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Effective Field Theory of Degenerate Higher-Order Inflation

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We extend the effective field theory of inflation to a general Lagrangian constructed from Arnowitt-Deser-Misner variables that encompasses the most general interactions with up to second derivatives of the scalar field whose background breaks temporal diffeomorphism invariance. Degeneracy conditions, corresponding to 8 distinct types – only one of which corresponds to known degenerate higher-order scalar-tensor models – provide necessary conditions for eliminating the Ostrogradsky ghost in a covariant theory at the level of the quadratic action in unitary gauge. Novel implications of the degenerate higher-order system for the Cauchy problem are illustrated with the phase space portrait of an explicit inflationary example: not all field configurations lead to physical solutions for the metric even for positive potentials; solutions are unique for a given configuration only up to a branch choice; solutions on one branch can apparently end at nonsingular points of the metric and their continuation on alternate branches lead to nonsingular bouncing solutions; unitary gauge perturbations can go unstable even when degenerate terms in the Lagrangian are infinitesimal. The attractor solution leads to an inflationary scenario where slow-roll parameters vary and running of the tilt can be large even with no explicit features in the potential far from the end of inflation, requiring the optimized slow-roll approach for predicting observables.

I. INTRODUCTION

Single-field scalar-tensor theories as inflationary models can be studied in a unified way in the framework of the effective field theory (EFT) of inflation, where the timelike scalar field is treated as a clock that breaks the time diffeomorphism invariance leaving spatial diffeomorphism invariance unbroken [1, 2]. In general, the EFT of inflation with higher-derivative operators contains extra ghost degrees of freedom, which may or may not propagate in the regime of validity of the EFT. To consider a regime where higher-derivative interactions also produce interesting observable phenomenology, one needs to rely on the framework where the ghost degrees of freedom are appropriately eliminated. Therefore, an EFT Lagrangian motivated by general ghost-free theories serve as a useful framework. In this context, the original EFT framework has been extended in subsequent works [3–7] to include derivative operators appearing in the Horndeski [8–13], Gleyzes-Langlois-Piazza-Vernizzi (GLPV) [14, 15] and Horava-Lifshitz [16–18] theories.

More general theories involving additional derivative of fields typically propagate ghost degrees of freedom. The ghosts associated with higher-order derivatives are known as the Ostrogradsky ghosts [19, 20], which makes the Hamiltonian unbounded due to its linear dependence on canonical momenta. Unlike the classically unbounded Hamiltonian of the hydrogen atom, the Ostrogradsky Hamiltonian remains unbounded quantum mechanically as well [21]. To eliminate the ghost degrees of freedom, one needs to evade the condition of the Ostrogradsky theorem that the Lagrangian is nondegenerate with respect to the highest-order derivatives. However, degeneracy with respect to the highest-order derivative is necessary but not sufficient to evade the unbounded Hamiltonian [22], which is the reason why one needs to impose a certain set of degeneracy conditions to eliminate all aspects of the Ostrogradsky ghosts [23–26]. This arguments can be also understood in a broader context in the language of constraints as a generalization of the Ostrogradsky theorem [27].

The degeneracy conditions were applied to a construction of degenerate higher-order scalar-tensor (DHOST) theories with quadratic [23] and cubic interactions [28] of second derivatives of a scalar field, which include the derivative of the lapse function. The EFT description of the quadratic and cubic DHOST theories was developed in [29], where the quadratic Lagrangian around the cosmological background was investigated. Cosmological evolution and the linear stability analysis were also investigated [30], focusing on de Sitter attractor in a shift symmetric quadratic DHOST model. In [29, 30], two assumptions on background dynamics were adopted: that the lapse remains unity and that the scalar field is proportional to time coordinate. The first can be imposed as a gauge condition, and the second should be satisfied dynamically. For general timelike scalar field evolution on a given phase space trajectory, one needs to perform a redefinition of the scalar field to make it proportional to time, which changes all of the DHOST coefficients (cf. [30] v2). A dynamical lapse is also taken into account in [31, 32] in the context of spatially covariant gravity [33], where the degeneracy conditions were also studied. However, the general EFT framework of degenerate theories including but not limited to quadratic and cubic DHOST and its application to cosmology have not been fully investigated yet.

In this paper, generalizing our previous work [7], we develop the EFT framework of general degenerate theories, and explore its peculiar phenomenology. This framework includes quadratic and cubic DHOST as a special subclass as well as theories where the lapse is nondynamical, e.g. those with second-order equations of motion for the scalar field, as in the Horndeski case, or the spatial metric in unitary gauge, as in the GLPV case. In §II, we consider the EFT action composed of Arnowitt-Deser-Misner (ADM) geometric quantities including the acceleration and lapse derivative and their couplings to intrinsic and extrinsic curvatures. This action includes operators appearing in covariant theories involving the most general combination of second-order derivatives of scalar field. It also includes Lorentz-violating theories such as Horava-Lifshitz gravity [16–18], as well as the scordatura degenerate theory [34] weakly violating the degeneracy condition. We derive the background and quadratic actions for various degeneracy classes, which include known DHOST cases, summarized in Appendix A. In §III, we investigate dynamics in degenerate higher-order inflation, which we dub "D-inflation", and clarify several novel features of degenerate models for both the background and perturbations. We provide a detailed study of a specific model, for which the optimized slow-roll (OSR) formalism [35] serves as a powerful tool as the EFT coefficients can exhibit variation on the several efold time scale. In §IV, we discuss conclusions.

II. EFT OF INFLATION

In this section we adopt ADM decomposition and consider the general EFT Lagrangian allowing the most general combination of second-order derivatives of scalar field. In §II A, we construct the EFT Lagrangian from geometric quantities including the acceleration and lapse derivative and their arbitrary couplings to intrinsic and extrinsic curvatures. In §II B, we write down the background and quadratic Lagrangians around cosmological background. Since vector and tensor perturbations are same as the previous work [7], in §II C we focus on the scalar perturbations, and reduce the quadratic Lagrangian for a specific degeneracy class. We provide the complete analysis of the construction of degeneracy conditions in Appendix A.

A. ADM EFT Lagrangian

We work in the $3+1$ ADM decomposition of the metric

$$ds^2 = -N^2dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt), \tag{1}$$

where N, N^i, h_{ij} are the lapse, shift, spatial metric, respectively. We define a timelike unit vector $n_\mu \equiv -Nt_{,\mu}$ orthogonal to constant t surfaces, the acceleration $a_\mu \equiv n^\nu n_{\mu;\nu}$, and the extrinsic curvature $K_{\mu\nu} = n_{\nu;\mu} + n_\mu a_\nu$, where semicolons on indices here and throughout denote covariant derivatives with respect to $g_{\mu\nu}$.

For general scalar-tensor theories, we can choose so-called unitary gauge, where $\phi = \phi(t)$ as long as the gradient of the scalar field is always timelike. In unitary gauge, any Lagrangian with up to second derivatives in the field can be expressed in terms of ADM quantities through

$$\begin{aligned} \phi_{,\mu} &= -\sqrt{-X}n_\mu, \\ \phi_{;\mu\nu} &= \sqrt{-X}(-K_{\nu\mu} + n_\nu a_\mu + n_\mu a_\nu - \beta n_\mu n_\nu). \end{aligned} \tag{2}$$

Here, we define β by

$$\beta = -\frac{1}{2}n^\mu(\ln X)_{,\mu} = -\frac{\ddot{\phi}}{N\dot{\phi}} + \frac{\dot{N} - N^i\partial_i N}{N^2}, \tag{3}$$

where $X \equiv g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} = -\dot{\phi}^2/N^2$ is the kinetic term for the scalar. In particular, if we take $\phi = t$, $X = -1/N^2$ and then β measures the fractional change in the lapse along the normal

$$\beta = n^\lambda(\ln N)_{,\lambda} = \frac{\dot{N} - N^i\partial_i N}{N^2}, \tag{4}$$

and more generally it determines the fractional change in the elapsed proper time in field coordinates and so (3) involves $\ddot{\phi}$. In the gauge where $\phi \propto t$, $\ddot{\phi} = 0$ and this has been used widely for the purpose of counting the number of degrees of freedom through the Hamiltonian analysis. However, to keep a normal perturbation analysis where the background lapse $\bar{N} = 1$, we use $\phi = \phi(t)$, and retain the $\ddot{\phi}$ term. After solving for a given trajectory $\phi(t)$, we

can always make a field redefinition $\varphi \propto t(\phi)$, which maintains $\bar{N} = 1$ at the expense of redefining the scalar field Lagrangian, but adopting this at the outset prevents a phase space analysis for background trajectories.

We seek to construct a general spatial diffeomorphism invariant EFT Lagrangian involving no more than second-order derivatives of the scalar field which a priori contains $N, K_{\mu\nu}, a_\mu, \beta$. So long as we consider theories involving up to $\phi_{;\mu\nu}$, a^i and a_j are the only quantities that have one spatial sub/superscript. Hence, a^i and a_j always appear together through

$$\alpha^i{}_j \equiv a^i a_j = h^{ik}(\ln N)_{,j}(\ln N)_{,k}, \quad \alpha \equiv \alpha^i{}_i. \tag{5}$$

We therefore consider the Lagrangian to be a spatially diffeomorphism invariant function of these quantities

$$S = \int d^4x N\sqrt{h} L(N, K^i{}_j, R^i{}_j, \alpha^i{}_j, \beta; t), \tag{6}$$

generalizing [7] to allow it to depend on $\alpha^i{}_j$ and β. Here $R^i{}_j$ is the Ricci 3-tensor on the spatial slice. Higher-derivative Lagrangians typically contain Ostrogradsky ghosts, but we shall see in the next section that for special combinations of $N, K^i{}_j, R^i{}_j, \alpha^i{}_j, \beta$ the Lagrangian only propagates one scalar and the usual two tensor degrees of freedom. Without the $\alpha^i{}_j$ dependence and allowing higher-order spatial derivatives, the Lagrangian (6) reduces to the one explored in [31, 32].

To explicitly relate this EFT Lagrangian to known ghost-free scalar-tensor theories, we begin with the most general covariant Lagrangian that is at most cubic in second derivatives of the field and coupled to the metric as

$$S = \int d^4x\sqrt{-g}\left[F_0 + F_1\Box\phi + F_2{}^{(4)}R + F_3{}^{(4)}G_{\mu\nu}\phi^{;\mu\nu} + \sum_{i=1}^{5} A_i L_i^{(2)} + \sum_{i=1}^{10} B_i L_i^{(3)}\right], \tag{7}$$

where F_i, A_i, B_i are general functions of ϕ, X and ${}^{(4)}R$ is the 4 dimensional Ricci scalar. The terms that are quadratic in second derivatives are

$$\begin{aligned}
L_1^{(2)} &= \phi_{;\mu\nu}\phi^{;\mu\nu} = \frac{\dot\phi^2}{N^2}(K^i{}_j K^j{}_i + \beta^2 - 2\alpha),\\
L_2^{(2)} &= (\Box\phi)^2 = \frac{\dot\phi^2}{N^2}(-K+\beta)^2,\\
L_3^{(2)} &= (\Box\phi)\phi^{;\mu}\phi_{;\mu\nu}\phi^{;\nu} = \frac{\dot\phi^4}{N^4}\beta(K-\beta),\\
L_4^{(2)} &= \phi^{;\mu}\phi_{;\mu\nu}\phi^{;\nu\rho}\phi_{;\rho} = \frac{\dot\phi^4}{N^4}(\alpha-\beta^2),\\
L_5^{(2)} &= (\phi^{;\mu}\phi_{;\mu\nu}\phi^{;\nu})^2 = \frac{\dot\phi^6}{N^6}\beta^2,
\end{aligned} \tag{8}$$

