}}{d - \Xi\sqrt{q}}, & v_{32} &\equiv \frac{p_2 - \omega\sqrt{q}}{d - \Xi\sqrt{q}}, \end{aligned} \quad (\text{B3})$$

and

$$\begin{aligned}
q &\equiv 16(6-\omega)^2(r^2-1)^2[\Xi^4+2\Xi^2+(2\omega-1)^2]\left(\frac{k}{aH}\right)^4 \\
&\quad - 8\omega(6-\omega)(r^2-1)(\Xi^2+\omega-6)[(\omega+6)\Xi^4+2(\omega+6)\Xi^2+(2\omega-1)(13\omega-6)]\left(\frac{k}{aH}\right)^2 \\
&\quad + \omega^2(\Xi^2+\omega-6)^2[(\omega+6)^2\Xi^4+2(\omega+6)^2\Xi^2+(13\omega-6)^2], \\
d &\equiv -4(6-\omega)(r^2-1)\Xi(\Xi^2+1)\left(\frac{k}{aH}\right)^2 + \omega\Xi(\Xi^2+\omega-6)[(\omega+6)\Xi^2+2\omega^2-\omega+6], \\
p_0 &\equiv 4(\omega-6)(r^2-1)(\Xi^2+2\omega+1)\left(\frac{k}{aH}\right)^2 + \omega(\Xi^2+\omega-6)[(\omega-6)\Xi^2-(11\omega+6)], \\
p_1 &\equiv 4(6-\omega)(r^2-1)[\Xi^4+(\omega+1)\Xi^2+\omega(2\omega-1)]\left(\frac{k}{aH}\right)^2 \\
&\quad - \omega(\Xi^2+\omega-6)[(\omega+6)\Xi^4+(\omega+1)(\omega+6)\Xi^2+\omega(13\omega-6)], \\
p_2 &\equiv -4\omega(6-\omega)(r^2-1)[\Xi^2+2\omega-1]\left(\frac{k}{aH}\right)^2 + \omega^2(\Xi^2+\omega-6)[(8-\omega)\Xi^2-13\omega+6].
\end{aligned} \tag{B4}$$

The following relations also help simplify the derivation:

$$\begin{aligned}
\Xi^2+\omega-6 &= -\frac{2}{H^2}\left(\Lambda+\Lambda_X+\frac{V}{M_{\text{Pl}}^2}\right) \\
&= -2\left(1-\frac{\omega}{6}\right)\frac{3H^2+\dot{H}}{H^2},
\end{aligned} \tag{B5}$$

from the background equations and

$$\begin{aligned}
v_{21}+v_{22}+\Xi &= 0, \\
v_{31}+v_{32}+\Xi &= 0, \\
v_{21}v_{31}+v_{22}v_{22}+1 &= 0,
\end{aligned} \tag{B6}$$

from the orthogonality of the eigenvectors.

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- [1] C. de Rham, G. Gabadadze, and A. J. Tolley, Phys.Rev.Lett. **106**, 231101 (2011), arXiv:1011.1232 [hep-th].
- [2] K. Koyama, G. Niz, and G. Tasinato, Phys.Rev.Lett. **107**, 131101 (2011), arXiv:1103.4708 [hep-th].
- [3] K. Koyama, G. Niz, and G. Tasinato, Phys.Rev. **D84**, 064033 (2011), arXiv:1104.2143 [hep-th].
- [4] T. Nieuwenhuizen, Phys.Rev. **D84**, 024038 (2011), arXiv:1103.5912 [gr-qc].
- [5] L. Berezhiani, G. Chkareuli, C. de Rham, G. Gabadadze, and A. Tolley, Phys.Rev. **D85**, 044024 (2012), arXiv:1111.3613 [hep-th].
- [6] P. Gratia, W. Hu, and M. Wyman, Phys.Rev. **D86**, 061504 (2012), arXiv:1205.4241 [hep-th].
- [7] M. S. Volkov, Phys.Rev. **D86**, 061502 (2012), arXiv:1205.5713 [hep-th].
- [8] M. S. Volkov, Phys.Rev. **D86**, 104022 (2012), arXiv:1207.3723 [hep-th].
- [9] G. D'Amico, C. de Rham, S. Dubovsky, G. Gabadadze, D. Pirtskhalava, *et al.*, Phys.Rev. **D84**, 124046 (2011), arXiv:1108.5231 [hep-th].
- [10] A. E. Gumrukcuoglu, C. Lin, and S. Mukohyama, JCAP **1111**, 030 (2011), arXiv:1109.3845 [hep-th].
- [11] A. E. Gumrukcuoglu, C. Lin, and S. Mukohyama, JCAP **1203**, 006 (2012), arXiv:1111.4107 [hep-th].
- [12] A. De Felice, A. E. Gumrukcuoglu, and S. Mukohyama, Phys.Rev.Lett. **109**, 171101 (2012), arXiv:1206.2080 [hep-th].
- [13] T. Kobayashi, M. Siino, M. Yamaguchi, and D. Yoshida, Phys.Rev. **D86**, 061505 (2012), arXiv:1205.4938 [hep-th].
- [14] A. E. Gumrukcuoglu, C. Lin, and S. Mukohyama, Phys.Lett. **B717**, 295 (2012), arXiv:1206.2723 [hep-th].
- [15] H. Motohashi and T. Suyama, Phys.Rev. **D86**, 081502 (2012), arXiv:1208.3019 [hep-th].
- [16] G. D'Amico, G. Gabadadze, L. Hui, and D. Pirtskhalava, Phys.Rev. **D87**, 064037 (2013), arXiv:1206.4253 [hep-th].
- [17] A. E. Gumrukcuoglu, K. Hinterbichler, C. Lin, S. Mukohyama, and M. Trodden, Phys.Rev. **D88**, 024023 (2013), arXiv:1304.0449 [hep-th].
- [18] G. D'Amico, G. Gabadadze, L. Hui, and D. Pirtskhalava, Class.Quant.Grav. **30**, 184005 (2013), arXiv:1304.0723 [hep-th].
- [19] A. De Felice and S. Mukohyama, Phys.Lett. **B728**, 622 (2014), arXiv:1306.5502 [hep-th].
- [20] J.-P. Bruneton and G. Esposito-Farese, Phys.Rev. **D76**, 124012 (2007), arXiv:0705.4043 [gr-qc].