and those that are cubic are

$$\begin{aligned}
L_1^{(3)} &= (\Box\phi)^3 = \frac{\dot\phi^3}{N^3}(-K+\beta)^3,\\
L_2^{(3)} &= (\Box\phi)\phi_{;\mu\nu}\phi^{;\mu\nu} = \frac{\dot\phi^3}{N^3}(-K+\beta)(K^i{}_j K^j{}_i + \beta^2 - 2\alpha),\\
L_3^{(3)} &= \phi_{;\mu\nu}\phi^{;\nu\rho}\phi^{;\mu}{}_{;\rho} = \frac{\dot\phi^3}{N^3}(-K^i{}_j K^j{}_k K^k{}_i + 3\alpha^i{}_j K^j{}_i + \beta^3 - 3\alpha\beta),\\
L_4^{(3)} &= (\Box\phi)^2\phi^{;\mu}\phi_{;\mu\nu}\phi^{;\nu} = -\frac{\dot\phi^5}{N^5}\beta(-K+\beta)^2,\\
L_5^{(3)} &= (\Box\phi)\phi^{;\mu}\phi_{;\mu\nu}\phi^{;\nu\rho}\phi_{;\rho} = \frac{\dot\phi^5}{N^5}(-K+\beta)(\alpha-\beta^2),\\
L_6^{(3)} &= \phi_{;\mu\nu}\phi^{;\mu\nu}\phi^{;\rho}\phi_{;\rho\sigma}\phi^{;\sigma} = \frac{\dot\phi^5}{N^5}\beta(-K^i{}_j K^j{}_i - \beta^2 + 2\alpha),\\
L_7^{(3)} &= \phi^{;\mu}\phi_{;\mu\nu}\phi^{;\nu\rho}\phi_{;\rho\sigma}\phi^{;\sigma} = \frac{\dot\phi^5}{N^5}(-\alpha^i{}_j K^j{}_i - \beta^3 + 2\alpha\beta),\\
L_8^{(3)} &= \phi^{;\mu}\phi_{;\mu\nu}\phi^{;\nu\rho}\phi_{;\rho}\phi^{;\sigma}\phi_{;\sigma\xi}\phi^{;\xi} = \frac{\dot\phi^7}{N^7}\beta(\beta^2-\alpha),
\end{aligned}$$

$$L_9^{(3)} = \Box\phi(\phi^{;\mu}\phi_{;\mu\nu}\phi^{;\nu})^2 = \frac{\dot\phi^7}{N^7}\beta^2(-K+\beta),$$
$$L_{10}^{(3)} = (\phi^{;\mu}\phi_{;\mu\nu}\phi^{;\nu})^3 = -\frac{\dot\phi^9}{N^9}\beta^3, \tag{9}$$

where we have used (2) to establish the correspondence with the ADM variables. Similarly, we can relate the coupling to the metric using the Gauss-Codazzi relation and integration by parts to rewrite up to boundary terms (see e.g. [3, 36])

$$\int d^4x\sqrt{-g}F_2{}^{(4)}R = \int d^4x\sqrt{-g}\left[F_2\left(R + K^i{}_jK^j{}_i - K^2\right) - 2\left(F_{2\phi}\frac{\dot\phi}{N}K + 2F_{2X}\frac{\dot\phi^2}{N^2}(\beta K - \alpha)\right)\right], \tag{10}$$

$$\int d^4x\sqrt{-g}F_3{}^{(4)}G_{\mu\nu}\phi^{;\mu\nu} = \int d^4x\sqrt{-g}\Bigg[\mathcal{F}_3\frac{\dot\phi}{N}\left(\frac{KR}{2} - K^i{}_jR^j{}_i\right) + \frac{\mathcal{F}_{3\phi} - F_{3\phi}}{2}\frac{\dot\phi^2}{N^2}R + \frac{F_{3\phi}}{2}\frac{\dot\phi^2}{N^2}(K^i{}_jK^j{}_i - K^2)$$
$$+ F_{3X}\frac{\dot\phi^3}{N^3}\left((K^i{}_jK^j{}_i - K^2)\beta + 2\alpha K - 2\alpha^i{}_jK^j{}_i)\right)\Bigg], \tag{11}$$

where

$$F_{3X} = \mathcal{F}_{3X} + \frac{\mathcal{F}_3}{2X}. \tag{12}$$

While in general, the appearance of $\alpha^i{}_j$ and β in the Lagrangian signals an extra degree of freedom since the lapse and shift no longer obey constraint equations, this general Lagrangian (7) contains classes that propagate only 3 degrees of freedom and avoids Ostrogradsky ghosts. First, there is the GLPV class which defines a special relationship between the A_i, B_i, F_i coefficients

$$A_1 = -A_2 = 2F_{2X} + XF_4, \quad A_3 = -A_4 = 2F_4,$$
$$B_1 = -\frac{B_2}{3} = \frac{B_3}{2} = \frac{F_{3X}}{3} + XF_5, \quad -2B_4 = B_5 = 2B_6 = -B_7 = 6F_5,$$
$$A_5 = B_8 = B_9 = B_{10} = 0. \tag{13}$$

Note that the F_4 and F_5 terms are also arbitrary functions of ϕ, X. In the Horndeski subclass of GLPV, where the scalar field equations themselves are explicitly second order, $F_4 = F_5 = 0$ and the remaining functions are more typically labeled $(G_2, G_3, G_4, G_5) = (F_0, F_1, F_2, F_3)$. It is easy to verify through Eqs. (8,9,10) that this relationship eliminates the dependence of $\alpha^i{}_j$ and β in the GLPV and Horndeski Lagrangians leaving the EFT Lagrangian of the form $L(N, K^i{}_j, R^i{}_j; t)$. More generally, $\alpha^i{}_j$ and β can appear in a Lagrangian which still only propagates 3 degrees of freedom if the functions A_i, B_i, F_i satisfy a certain set of degeneracy conditions (see (40), (41) and [23, 28]). This is the DHOST class of models. The Lagrangian (6) also includes the scordatura degenerate theory [34] with a weak violation of the degeneracy condition.

In §II C we generalize these degeneracy conditions to the full EFT Lagrangian (6). Since there the dependence on $\alpha^i{}_j, \beta$ is arbitrary it encompasses theories with further higher-order products of $\phi_{;\mu\nu}$ beyond (7). The Lagrangian (6) thus can represent any fully covariant or Lorentz-violating theory involving up to second derivatives of metric and scalar field in the unitary gauge. Furthermore in comparison to EFTs that are explicitly built to encompass DHOST, it allows terms like βR that would only appear with different couplings between the field and the metric than represented in (7) (cf. [23, 28, 29]).

B. Background and Quadratic Lagrangian

We next consider the expansion of the ADM EFT Lagrangian (6) to quadratic order in metric perturbations around a spatially flat Friedmann-Lemaître-Robertson-Walker (FLRW) background

$$\bar N = 1, \quad \bar N^i = 0, \quad \bar h_{ij} = a^2\delta_{ij}, \tag{14}$$

for which

$$\bar K^i{}_j = H\delta^i{}_j, \quad \bar R^i{}_j = 0, \quad \bar\alpha^i{}_j = 0, \quad \bar\beta = -\frac{\ddot\phi}{\dot\phi}, \tag{15}$$

where $H \equiv d\ln a/dt$ is the Hubble parameter. Following [7] we define the Taylor coefficients as

$$\begin{aligned} L\Big|_{\rm b} &= \mathcal{C}, \\ \frac{\partial L}{\partial Y^i{}_j}\Big|_{\rm b} &= \mathcal{C}_Y \delta^j{}_i, \\ \frac{\partial^2 L}{\partial Y^i{}_j \partial Z^k{}_\ell}\Big|_{\rm b} &= \mathcal{C}_{YZ}\delta^j{}_i\delta^\ell{}_k + \frac{\tilde{\mathcal{C}}_{YZ}}{2}(\delta^\ell{}_i\delta^j{}_k + \delta_{ik}\delta^{j\ell}), \end{aligned} \tag{16}$$

where "b" denotes that the quantities are evaluated on the background, $Y, Z \in \{N, K, R, \alpha, \beta\}$ and the index structure is determined by the symmetry of the background. For notational simplicity we treat scalars and traces with the same notation; thus implicitly $N^i{}_i \equiv N$, $\beta^i{}_i \equiv \beta$ and $\tilde{\mathcal{C}}_{NZ} = \tilde{\mathcal{C}}_{\beta Z} = 0$.

We can further eliminate terms linear in $\delta K = K - 3H$ through the identity

$$\int d^4x\sqrt{-g}F(t)K = -\int d^4x\sqrt{-g}n^\mu F_{;\mu} = -\int d^4x\sqrt{-g}\frac{\dot{F}}{N}, \tag{17}$$

which follows from $K = n^\mu{}_{;\mu}$ ignoring boundary terms. The Lagrangian $\mathcal{L} = N\sqrt{h}L$ up to quadratic order in metric perturbations becomes

$$\mathcal{L} = N\sqrt{h}(\mathcal{C} - 3H\mathcal{C}_K) - \sqrt{h}\dot{\mathcal{C}}_K + N\sqrt{h}(\mathcal{C}_N\delta N + \mathcal{C}_R\delta R + \mathcal{C}_\alpha\delta\alpha + \mathcal{C}_\beta\delta\beta) + \frac{a^3}{2}\sum_{Y,Z}(\mathcal{C}_{YZ}\delta Y\delta Z + \tilde{\mathcal{C}}_{YZ}\delta Y^i{}_j\delta Z^j{}_i). \tag{18}$$

In this form δK only shows up in the quadratic-order terms, and we need only its first-order perturbation. In contrast, we expand $\delta R^i{}_j, \delta\alpha^i{}_j, \delta\beta$ up to quadratic order

$$\delta Y^i{}_j = \delta_1 Y^i{}_j + \delta_2 Y^i{}_j + \cdots. \tag{19}$$

We note that $\delta_1 R^i{}_j, \delta_2 R^i{}_j$ do not involve $\delta N, \delta N^i$, whereas

$$\begin{aligned} \delta_1\alpha^i{}_j &= 0, \\ \delta_2\alpha^i{}_j &= \frac{1}{a^2}\delta^{ik}\delta N_{,j}\delta N_{,k}, \\ \delta_1\beta &= -\bar{\beta}\delta N + \delta\dot{N}, \\ \delta_2\beta &= \bar{\beta}\delta N^2 - 2\delta N\delta\dot{N} - \delta N^i\delta N_{,i}, \end{aligned} \tag{20}$$

since

$$\begin{aligned} n^\mu &\approx (1 - \delta N + (\delta N)^2, -\delta N^i(1 - \delta N)), \\ (\ln N)_{,\mu} &\approx \delta N_{,\mu}(1 - \delta N), \end{aligned} \tag{21}$$

up to quadratic order. Note that for the perturbed FLRW metric, $\alpha^i{}_j$ is a quadratic-order quantity. The background equations are given by varying the action with respect to N and $\sqrt{h}$

$$\begin{aligned} \mathcal{C} - 3H\mathcal{C}_K + \mathcal{C}_N - (3H + \bar{\beta})\mathcal{C}_\beta - \dot{\mathcal{C}}_\beta &= 0, \\ \mathcal{C} - 3H\mathcal{C}_K - \dot{\mathcal{C}}_K &= 0. \end{aligned} \tag{22}$$

With the background equation and integration by parts, the quadratic Lagrangian becomes

$$\begin{aligned} \mathcal{L}_2 = a^3\Bigg[&\mathcal{C}_R\left(\delta_1 R\frac{\delta_1\sqrt{h}}{a^3} + \delta_2 R\right) - \mathcal{C}_\beta\delta N\left(\frac{\delta_1\sqrt{h}}{a^3}\right)^{\cdot} - \mathcal{C}_\beta\delta N^i\delta N_{,i} + \frac{1}{2}\mathcal{C}_{\beta\beta}\delta\dot{N}^2 + \mathcal{C}_\alpha\frac{1}{a^2}\delta^{ij}\delta N_{,i}\delta N_{,j} \\ &+ \frac{1}{2}\left(2\mathcal{C}_N + \mathcal{C}_{NN} - 2\bar{\beta}\mathcal{C}_{\beta N} + \bar{\beta}^2\mathcal{C}_{\beta\beta} + \frac{\left(a^3\mathcal{C}_\beta\right)^{\cdot}}{a^3} - \frac{\left(a^3\mathcal{C}_{\beta N}\right)^{\cdot}}{a^3} + \frac{\left(a^3\bar{\beta}\mathcal{C}_{\beta\beta}\right)^{\cdot}}{a^3}\right)\delta N^2 + (\mathcal{C}_{\beta K}\delta_1 K + \mathcal{C}_{\beta R}\delta_1 R)\delta\dot{N} \\ &+ [(\mathcal{C}_{NK} - \bar{\beta}\mathcal{C}_{\beta K})\delta_1 K + (\mathcal{C}_{NR} + \mathcal{C}_R - \bar{\beta}\mathcal{C}_{\beta R})\delta_1 R]\delta N + \frac{1}{2}\sum_{Y=K,R}\sum_{Z=K,R}(\mathcal{C}_{YZ}\delta_1 Y\delta_1 Z + \tilde{\mathcal{C}}_{YZ}\delta_1 Y^i_j\delta_1 Z^j_i)\Bigg]. \end{aligned} \tag{23}$$

This expansion can be continued to higher order for the computation of non-Gaussianity (see e.g. [37]).

C. Scalar perturbations

From the quadratic Lagrangian (23), we note that the new terms $\mathcal{C}_\alpha, \mathcal{C}_\beta, \mathcal{C}_{\beta Y}$ are always accompanied by δN, which is a natural consequence of (20). Therefore, $\alpha^i{}_j$ and β dependencies of the Lagrangian change the dynamics of scalar but not vector or tensor perturbations which are given explicitly in [7].

For scalar perturbations

$$N = 1 + \delta N, \quad N_i = \partial_i \psi, \quad h_{ij} = a^2 e^{2\zeta} \delta_{ij}. \tag{24}$$

Following [7], we use

$$\begin{aligned}
\delta\sqrt{h} &= 3a^3\zeta, \\
\delta K^i{}_j &= (\dot\zeta - H\delta N)\delta^i{}_j - \frac{1}{a^2}\delta^{ik}\partial_k\partial_j\psi, \\
\delta K &= 3(\dot\zeta - H\delta N) - \frac{\partial^2\psi}{a^2}, \\
\delta_1 R^i{}_j &= -\frac{1}{a^2}(\delta^i{}_j\partial^2\zeta + \delta^{ik}\partial_k\partial_j\zeta), \\
\delta_2 R &= -\frac{2}{a^2}[(\partial\zeta)^2 - 4\zeta\partial^2\zeta] \sim -\frac{10}{a^2}(\partial\zeta)^2,
\end{aligned} \tag{25}$$

where the last equality for $\delta_2 R$ holds up to a total derivative, to obtain the quadratic Lagrangian in Fourier space as

$$\begin{aligned}
\mathcal{L}_2 = &\frac{1}{2}c_1\dot\zeta^2 + c_2\dot\zeta\delta\dot N + \frac{1}{2}c_3\delta\dot N^2 + \left(c_4 + c_5\frac{k^2}{a^2}\right)\dot\zeta\delta N \\
&+ \frac{1}{2}\left(c_6 + c_7\frac{k^2}{a^2}\right)\frac{k^2}{a^2}\zeta^2 + c_8\frac{k^2}{a^2}\zeta\delta N + \frac{1}{2}\left(c_9 + c_{10}\frac{k^2}{a^2}\right)\delta N^2 \\
&+ \frac{1}{2}c_{11}\frac{k^4}{a^4}\psi^2 + \frac{k^2}{a^2}\psi\left(\frac{c_1}{3}\dot\zeta + \frac{c_2}{3}\delta\dot N + \frac{c_4}{3}\delta N + c_{12}\frac{k^2}{a^2}\zeta\right),
\end{aligned} \tag{26}$$

where

$$\begin{aligned}
&c_1 = 3a^3(3\mathcal{C}_{KK} + \tilde{\mathcal{C}}_{KK}), && c_2 = 3a^3\mathcal{C}_{\beta K}, && c_3 = a^3\mathcal{C}_{\beta\beta}, && c_4 = -3a^3\Theta, \\
&c_5 = -4a^3\mathcal{C}_{\beta R}, && c_6 = 4a^3\Psi, && c_7 = 2a^3(8\mathcal{C}_{RR} + 3\tilde{\mathcal{C}}_{RR}), && c_8 = 4a^3\Xi, \\
&c_9 = a^3\Phi, && c_{10} = 2a^3\mathcal{C}_\alpha, && c_{11} = a^3(\mathcal{C}_{KK} + \tilde{\mathcal{C}}_{KK}), && c_{12} = 2a^3(2\mathcal{C}_{KR} + \tilde{\mathcal{C}}_{KR}),
\end{aligned} \tag{27}$$

and

$$\begin{aligned}
\Phi \equiv{}& 2\mathcal{C}_N + \mathcal{C}_{NN} - 2\bar\beta\mathcal{C}_{\beta N} + \bar\beta^2\mathcal{C}_{\beta\beta} + (\mathcal{C}_\beta - \mathcal{C}_{\beta N} + \bar\beta\mathcal{C}_{\beta\beta})^\cdot \\
&+ 3H[\mathcal{C}_\beta - \mathcal{C}_{\beta N} + \bar\beta\mathcal{C}_{\beta\beta} - 2(\mathcal{C}_{NK} - \bar\beta\mathcal{C}_{\beta K}) + \dot{\mathcal{C}}_{\beta K}] + 3\mathcal{C}_{\beta K}\dot H + 3H^2[3(\mathcal{C}_{\beta K} + \mathcal{C}_{KK}) + \tilde{\mathcal{C}}_{KK}], \\
\Psi \equiv{}& \mathcal{C}_R - 3\dot{\mathcal{C}}_{KR} - \dot{\tilde{\mathcal{C}}}_{KR} - H(3\mathcal{C}_{KR} + \tilde{\mathcal{C}}_{KR}), \\
\Xi \equiv{}& \mathcal{C}_{NR} + \mathcal{C}_R - \bar\beta\mathcal{C}_{\beta R} - \dot{\mathcal{C}}_{\beta R} - H(\mathcal{C}_{\beta R} + 3\mathcal{C}_{KR} + \tilde{\mathcal{C}}_{KR}), \\
\Theta \equiv{}& \mathcal{C}_\beta - (\mathcal{C}_{NK} - \bar\beta\mathcal{C}_{\beta K}) + H(3\mathcal{C}_{KK} + \tilde{\mathcal{C}}_{KK}).
\end{aligned} \tag{28}$$

We highlight these four combinations as they involve time derivatives or the Hubble parameter and degeneracy conditions involving them would typically need to arise from integration by parts on the Lagrangian [see e.g. (B1)].

In Appendix A, generalizing Ref. [29], we provide a complete analysis of the construction of degeneracy conditions imposed on the various c_i coefficients which we briefly summarize here. The result is 8 types of degeneracy conditions, cases $1a \ldots 3c$ (with 3a impossible to satisfy), each of which may be realized by the c_i or equivalently the $\mathcal{C}_i$ coefficients in various ways. The degeneracy conditions we derive apply for any theory involving second-order derivatives in any form in the Lagrangian for unitary gauge (6). Since the Lagrangian (6) allows any dependence on $(N, K^i{}_j, R^i{}_j, \alpha^i{}_j, \beta)$, or equivalently on second derivatives (2), this degeneracy conditions applies beyond the quadratic and cubic DHOST theories. Furthermore, in general it also applies to Lorentz-violating theories.

The first condition required for the single scalar propagating degree of freedom is degeneracy in the temporal structure of (26). Of the three possibilities, we focus on the case 1 type where the condition $c_3 = c_2/c_1^2$ is satisfied and the combination

$$\tilde\zeta = \zeta + \frac{c_2}{c_1}\delta N, \tag{29}$$

alone carries the temporal derivatives. The other two cases have the lapse δN as the propagating degree of freedom and would cause difficulties in recovering an observationally viable theory of gravity. More generally our linear degeneracy conditions should be viewed as necessary, but not necessarily sufficient, conditions for a viable nonlinear scalar-tensor theory of gravity.

Under this $c_3 = c_2/c_1^2$ condition, the quadratic Lagrangian (26) for scalar perturbation in unitary gauge would appear to propagate only 1 degree of freedom. This degeneracy condition applies to the scalar quadratic Lagrangian in any theory involving second-order derivative of any form in Lagrangian in the unitary gauge (6) and includes the DHOST models as well as the Horndeski or GLPV models where $c_2 = 0$ and $\tilde{\zeta} = \zeta$.

However, the temporal degeneracy condition alone is not sufficient to guarantee that there is only a single degree of freedom. In terms of the Euler-Lagrange equations, it only removes the fourth-order derivatives and third-order derivatives still need to be removed to avoid unbounded Hamiltonian [22]. Furthermore, if unitary gauge defines a foliation that corresponds to characteristic surfaces of the second degree of freedom then its dynamics are hidden from this temporal structure. Since such a degree of freedom propagates instantaneously on this surface, it is not a Cauchy surface upon which initial conditions can be propagated forwards in time. Hence its temporal kinetic terms vanish. However on a noncharacteristic surface, temporal kinetic terms reappear and can possess a well-posed Cauchy problem as discussed in detail in [38, 39]. One should therefore not take the apparent lack of an extra degree of freedom in unitary gauge as a definitive absence (cf. [33]). Of course, the counting of degrees of freedom cannot depend on the gauge or ADM slicing and so we expect additional degeneracy conditions that involve the spatial derivatives of the kinetic matrix in unitary gauge.

For a $1+1$ dimensional system of linear partial differential equations, including the plane parallel Fourier modes considered below, one can exploit the algorithm [39] based on the Kronecker form of a matrix pencil which includes all possible linear combinations of temporal and spatial derivatives to count degrees of freedom and find characteristic curves in the presence of any hidden constraints (see Appendix of [39]). However for the quadratic and cubic DHOST theories, the full covariant and nonlinear degeneracy conditions are already known. As shown in the Appendix, we can obtain the remaining conditions for the quadratic action by demanding that the dispersion relation of remaining degree of freedom take their normal linear form in unitary gauge. This logic also applies to the wider class of degenerate theories that originate from a covariant action and so we retain terms that are absent in the quadratic and cubic DHOST Lagrangian in Appendix A.

We now focus in particular on the degeneracy conditions given in (A14) in case 1a, as other branches may not have phenomenologically viable theories of gravity associated with them. We emphasize though that this same procedure can be carried out for any of the branches. In this case, the conditions on the c_i coefficients are

$$c_3 = \frac{c_2^2}{c_1}, \quad c_5 = c_7 = c_{11} = c_{12} = 0, \quad c_{10} = 2c_8 x - c_6 x^2, \tag{30}$$

where $x = c_2/c_1$, and these conditions imply

$$\mathcal{C}_{\beta\beta} = \frac{3\mathcal{C}_{\beta K}^2}{3\mathcal{C}_{KK} + \tilde{\mathcal{C}}_{KK}}, \quad \tilde{\mathcal{C}}_{KK} = -\mathcal{C}_{KK}, \quad \tilde{\mathcal{C}}_{KR} = -2\mathcal{C}_{KR}, \quad \tilde{\mathcal{C}}_{RR} = -\frac{8}{3}\mathcal{C}_{RR}, \quad \mathcal{C}_\alpha = \frac{2\mathcal{C}_{\beta K}}{\mathcal{C}_{KK}}\Xi - \frac{\mathcal{C}_{\beta K}^2}{2\mathcal{C}_{KK}^2}\Psi, \quad \mathcal{C}_{\beta R} = 0. \tag{31}$$

This branch includes the ^{2}N-I/Ia class of quadratic and cubic DHOST, GLPV and Horndeski theories.

Under these conditions we can simplify (26) as

$$\begin{aligned}\mathcal{L}_2 = {}& \frac{1}{2}c_1\dot{\tilde{\zeta}}^2 + (c_4 - c_1\dot{x})\,\dot{\tilde{\zeta}}\delta N + \frac{1}{2}c_6\frac{k^2}{a^2}\tilde{\zeta}^2 + (c_8 - c_6 x)\frac{k^2}{a^2}\tilde{\zeta}\delta N \\ & + \frac{1}{2}\left(c_9 + \dot{c}_4 x + c_1\dot{x}^2 - c_4\dot{x}\right)\delta N^2 + \frac{k^2}{a^2}\psi\left(\frac{c_1}{3}\dot{\tilde{\zeta}} + \frac{c_4 - c_1\dot{x}}{3}\delta N\right).\end{aligned} \tag{32}$$

The equation of motion for ψ and δN yields the constraints

$$\begin{aligned}\delta N &= \frac{c_1}{c_1\dot{x} - c_4}\dot{\tilde{\zeta}}, \\ \frac{k^2}{a^2}\psi &= \frac{3}{c_1\dot{x} - c_4}\left[(c_4 - c_1\dot{x})\,\dot{\tilde{\zeta}} + (c_8 - c_6 x)\frac{k^2}{a^2}\tilde{\zeta} + \left(c_9 + \dot{c}_4 x + c_1\dot{x}^2 - c_4\dot{x}\right)\delta N\right],\end{aligned} \tag{33}$$

where we have assumed

$$0 < |\Omega| < \infty, \tag{34}$$

with $\Omega \equiv c_1\dot{x} - c_4$, which generalizes the condition $2H\mathcal{C}_{KK} \neq \mathcal{C}_{NK}$ employed in Eq. (33) of [7] to cases where the Lagrangian depends $\alpha^i{}_j, \beta$. Violation of this condition makes unitary gauge perturbations ill-defined. For singular Ω, the kinetic term vanishes and hence the system is strongly coupled. On the other hand, for $\Omega = 0$, unitary gauge itself is ill-defined. To see this, we follow [7] and move to comoving gauge defined by the condition that for the perturbed Einstein tensor $\delta G^0{}_i = 0$ for a general metric theory of gravity [40]. The gauge transformation from unitary gauge to comoving gauge is characterized by the time shift $T = -\Delta/\dot{H}$, where $\Delta \equiv H\delta N - \dot{\zeta}$ (see Eq. (B14) of [7]). Using (33), we have

$$\Delta = \frac{1}{\Omega}\left\{\left[c_1(Hx - \dot{x}) + c_4\right]\dot{\zeta} + Hc_1x(\dot{\tilde{\zeta}} - \dot{\zeta})\right\}, \tag{35}$$

so that $\Omega = 0$ makes Δ diverge, implying that the gauge transformation between the two gauges requires an infinite time shift and hence is ill-defined. Note that this is not necessarily a problem if the original system of equations in $(\delta N, \tilde{\zeta}, \psi)$ possesses only regular singular points and is hence integrable without first imposing the constraint equation (see [41] for a related discussion). Furthermore if $\tilde{\zeta}$ freezes out but $\tilde{\zeta} - \zeta$ continues to evolve outside the horizon, then the two gauges will differ. We construct an explicit model where this occurs in §III (see [42] for a discussion of related cases).

Note also that if $c_8 = c_6x$ the $k^2\tilde{\zeta}$ term vanishes in the Euler-Lagrange equation (33) for δN, and prevents the recovery of Newtonian gravity for non-relativistic matter, as found in [29] for one of the DHOST subclasses (see Appendix A for more details).

Substituting the constraints (33) into the Lagrangian (32) and integrating by parts give the usual Mukhanov-Sasaki form for the quadratic Lagrangian

$$\mathcal{L}_2 = A_\zeta\dot{\tilde{\zeta}}^2 - B_\zeta\frac{k^2}{a^2}\tilde{\zeta}^2, \tag{36}$$

where

$$\begin{aligned} A_\zeta &= \frac{c_1}{2}\frac{c_1(c_9 + \dot{c}_4x) + c_4(c_1\dot{x} - c_4)}{(c_1\dot{x} - c_4)^2}, \\ B_\zeta &= a^2\left(\frac{c_1}{2a^2}\frac{c_8 - c_6x}{c_1\dot{x} - c_4}\right)^{\cdot} - \frac{c_6}{2}. \end{aligned} \tag{37}$$

From these terms, we can define the scalar sound speed c_s^2 and the normalization parameter b_s as

$$c_s^2 = \frac{B_\zeta}{A_\zeta}, \quad b_s = \frac{B_\zeta}{a^3\epsilon_H}. \tag{38}$$

For a canonical scalar field $c_s^2 = b_s = 1$. These expressions are generalizations of Eq. (37) of [7].

III. D-INFLATION WITH TIME VARYING EFT COEFFICIENTS

In this section we consider models of degenerate higher-order inflation (D-inflation) with time varying EFT coefficients, specifically in the quadratic DHOST class. In general, EFT coefficients can vary in time and one needs to evaluate carefully the slow-roll hierarchy of all dynamical parameters, for which the generalized slow-roll approximation developed in [7] provides a systematic framework, based on the evolution of H, ϵ_H, b_s and c_s in (38) as well as the analogous quantities for tensors

$$b_t = 2\mathcal{C}_R, \quad c_t^2 = \frac{2\mathcal{C}_R}{\tilde{\mathcal{C}}_{KK}}. \tag{39}$$

D-inflation provides an additional motivation for these time-varying considerations in that one might seek to construct models where their novel features are only present during inflation and are absent thereafter where they would otherwise impact cosmological and astrophysical observables. We construct our model in §III A, and elucidate the novel features on background dynamics and evolution of perturbations in §III B and §III C, respectively.

A. D-Inflation model

As a concrete example, let us require the inflationary model to satisfy $c_t^2-1=0$ and $\tilde{\zeta}-\zeta=0$ at the end of inflation. The former is the requirement for tensor sound speed to be the speed of light, imposed at least at the present epoch by observation of gravitational waves from binary neutron star merger and its electromagnetic counterpart [43, 44]. The latter is imposed as we would like inflation to become fully canonical so that by reheating everything is as usual. We shall see that enforcing this requirement for all perturbation quantities allows us to avoid instabilities caused by derivative couplings [45].

We can concretely implement these requirements using the EFT of a quadratic DHOST model starting with the case 1a degeneracy conditions (31). Since $\tilde{\mathcal{C}}_{KR}=\mathcal{C}_{KR}=\tilde{\mathcal{C}}_{RR}=\mathcal{C}_{RR}=0$ in this case, the third and the fourth conditions in (31) identically hold. From the second condition in (31) we obtain

$$A_1=-A_2. \tag{40}$$

Plugging it into the first and fifth conditions in (31) and solving the two equations for A_4 and A_5, we obtain

$$\begin{aligned} A_4&=\frac{2(A_2+2F_{2X})}{X}-\frac{(2A_2+4F_{2X}+XA_3)[8F_2^2+2XA_2(5F_2-8XF_{2X})+X^2A_3F_2-12XF_2F_{2X}]}{8X(F_2+XA_2)^2},\\ A_5&=\frac{(2A_2+4F_{2X}+XA_3)[4A_3F_2-A_2(2A_2+4F_{2X}-3XA_3)]}{8(F_2+XA_2)^2}, \end{aligned} \tag{41}$$

which matches Eqs. (5.1) and (5.2) in [23]. In general a degenerate theory in this class is identified by A_2, A_3, F_2 and one can choose F_0, F_1 as free functions without affecting the degeneracy structure. Note that if $\mathcal{C}_{\beta K}=0$ then

$$2A_2+4F_{2X}+XA_3=0, \tag{42}$$

so that A_2 and A_3 are no longer independent; this case corresponds to Eq. (5.3) in [23] and reproduces the GLPV restriction for quadratic terms in (13).

Next, from (8), we obtain the tensor sound speed as

$$c_t^2-1=\frac{2\mathcal{C}_R}{\tilde{\mathcal{C}}_{KK}}-1=\frac{XA_1}{F_2-XA_1}=-\frac{XA_2}{F_2+XA_2}, \tag{43}$$

where we used (40) and

$$\tilde{\zeta}-\zeta=\frac{\mathcal{C}_{\beta K}}{3\mathcal{C}_{KK}+\tilde{\mathcal{C}}_{KK}}=-\frac{X(2A_2+4F_{2X}+XA_3)}{4F_2+2X(A_1+3A_2)}. \tag{44}$$

For the Horndeski theory, requiring the right-hand side of (43) to vanish implies that $F_2=F_2(\phi)$. Note also that the right-hand side of (44) identically vanishes for Horndeski and GLPV theories.

We would like to choose the functions A_2, A_3, F_2 to make these two quantities (43), (44) be nonzero during inflation and evolve to zero by the end of inflation. As a simple example, we set

$$F_0=-\frac{X}{2}-V(\phi),\quad F_1=0,\quad F_2=\frac{1}{2},\quad A_3=0, \tag{45}$$

where we work in natural units $M_{\rm Pl}\equiv(8\pi G)^{-1/2}=1$, and for which the degeneracy conditions (40), (41) yield

$$A_1=-A_2,\quad A_4=\frac{A_2^2(3+8XA_2)}{(1+2XA_2)^2},\quad A_5=-\frac{2A_2^3}{(1+2XA_2)^2}, \tag{46}$$

and (43), (44) read

$$c_t^2-1=2(\tilde{\zeta}-\zeta)=\frac{1}{1+2XA_2}-1\equiv\theta(\phi,X). \tag{47}$$

Here, we are interested in a function θ such that it starts from finite value and evolves to zero either from the evolution of X or an appropriate form for $A_2(\phi,X)$.

Under these assumptions, the action is given by

$$S = \int d^4x N\sqrt{h}\Bigg[\frac{1}{2}(R + K^i_j K^j_i - K^2) + \frac{\dot\phi^2}{2N^2} - V(\phi) + A_2\frac{\dot\phi^2}{N^2}(K^2 - K^i{}_j K^j{}_i - 2\beta K + 2\alpha) + \frac{A_2^2}{1 - 2\frac{\dot\phi^2}{N^2}A_2}\frac{\dot\phi^4}{N^4}\left(\frac{3 - 8\frac{\dot\phi^2}{N^2}A_2}{1 - 2\frac{\dot\phi^2}{N^2}A_2}\alpha - 3\beta^2\right)\Bigg]. \tag{48}$$

For simplicity, we will illustrate this model with $A_2 = $ const., or at least nearly so during most of the ~ 60 efolds before the end of inflation. We shall see that in models where the field oscillates at reheating, A_2 needs to vanish before this point to avoid gradient or ghost instabilities. However, any late-time change does not affect large-scale observables which are well outside the horizon at that point.

B. Background dynamics

From (48) we can calculate EFT coefficients. For instance, those which are necessary for the background equations (22) are

$$\begin{aligned}
\mathcal{C} &= \frac{\dot\phi^2}{2} - V - 3H_b^2(1 - 2\dot\phi^2 A_2),\\
\mathcal{C}_K &= -2H_b(1 - 2\dot\phi^2 A_2),\\
\mathcal{C}_N &= -\dot\phi^2 - 12H_b[H - H_b(1 - \dot\phi^2 A_2)],\\
\mathcal{C}_\beta &= -6A_2\dot\phi^2 H_b,
\end{aligned} \tag{49}$$

and those for the perturbations are

$$\begin{aligned}
&\Xi = \Psi = \mathcal{C}_R = \frac{1}{2}, \qquad \tilde{\mathcal{C}}_{KK} = -\mathcal{C}_{KK} = 1 - 2\dot\phi^2 A_2, \qquad \mathcal{C}_{\beta K} = -2\dot\phi^2 A_2,\\
&\mathcal{C}_{\beta N} = \frac{12\dot\phi^2 A_2[2(1 - \dot\phi^2 A_2)H_b - H]}{1 - 2\dot\phi^2 A_2}, \qquad \mathcal{C}_{NK} = 4[(1 - 2\dot\phi^2 A_2)H_b - (1 + \dot\phi^2 A_2)H],\\
&\mathcal{C}_{NN} = 3(1 + 12A_2H_b^2)\dot\phi^2 + 36(H - H_b)H_b - \frac{24(H - H_b)^2}{1 - 2\dot\phi^2 A_2},
\end{aligned} \tag{50}$$

where [30]

$$H_b \equiv H - \frac{A_2\ddot\phi\dot\phi}{1 - 2A_2\dot\phi^2}. \tag{51}$$

The background equations (22) are then given by

$$6\dot\phi^2 A_2(\dot H_b - HH_b) + 3(1 + 2\dot\phi^2 A_2)H_b^2 - \frac{1}{2}\dot\phi^2 - V = 0, \tag{52}$$

$$2(1 - 2\dot\phi^2 A_2)(\dot H_b - HH_b) + 5H_b^2 - 10\dot\phi^2 A_2 H_b^2 + \frac{1}{2}\dot\phi^2 - V = 0. \tag{53}$$

Note that $A_2 = 0$ recovers the Einstein equations in canonical inflation. While the system involves $\dddot\phi$ via $\dot H_b$, by virtue of the degeneracy conditions, it is equivalent to a system whose evolution is determined by initial data in a single degree of freedom e.g. for the background, the position of the field in phase space $(\phi, \dot\phi)$.

The first step in establishing this equivalence is to eliminate $\dot H_b$ from (52) and (53) to obtain:

$$6(1 - 5\dot\phi^2 A_2 + 6\dot\phi^4 A_2^2)H_b^2 - \dot\phi^2(1 + \dot\phi^2 A_2) - 2(1 - 5\dot\phi^2 A_2)V = 0. \tag{54}$$

Hence, there are two branches for H_b. In general, if $A_2 \neq$ const. we would have a term linear in H_b but here, we obtain simple positive and negative roots

$$H_b = \sigma\sqrt{\frac{\dot\phi^2(1 + \dot\phi^2 A_2) + 2(1 - 5\dot\phi^2 A_2)V}{6(1 - 5\dot\phi^2 A_2 + 6\dot\phi^4 A_2^2)}}, \tag{55}$$

where $\sigma = \pm 1$. Next, we choose one of the two branches of $H_b = H_b(\phi, \dot\phi)$ and take its time derivative $\dot H_b = \dot H_b(\phi, \dot\phi, \ddot\phi)$. Substituting $H_b = H_b(\phi, \dot\phi)$ and $\dot H_b = \dot H_b(\phi, \dot\phi, \ddot\phi)$ into (51) and either of (52) or (53), we obtain two equations for $H = H(\phi, \dot\phi, \ddot\phi)$. Finally eliminating H from the two equations, we obtain an equation for $\ddot\phi = \ddot\phi(\phi, \dot\phi)$ governing the evolution of the system from a point in phase space. From this evolution we can then define $H = H(\phi, \dot\phi)$ and other background quantities which define the slow-roll parameters. Note that equations depend only on $m^2 A_2$ if one rescales time to mt. Hence, so long as $m^2 A_2$ is fixed, the relative evolution in mt is the same for various values of m with only the amplitudes $H \propto m$ and $\dot\phi \propto m$ changing. Given the inflationary dynamics, we can check the condition (34) for whether unitary gauge perturbations are well-defined. In our model,

$$\Omega = 6a^3[-H_b + \dot\phi^2 A_2(2H + 3H_b)], \tag{56}$$

which should be a finite value.

Even at the background level, this procedure produces novel behavior in phase space. First, not all phase space positions are allowed, even for a positive potential, and allowed positions can evolve into or from disallowed regions. For definiteness consider the quadratic potential $V(\phi) = m^2\phi^2/2$. With this potential, for both branches of H_b in (55), $H(\phi, \dot\phi)$ is singular at

$$\phi = \pm\sqrt{\frac{1 + \dot\phi^2 A_2 - 9\dot\phi^4 A_2^2 + 3\dot\phi^6 A_2^3}{m^2 A_2(1 + 4\dot\phi^2 A_2 - 15\dot\phi^4 A_2^2)}}, \quad \text{or} \quad \frac{\dot\phi}{m} = \pm\frac{1}{\sqrt{3m^2 A_2}}. \tag{57}$$

For instance, plugging $\frac{\dot\phi}{m} = \frac{1}{\sqrt{3m^2A_2}} - \delta$ with an infinitesimal variable δ into $H(\phi, \dot\phi)$ and Taylor expanding around $\delta = 0$ yields

$$\frac{H}{m} = 3^{-5/4}(m^2 A_2)^{-3/4}\sqrt{1 - \frac{3}{2}m^2 A_2\phi^2}\,\frac{\sigma}{\sqrt{\delta}} + \mathcal{O}(\delta^{1/2}), \tag{58}$$

which is indeed singular at $\delta = 0$ for both $\sigma = \pm 1$ branches so long as $1 - \frac{3}{2}m^2A_2\phi^2 \neq 0$. For $1 - \frac{3}{2}m^2A_2\phi^2 > 0$, the Hubble parameter is real for the $\delta > 0$ side of the boundary, and imaginary for the $\delta < 0$ side. On the other hand $H = \mathcal{O}(\delta^{-1}) \to \pm\infty$ on alternate sides of the ϕ values of the first case in (57).

Also, there are boundaries across which $H(\phi, \dot\phi)$ changes from real to complex value while remaining finite:

$$\phi = \pm\frac{\dot\phi}{m}\sqrt{\frac{1 + \dot\phi^2 A_2}{-1 + 5\dot\phi^2 A_2}}, \quad \text{or} \quad \frac{\dot\phi}{m} = \pm\frac{1}{\sqrt{2m^2 A_2}}, \tag{59}$$

where H_b is zero or singular respectively. For instance, plugging $\frac{\dot\phi}{m} = \frac{1}{\sqrt{2m^2A_2}} + \delta$ into $H(\phi, \dot\phi)$ and $\ddot\phi(\phi, \dot\phi)$ and Taylor expanding around $\delta = 0$ yields

$$\begin{aligned}
\frac{H}{m} &= \frac{\sqrt{2m^2A_2}\phi}{1 - 2m^2A_2\phi^2} + \frac{2^{3/4}(7 - 10m^2A_2\phi^2)}{3(m^2A_2)^{1/4}\sqrt{1 - 2m^2A_2\phi^2}}\sigma\sqrt{\delta} + \mathcal{O}(\delta),\\
\frac{\ddot\phi}{m^2} &= \frac{2^{1/4}}{(m^2A_2)^{3/4}}\sqrt{1 - 2m^2A_2\phi^2}\,\sigma\sqrt{\delta} + \mathcal{O}(\delta).
\end{aligned} \tag{60}$$

Therefore, for $A_2 > 0$ and $1 - 2m^2A_2\phi^2 > 0$, approaching the boundary $\frac{\dot\phi}{m} = \frac{1}{\sqrt{2m^2A_2}}$ from the positive δ side, $H(\phi, \dot\phi)$ changes from real to complex value for both branches. Furthermore, as can be seen from the slope $\ddot\phi/\dot\phi$, the $\sigma = \pm 1$ trajectories form two halves of a parabola whose minimum intersects the boundary. Taylor expansion around another boundary in (59) also has a similar structure.

Another interesting point is that the branches of H_b are not in general related by time reversal as they would be for H in GR beyond the $A_2 =$ const. case, where there would be terms linear in H_b in (54). This means that for a given initial position in the field phase space, evolution is not unique without specifying the branch choice for the metric. This feature is shared by the class of galileon or G-inflation models as well [13, 46].

We take an example parameter set based on the following rough estimation. Proceeding backwards from the end of inflation on the $\sigma = +1$ branch when the field approaches the origin $(\phi, \dot\phi/m) = (0, 0)$, we see that since D-inflation effects for the background scale as $\dot\phi^2 A_2$, the background field behaves close to the canonical model with $\epsilon_H \approx 2/\phi^2$ and $\dot\phi/m \approx \sqrt{2/3}$ on the slow-roll attractor. Thus for the canonical phase to last at least ~ 60 efolds, we require $\phi \approx 15$ to be in the canonical phase. This phase is bounded at some maximum $|\phi|$ by encountering the first of

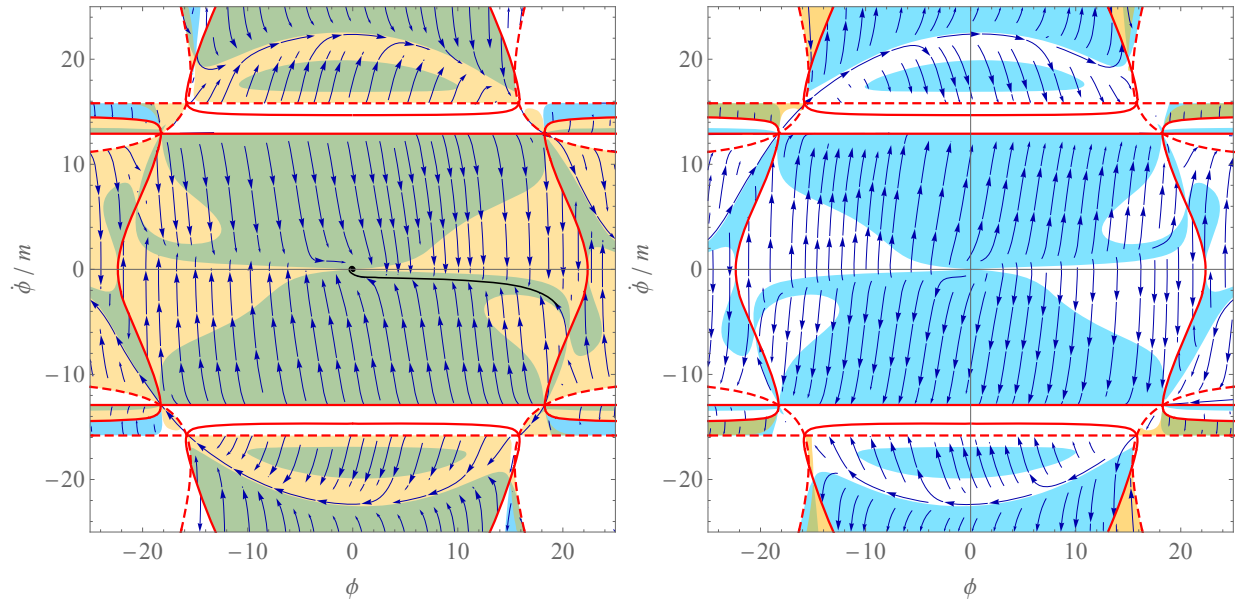

FIG. 1. Phase space portrait of the D-inflation model (48) with $V = m^2\phi^2/2$ and $m^2A_2 = 0.002$ and branches $\sigma = +1$ (left), $\sigma = -1$ (right). Shown are: general trajectories (blue arrows where $H^2 > 0$); main attractor trajectory (black) starting from 60 efolds before the end of inflation; regions with $H > 0$ (yellow), $\epsilon_H > 0$ (blue) or both (green); curves with H singular (solid red) or changing from real to complex (dashed red). The attractor trajectory inevitably crosses into a region where $\epsilon_H < 0$ while spiraling around the origin after the end of inflation.

the boundaries (57) where H is singular. For small $\dot\phi/m$ this occurs at $\phi \approx \pm(m^2A_2)^{-1/2}$, and hence we require $|m^2A_2\phi^2| < 1$ or $m^2A_2 < 1/15^2$. Thus, as an example, we set $m^2A_2 = 0.002$.

The phase space portrait for this set of parameters is depicted in Fig. 1 for both branches. Singularities in H from (57) (red solid lines) separate the phase space into disconnected regions with regions where $H > 0$ (yellow), $\epsilon_H > 0$ (blue) or both (green) shaded. Trajectories (blue arrows) flowing from the boundary given by the first of the conditions in (57) (curved solid red) with $\epsilon_H > 0$ begin at $H = +\infty$, whereas with $\epsilon_H < 0$ begin in a collapsing phase with $H \to -\infty$ at the boundary but then bounces without a curvature singularity when $H = 0$ and becomes an expanding phase $H > 0$ too rapidly to be resolved in Fig. 1. The same is true for trajectories flowing into the boundaries but with reversed signs for H. Trajectories flowing from the boundary given by the second of the conditions in (57) (horizontal solid red) originate from $H = +\infty$, whereas H is complex on the other side of the boundary.

From Fig. 1, we see several other novel features of this model. First, physical solutions do not exist for all possible initial phase space points: there are regions where no real solution of H exists on either branch. This occurs outside the boundaries (59) (dashed red, no trajectories), e.g. $\phi = \dot\phi/m = 20$.

Furthermore, some trajectories in the upper and lower disconnected regions of Fig. 1 appear to end at boundaries across which H becomes complex by satisfying either the first (dashed red curves) or the second condition (horizontal dashed red lines) in (59). Note that at these boundaries H is finite so that they do not represent curvature singularities. In these cases, as mentioned below (60), the trajectories actually sharply turn so as to be tangent to the boundary at intersection. At intersection, the two branches become degenerate and so solutions continue on the opposite branch, forming a parabola around this point. In other words, trajectories staring on one branch bounce off the boundary into the opposite branch so as to never enter the phase space region where only complex solutions exist. Additionally, near this bounce the contracting solution $H < 0$ bounces to expansion $H > 0$ as well. This nonsingular bounce is generally accompanied by a ghost or gradient instability in the scalar or tensor sector of unitary gauge. On the other hand, the condition (34) for the well-definedness of unitary gauge perturbations is itself violated around bounce solutions where the field transits a region where $\Omega = 0$ or $\pm\infty$. In these cases, a covariant treatment or full numerical solution is required to assess perturbation pathologies (see also [41]). For instance, on the second boundary of (59) where H_b diverges at finite H, $\Omega = \pm\infty$. Note that at this boundary, H_b and the original higher-order equations of motion (52), (53) appear discontinuous between the branches but when reduced to a second-order system, the two branches of $\ddot\phi(\phi, \dot\phi)$ join. This property is unique to degenerate models. Finally, there is also a novel feature that some trajectories have the field roll up hill, but we shall see that in general these regions are associated with ghost or gradient instability as well.

On the other hand, the trajectories in the central region of Fig. 1 for $\sigma = +1$ are similar to the canonical ones. Also

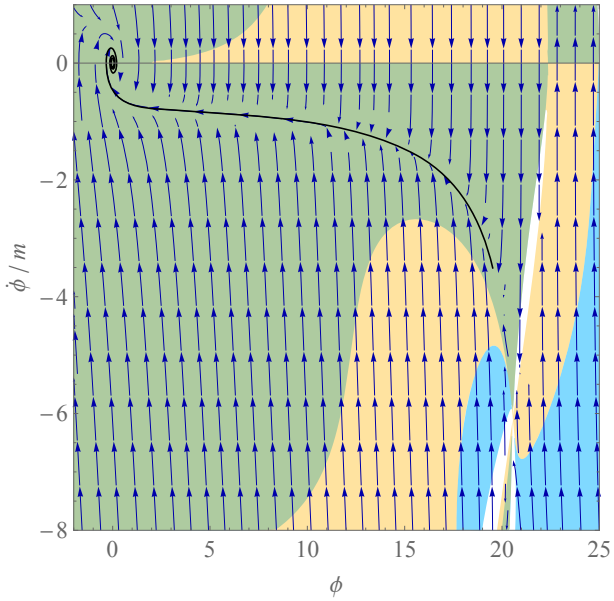

FIG. 2. Phase space regions for the $\sigma = +1$ branch where $b_s\epsilon_H/c_s^2 > 0$ (yellow), $c_s^2 > 0$ (blue), or both (green), with general trajectories (blue arrows) and attractor trajectory (black curve). The attractor trajectory inevitably crosses into a regime of gradient instability when spiraling around the origin (see also Taylor expansions (63) and Fig. 3).

as in GR, trajectories start or end on singularities (solid red curves), albeit here at finite field values. This region also exhibits an attractor solution which is visually apparent from the converging flows in Fig. 2. To isolate this trajectory we numerically integrate the reduced evolution equation $\ddot\phi = \ddot\phi(\phi, \dot\phi)$. For the initial condition, we adopt $(\phi, \dot\phi/m) = (20.3, -6)$ at $t = 0$, which is close to the intersection of the singularity and the attractor and rapidly evolves onto the attractor. The numerical solution $\phi = \phi(t)$ on the attractor (black curve, the left panel of Fig. 1) is shown for the 60 efolds before the end of inflation. Time t is converted to efolds $\mathcal{N}$ by plugging the numerical solution $\phi = \phi(t)$ into the equation $H = H(\phi, \dot\phi)$ and numerically integrating it. We place the zero point of efolds at the end of inflation, i.e. $\epsilon_H = 1$ at $\mathcal{N} = 0$. Note that $\mathcal{N} = 0$ at $(\phi, \dot\phi/m) \approx (1.0, -0.71)$ and $\mathcal{N} = -60$ at $(\phi, \dot\phi/m) \approx (20, -3.5)$. During inflation on the attractor unitary gauge perturbations are well-defined since Ω given in (56) is finite and nonzero.

After inflation when $\mathcal{N} > 0$, the attractor trajectory spirals around the origin and inevitably crosses into a region of non-canonical behavior where $\epsilon_H < 0$. We shall see next that this region is associated with gradient instabilities.

C. Perturbations

The central region of Fig. 1, with its attractor solution on the $\sigma = +1$ branch, provides a potentially viable inflationary regime and we therefore focus on it for the perturbation analysis. From the EFT coefficients (49), (50) and their time derivatives, we can construct b_s, c_s, b_t, c_t and their associated slow-roll parameters. First, the tensor sector is simple. From (39), $b_t = 1$ and $c_t^2 = (1 - 2\dot\phi^2 A_2)^{-1}$ and hence the stability condition $b_t > 0$ and $c_t^2 > 0$ for the present case is satisfied if $|\dot\phi/m| < (2m^2 A_2)^{-1/2} \approx 16$ as it is in the central region.

The scalar sector, parameterized by b_s and c_s, is more complicated. Their explicit forms are too cumbersome to provide here, but straightforward to obtain. In Fig. 2, we show the regions where $\epsilon_H b_s/c_s^2 > 0$ (yellow), $c_s^2 > 0$ (blue), or both (green, instability free) near the attractor of the central region. Note that in our example these are invariant for $(\phi, \dot\phi) \to (-\phi, -\dot\phi)$ so we only display the lower right quadrant. The attractor itself (black line) remains in the stable region from the end of inflation to ~ 60 efolds prior, but approaching it especially from small velocities may require crossing from a region of gradient instability.

At reheating, trajectories spiral around the origin and cross $\dot\phi = 0$. Here the field will inevitably enter into an unstable regime as we can analytically check as follows. For both branches of H_b and a general potential, the Taylor

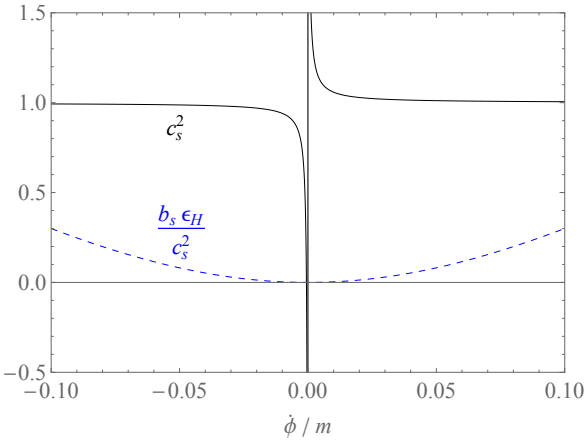

FIG. 3. Behavior of c_s^2 (solid black) and $b_s\epsilon_H/c_s^2$ (dashed blue) and as a function of $\dot\phi/m$ near the origin at fixed $\phi=-0.3$. At $\dot\phi=0$, $c_s^2\to\pm\infty$ when approached from alternate sides. Similar behaviors can be observed for $\phi=0.3$. There exists gradient instability near the origin, while no ghost instability.

expansions of H_b/H and $\dot H$ around $\dot\phi=0$ are given by

$$\begin{aligned}
H &= \sigma\sqrt{\frac{V}{3}}\left[1-\sigma a_1 A_2\dot\phi+\mathcal{O}(\dot\phi^2)\right],\\
H_b &= \sigma\sqrt{\frac{V}{3}}\left[1+\frac{\dot\phi^2}{4V}+\mathcal{O}(\dot\phi^4)\right],\\
\dot H &= \frac{A_2V}{3}\left[a_1^2+8\sigma a_1\dot\phi+\mathcal{O}(\dot\phi^2)\right],\\
\dot H_b &= -\frac{A_2V}{3}\sigma a_1\dot\phi-\frac{\dot\phi^2}{2}+\mathcal{O}(\dot\phi^3),
\end{aligned}\tag{61}$$

where $a_1\equiv\sqrt{\frac{3}{V}}\frac{V'}{1-2A_2V}$. Note that the Taylor expansion of H_b does not include odd powers of $\dot\phi$ since in our model H_b in (55) is a function of $\dot\phi^2$. The leading order behavior of $\dot H$ is a constant as $\dot\phi\to0$ which vanishes if $A_2\to0$. In our model, this is a positive constant so unlike a canonical field $\epsilon_H<0$ as $\dot\phi\to0$ as shown in Fig. 1.

Using (61), we can expand the sound speed and normalization as

$$\begin{aligned}
\lim_{\dot\phi\to0}c_s^2 &= \frac{(H-H_b)(4H+H_b)-\dot H_b}{6(H-H_b)^2},\\
&= -\frac{2\sigma}{3a_1A_2\dot\phi}\left[1+\mathcal{O}(\dot\phi)\right],\\
\lim_{\dot\phi\to0}\frac{b_s\epsilon_H}{c_s^2} &= \frac{6(H-H_b)^2}{H_b^2},\\
&= 6a_1^2A_2^2\dot\phi^2\left[1+\mathcal{O}(\dot\phi)\right].
\end{aligned}\tag{62}$$

The normalization $b_s\epsilon_H/c_s^2$ remains positive but approaches zero as $\dot\phi\to0$, while c_s^2 diverges in amplitude. Notice that this divergence occurs even as $A_2\to0$, despite the fact that $c_s^2=1$ for $A_2=0$, which indicates a discontinuous limit. For the potential $V(\phi)=m^2\phi^2/2$,

$$\lim_{\dot\phi\to0}c_s^2=-\sqrt{\frac{2}{3}}\frac{m}{\dot\phi}\frac{\sigma\phi}{|\phi|}\frac{1-A_2m^2\phi^2}{3m^2A_2}\left[1+\mathcal{O}(\dot\phi)\right].\tag{63}$$

Therefore, for both branches, near the origin where $1-A_2m^2\phi^2>0$ is satisfied, the sign of c_s^2 is determined by $\phi/\dot\phi$. Hence, for $\sigma=+1$ branch, $c_s^2\to-\infty$ at the first and third quadrants and $c_s^2\to+\infty$ in the second and fourth quadrants, where the attractor originates. In the former, $c_s^2<0$ occurs only in a small neighborhood around $\dot\phi=0$ as shown in Fig. 3.

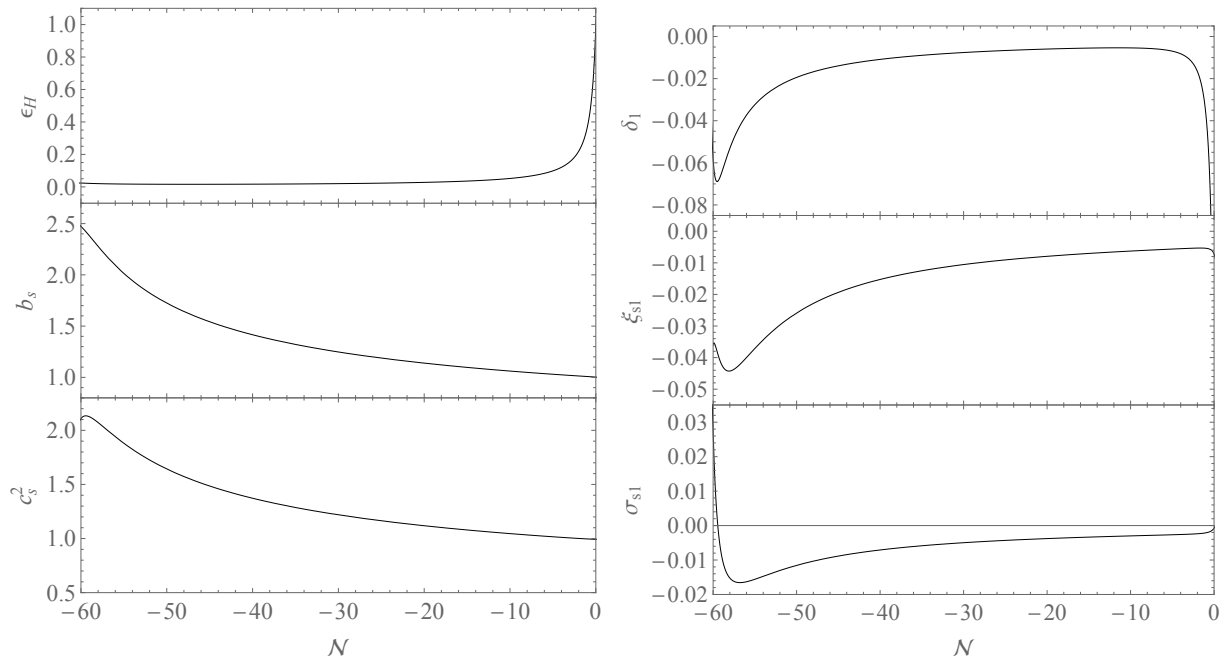

FIG. 4. Variation of ϵ_H, b_s, c_s^2, and corresponding slow-roll parameters $\delta_1, \xi_{s1}, \sigma_{s1}$ along the attractor.

Furthermore, from (56),

$$\lim_{\dot\phi\to 0}\frac{\Omega}{6a^3} = -H_b = -\sigma\sqrt{\frac{V}{3}}\left[1+\mathcal{O}(\dot\phi^2)\right], \tag{64}$$

and hence exactly at the origin $(\phi,\dot\phi)=(0,0)$ of our model, the condition (34) is violated with $\Omega=0$ and the unitary gauge becomes ill-defined.

To avoid gradient instability and unitary gauge being ill-defined, we can relax the assumption that $A_2 = $ const. and choose $A_2(\phi, X) \to 0$ at the origin. Provided this occurs only for $|\phi|, |\dot\phi/m| \lesssim 1$, the dynamics of perturbations during inflation will not be affected.

Finally we can examine the evolution of b_s and c_s^2 along the attractor during inflation. In Fig. 4 we show variation of ϵ_H, b_s, c_s^2, and corresponding slow-roll parameters [7]

$$\delta_1 \equiv \frac{1}{2}\frac{d\ln\epsilon_H}{d\mathcal{N}} - \epsilon_H, \quad \xi_{s1}\equiv\frac{d\ln b_s}{d\mathcal{N}}, \quad \sigma_{s1}\equiv\frac{d\ln c_s}{d\mathcal{N}}, \tag{65}$$

along the attractor. They are defined based on the quadratic action (36) for $\tilde\zeta$, but as we mentioned above in our model $\tilde\zeta-\zeta$ evolves to zero so $\ln\Delta^2_{\tilde\zeta} = \ln\Delta^2_\zeta$ after inflation. Notice that while all remain perturbative, σ_{s1} in particular can become moderately large around $\mathcal{N}\sim -60$ and moreover evolves on the several efold time scale.

Such cases can be treated in the optimized slow-roll (OSR) formalism [7, 35], where the slow-roll (SR) result for the curvature power spectrum after inflation when $\tilde\zeta=\zeta$

$$\ln\Delta^2_\zeta\Big|_{\rm SR} = \ln\left(\frac{H^2}{8\pi^2 b_s c_s \epsilon_H}\right), \tag{66}$$

is corrected by the slow-roll parameters

$$\ln\Delta^2_\zeta \approx \ln\Delta^2_\zeta\Big|_{\rm SR} - \frac{10}{3}\epsilon_H - \frac{2}{3}\delta_1 - \frac{7}{3}\sigma_{s1} - \frac{1}{3}\xi_{s1}, \tag{67}$$

and evaluated at freezeout where $k\int_{\mathcal{N}}^0 d\mathcal{N} c_s/aH \approx e^{1.06}$, contrary to $k\int_{\mathcal{N}}^0 d\mathcal{N} c_s/aH = 1$ in the SR approximation.

These approximations are compared in Fig. 5 for the same k, where k_0 is the mode that freezes out at $\mathcal{N}=-60$ in OSR. Here we additionally choose $m=10^{-5}$ to fix the normalization of H in Planck units and hence that of Δ^2_ζ

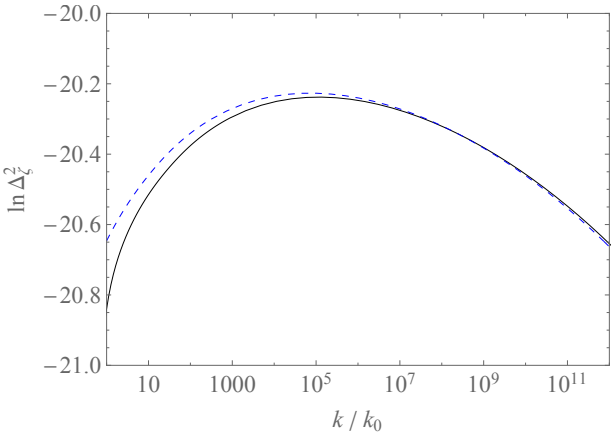

FIG. 5. The curvature power spectrum $\ln \Delta^2_\zeta$ evaluated under the improved OSR approximation (67) (solid black) compared with the SR approximation (dashed blue). k_0 represents the mode that freezes out in the OSR approximation at $\mathcal{N} = -60$.

to be roughly compatible with observations. Notice that there is a significant running of the tilt pivoting around $k/k_0 \sim 10^4$ or $\mathcal{N} \sim -50$ despite being far from the end of inflation and containing no features in the potential there. In this region, the OSR and SR results differ in shape and OSR itself breaks down as an approximation for some $\mathcal{N} < -60$ where the corrections become order unity. The OSR approximation thus extends the regime of validity for the calculation into the range $-60 \lesssim \mathcal{N} \lesssim -50$, which is relevant for the CMB, and is useful in observationally constraining D-inflation. We leave such a study and the construction of an observationally viable model to a future work.

IV. CONCLUSION

In this work, we developed the EFT of inflation for a general Lagrangian constructed from ADM variables, which encompasses the most general interactions with up to second derivatives of a scalar field whose background spontaneously breaks temporal diffeomorphism symmetry. The Ostrogradsky ghost usually implied by such higher-order terms is eliminated by degeneracy conditions, leading to degenerate higher-order (or D-)inflation. We identify 8 types of degeneracy conditions, one of which corresponds to known DHOST models. For the other cases, which include curvature couplings not considered in DHOST, we provide necessary conditions for a covariant scalar-tensor theory based on the dispersion relation of the quadratic action and leave a full assessment of their viability to future work.

Higher-order theories imply equations of motions that are higher than second order in the scalar field and typically lead to an ill-posed Cauchy problem. The degeneracy conditions, which involve the metric as well, restores a well-defined forwards or backwards evolution from initial field and field derivative data on a Cauchy surface but with novel features.

We illustrate these features with an explicit example of D-inflation. First, not all field configurations lead to physical solutions for the metric as illustrated by values where all solutions for the Hubble parameter become complex even for positive potentials and timelike field gradients. Second, evolution is only uniquely defined up to a branch choice since the same field configurations lead to distinct expansion histories that are not related by time reversal as they would be in GR. This feature is present in Horndeski theory as well. Third, trajectories can sharply turn to avoid phase space regions where real solutions fail to exist leading to highly complicated phase space portraits where contraction can turn to expansion without encountering a curvature singularity. These bouncing solutions generally traverse regions of ghost or gradient instabilities in unitary gauge but also cross coordinate singularities in defining its metric perturbations (see also [41]). Finally, perturbations can go unstable even in the limit that the additional degenerate terms in the Lagrangian are infinitesimal. In our example this occurs for curvature perturbations in the simplest model of constant, but arbitrarily small, higher-order coefficients during reheating when the inflaton oscillates around the minimum of its potential.

Our D-inflation model also has novel phenomenology. While the model possesses an attractor which leads to nearly scale invariant fluctuations across a sufficient number of efolds of inflation, it also can produce substantial running of the tilt on CMB scales despite having no features in the potential there and being far from the end of inflation. In this case, EFT coefficients vary on the several efold timescale and require an approach that goes beyond the usual

slow-roll formalism. We show that corrections captured by the optimized slow-roll approach extends the validity of predictions into the large running regime of interest and should be useful in observational tests of the D-inflation scenario.

ACKNOWLEDGMENTS

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Appendix A: Degeneracy Conditions

We can determine the necessary conditions for degeneracy by examining the high k or high frequency limit of the quadratic Lagrangian. We can then find the number of propagating modes and their dispersion relation by assuming solutions of the form $u(x,t) = u(k)e^{i(\omega t+kx)}$ where $u = (\delta N, \zeta, \psi)^T$ [29].

In the limit, we can neglect evolution on the Hubble time scale of the background and the EFT coefficients up to corrections of order $(k/aH)^2$, which as we detail below is sufficient to establish degeneracy conditions for most solutions and easily supplemented in the remaining ones. The quadratic Lagrangian (26) for scalars can be then written as

$$\mathcal{L}_2 = \frac{1}{2}u^\dagger K u, \tag{A1}$$

with the kinetic matrix

$$K \equiv \begin{pmatrix} c_9 + \omega^2 c_3 + c_{10}\frac{k^2}{a^2} & \omega^2 c_2 + i\omega\left(c_4 + c_5\frac{k^2}{a^2}\right) + c_8\frac{k^2}{a^2} & -i\omega\frac{c_2}{3}\frac{k^2}{a^2} + \frac{c_4}{3}\frac{k^2}{a^2} \\ \omega^2 c_2 - i\omega\left(c_4 + c_5\frac{k^2}{a^2}\right) + c_8\frac{k^2}{a^2} & \omega^2 c_1 + c_6\frac{k^2}{a^2} + c_7\frac{k^4}{a^4} & -i\omega\frac{c_1}{3}\frac{k^2}{a^2} + c_{12}\frac{k^4}{a^4} \\ i\omega\frac{c_2}{3}\frac{k^2}{a^2} + \frac{c_4}{3}\frac{k^2}{a^2} & i\omega\frac{c_1}{3}\frac{k^2}{a^2} + c_{12}\frac{k^4}{a^4} & c_{11}\frac{k^4}{a^4} \end{pmatrix}. \tag{A2}$$

For nontrivial solutions of the equation of motion $Ku = 0$ to exist, we require $\det K = 0$, which can be written as

$$f_1\omega^4 + \left\{f_2\left(\frac{k}{a}\right)^4 + f_3\left(\frac{k}{a}\right)^2 + f_4\right\}\omega^2 + f_5\left(\frac{k}{a}\right)^6 + f_6\left(\frac{k}{a}\right)^4 + f_7\left(\frac{k}{a}\right)^2 = 0, \tag{A3}$$

where

$$\begin{aligned} f_1 &\equiv -\left(\frac{c_1}{9} - c_{11}\right)(c_1c_3 - c_2^2), \\ f_2 &\equiv c_3(c_7c_{11} - c_{12}^2) - \frac{1}{9}c_2^2c_7 - \frac{2}{3}c_2c_5c_{12} - c_5^2c_{11}, \\ f_3 &\equiv \frac{c_6}{9}(c_1c_3 - c_2^2) - \left(\frac{c_1}{9} - c_{11}\right)(c_1c_{10} - 2c_2c_8 + c_3c_6 - 2c_4c_5), \\ f_4 &\equiv \left(\frac{c_1}{9} - c_{11}\right)(c_4^2 - c_1c_9), \\ f_5 &\equiv c_{10}(c_7c_{11} - c_{12}^2), \\ f_6 &\equiv c_9(c_7c_{11} - c_{12}^2) + c_6c_{10}c_{11} - \frac{1}{9}c_4^2c_7 + \frac{2}{3}c_4c_8c_{12} - c_8^2c_{11}, \\ f_7 &\equiv c_6\left(c_9c_{11} - \frac{c_4^2}{9}\right). \end{aligned} \tag{A4}$$

In general, this is a fourth order system for ω representing two propagating modes. To remove the second propagating mode in unitary gauge, we demand $f_1 = 0$, for which there are several possibilities:

1. $c_3 = c_2^2/c_1$ or equivalently $\mathcal{C}_{\beta\beta} = 3\mathcal{C}_{\beta K}^2/(3\mathcal{C}_{KK} + \tilde{\mathcal{C}}_{KK})$. The kinetic terms organize into a single term for $\tilde{\zeta} = \zeta + (c_2/c_1)\delta N$, which is the propagating degree of freedom.

2. $c_1 = c_2 = 0$ or equivalently $3\mathcal{C}_{KK} + \tilde{\mathcal{C}}_{KK} = \mathcal{C}_{\beta K} = 0$. The kinetic term for ζ vanishes and δN is the propagating degree of freedom.

3. $c_{11} = c_1/9$ or equivalently $\tilde{\mathcal{C}}_{KK} = 0$. The constraint equation for ψ eliminates the kinetic term for ζ and δN is again the propagating degree of freedom.

Below we shall consider each case in Appendix A 1, A 2, A 3, respectively.

Furthermore, retaining a higher order in spatial derivatives (or k) compared with temporal derivatives (or ω) in a fully covariant theory corresponds to the reappearance of the second mode when changing the gauge. Therefore to find covariant degeneracy conditions, we seek solutions of Eq. (A3) that correspond to a normal dispersion relation $\omega^2 = c_s^2 k^2$. The possible cases are:

a. $f_4 f_7 \neq 0$, others $= 0 \;\Rightarrow\; f_4\omega^2 + f_7(k/a)^2 = 0$.

b. $f_3 f_6 \neq 0$, others $= 0 \;\Rightarrow\; f_3\omega^2 + f_6(k/a)^2 = 0$.

c. $f_2 f_5 \neq 0$, others $= 0 \;\Rightarrow\; f_2\omega^2 + f_5(k/a)^2 = 0$.

There are several caveats regarding this technique that need to be borne in mind. Since we neglect Hubble scale evolution, we work in the limit $\omega \gg H$ and since the c_i coefficients generically carry a mass dimension $M_i^{n_i}$ for some n_i and can vary on the Hubble time scale we assume $\dot{c}_i/c_i \sim H \ll M_i$ as well. Whereas the former condition corresponds to $c_s k/aH \gg 1$ for a linear dispersion relation, the latter need not necessarily be small in practice. Since we are mainly interested in this technique for deriving degeneracy conditions and the form of the dispersion relation, rather than the exact coefficients in the dispersion relation, this technique suffices. The exception is when $(c_s k/aH)^{-2}$ corrections spoil the form of the dispersion relation at $c_s k/aH \gg 1$. This can occur in the "a" case through corrections to f_2 and f_5 which can then dominate over the terms from f_4 and f_7 which form the desired linear dispersion relation. For the "b" case these corrections can change the coefficients but not the leading order form and for the "c" case, the corrections from the other terms are entirely negligible for $c_s k/aH \gg 1$. We therefore further check for supplemental degeneracy conditions in the "a" or $f_4 f_7 \neq 0$ case. Note that since the coefficients in the dispersion relation can also change in the "a" and "b" cases from those given by this static technique, the full quadratic Lagrangian should be used to check for ghost and gradient instabilities in those cases.

We now consider the various kinetic structures $1, 2, 3$ and their degeneracy conditions under a, b, c respectively. We treat "1a" in more detail as it serves both as an example of the technique and includes the known DHOST models.

1. $c_3 = c_2^2/c_1$ Case

Plugging $c_3 = c_2^2/c_1$ to (A4) we have $f_1 = 0$ and reduced forms for $f_{2\ldots7}$ which imply there is a single propagating degree of freedom

$$\tilde{\zeta} = \zeta + \frac{c_2}{c_1}\delta N \tag{A5}$$

in unitary gauge. The degeneracy classes a, b, c where this degree of freedom obeys a linear dispersion relation for $c_s k/aH \gg 1$ are defined by the pair of f coefficients that remain non-zero. Therefore in each case, we have 5 degeneracy conditions between the various EFT coefficients represented by c_i in the static limit. Case 1a can have supplementary degeneracy conditions beyond the static limit as discussed above.

a. First, let us consider the case $f_4 f_7 \neq 0$ and all other f's zero. Requiring first that $f_5 = 0$ leads to two branches

$$c_{12}^2 = c_7 c_{11} \quad \text{or} \quad c_{10} = 0, \tag{A6}$$

For each case, $f_2 = f_3 = f_6 = 0$ should be satisfied. While in general $f_3 = 0$ has two branches, since $f_4 \neq 0$ implies $c_{11} \neq c_1/9$, only one solution remains

$$c_1^2 c_{10} = 2c_1(c_2 c_8 + c_4 c_5) - c_2^2 c_6. \tag{A7}$$

Also, $f_2 = 0$ yields

$$c_5^2 c_{11} + \frac{1}{9}c_2^2 c_7 + \frac{2}{3}c_2 c_5 c_{12} = \frac{c_2^2}{c_1}(c_7 c_{11} - c_{12}^2), \tag{A8}$$

and $f_6 = 0$ yields

$$c_9(c_7c_{11} - c_{12}^2) + c_6c_{10}c_{11} - c_8^2c_{11} - \frac{1}{9}c_4^2c_7 + \frac{2}{3}c_4c_8c_{12} = 0. \tag{A9}$$

These are the degeneracy conditions for the static, high k limit.

More generally, this "1a" case is subject to corrections which require supplementary degeneracy conditions as described above. Using only the first condition $c_3 = c_2^2/c_1$, the quadratic Lagrangian (26) reduces to

$$\begin{aligned}\mathcal{L}_2 = {} & \frac{1}{2}c_1\dot{\tilde{\zeta}}^2 + \left(c_4 - c_1\dot{x} + c_5\frac{k^2}{a^2}\right)\dot{\tilde{\zeta}}\delta N + \frac{1}{2}\left(c_6 + c_7\frac{k^2}{a^2}\right)\frac{k^2}{a^2}\tilde{\zeta}^2 + \left(c_8 - c_6x - c_7x\frac{k^2}{a^2}\right)\frac{k^2}{a^2}\tilde{\zeta}\delta N \\ & + \frac{1}{2}\left(\tilde{c}_9 + \tilde{c}_{10}\frac{k^2}{a^2} + c_7x^2\frac{k^4}{a^4}\right)\delta N^2 + \frac{1}{2}c_{11}\frac{k^4}{a^4}\psi^2 + \frac{k^2}{a^2}\psi\left[\frac{c_1}{3}\dot{\tilde{\zeta}} + \frac{c_4 - c_1\dot{x}}{3}\delta N + c_{12}\frac{k^2}{a^2}(\tilde{\zeta} - x\delta N)\right],\end{aligned} \tag{A10}$$

where $x \equiv c_2/c_1$ and

$$\begin{aligned}\tilde{c}_9 &\equiv c_9 + \dot{c}_4x + c_1\dot{x}^2 - c_4\dot{x}, \\ \tilde{c}_{10} &\equiv c_{10} + (\dot{c}_5 - 2Hc_5 - 2c_8)x + c_6x^2 - c_5\dot{x}.\end{aligned} \tag{A11}$$

Potentially problematic terms are those where the fields have time derivatives and the coefficients carry additional factors of k^2/a^2. In the static limit, these terms are arranged to cancel, but beyond the static limit the time-variation of the coefficients breaks this degeneracy relation and changes the dispersion relation even for $c_sk/aH \gg 1$. The only term of this form is $c_5(k/a)^2\dot{\tilde{\zeta}}\delta N$. Therefore $c_5 = 0$ is sufficient as a supplemental degeneracy condition to ensure a linear dispersion relation for the the single propagating degree of freedom $\tilde{\zeta}$. This condition may be generalized to nonvanishing c_5 but would then involve tuning between c_5, a and the other c_i coefficients. Due to the appearance of the scale factor a in the generalized degeneracy condition this tuning is unlikely to be preserved in a fully covariant scalar-tensor theory. We therefore take $c_5 = 0$ and the complete set of degeneracy conditions for case 1a has two branches

$$\begin{aligned}i)\quad & c_3 = \frac{c_2^2}{c_1}, \quad c_{12}^2 = c_7c_{11}, \quad c_1^2c_{10} = 2c_1c_2c_8 - c_2^2c_6, \quad c_2^2c_7 = \frac{9c_2^2}{c_1}(c_7c_{11} - c_{12}^2), \\ & c_9(c_7c_{11} - c_{12}^2) + c_6c_{10}c_{11} - c_8^2c_{11} - \frac{1}{9}c_4^2c_7 + \frac{2}{3}c_4c_8c_{12} = 0, \quad c_5 = 0,\end{aligned} \tag{A12}$$

$$\begin{aligned}ii)\quad & c_3 = \frac{c_2^2}{c_1}, \quad c_{10} = 0, \quad 2c_1c_2c_8 - c_2^2c_6 = 0, \quad c_2^2c_7 = \frac{9c_2^2}{c_1}(c_7c_{11} - c_{12}^2), \\ & c_9(c_7c_{11} - c_{12}^2) + c_6c_{10}c_{11} - c_8^2c_{11} - \frac{1}{9}c_4^2c_7 + \frac{2}{3}c_4c_8c_{12} = 0, \quad c_5 = 0.\end{aligned} \tag{A13}$$

With this complete set of degeneracy conditions, one can explicitly verify that the Euler-Lagrange equations that result from Eq. (A10) describe a single propagating degree of freedom $\tilde{\zeta}$ with a linear dispersion relation at $c_sk/aH \gg 1$.

While the degeneracy conditions in Eq. (A12) or (A13) are complete, they allow for a variety of ways that the EFT coefficients can satisfy them. To make the connection with DHOST models, we can further examine these explicit solutions. For instance, for 1a-i we can have

$$c_3 = \frac{c_2^2}{c_1}, \quad c_5 = c_7 = c_{11} = c_{12} = 0, \quad c_{10} = \frac{2c_1c_2c_8 - c_2^2c_6}{c_1^2}, \tag{A14}$$

or

$$c_3 = \frac{c_2^2}{c_1}, \quad c_5 = c_7 = c_{12} = 0, \quad c_8 = \frac{c_2c_6}{c_1}, \quad c_{10} = \frac{c_2^2c_6}{c_1^2}, \tag{A15}$$

and for 1a-ii

$$c_3 = \frac{c_2^2}{c_1}, \quad c_5 = c_{10} = 0, \quad c_6 = \frac{2c_1c_8}{c_2}, \quad c_{11} = \frac{c_1}{9} + \frac{c_{12}^2}{c_7}, \quad c_9 = \frac{(c_4c_7 - 3c_8c_{12})^2 + c_1c_7c_8^2}{c_1c_7^2}. \tag{A16}$$

Models where $c_{11}=0$ on the (A15) branch are also members of the (A14) branch. On the other hand the conditions $c_8=c_2c_6/c_1$, $\tilde{c}_{10}=0$ ($c_{10}=c_2^2c_6/c_1^2$, $c_5=0$) must be satisfied for any model on the (A15) branch, including those that are part of the (A14) branch. As pointed out by Ref. [29], this presents a problem if one wants to recover Newtonian gravity for non-relativistic matter. Since these conditions and $c_7=0$ zero out the $k^2\tilde{\zeta}\delta N$ and $k^2\delta N^2$ terms in (A10), the Euler-Lagrange equation for δN which usually provides a source to the Poisson equation through the matter density is absent on this branch. Instead the $k^2\tilde{\zeta}$ term in its equation of motion comes from its own Euler-Lagrange equation and has a source in matter pressure. For this reason, in the main text we focus on the (A14) branch. This branch also includes the 2N-I/Ia class of DHOST models [29].

The case 1a-i with (A14) or (A15) corresponds to DHOST class I or II, respectively, and the latter was known to suffer from gradient instability. On the other hand, the case 1a-ii with (A16) is not included in DHOST theories, as it requires $c_7\neq 0$, namely $8\mathcal{C}_{RR}+3\tilde{\mathcal{C}}_{RR}\neq 0$, which can originate from the existence of the quadratic curvature terms in the covariant Lagrangian.

b. Next, we consider $f_3f_6\neq 0$ and all other f's zero. By following the same procedure as in case 1a, we obtain the following four sets of degeneracy conditions

$$
\begin{aligned}
&i) \quad c_3=\frac{c_2^2}{c_1}, \quad c_{12}^2=c_7c_{11}, \quad c_4=c_9=0, && 9c_5^2c_{11}+6c_2c_5c_{12}+c_2^2c_7=0,\\
&ii) \quad c_3=\frac{c_2^2}{c_1}, \quad c_{12}^2=c_7c_{11}, \quad c_6=0,\ c_9=\frac{c_4^2}{c_1}, && 9c_5^2c_{11}+6c_2c_5c_{12}+c_2^2c_7=0,\\
&iii) \quad c_3=\frac{c_2^2}{c_1}, \quad c_{10}=0, \quad c_4=c_9=0, && 9c_5^2c_{11}+6c_2c_5c_{12}+c_2^2c_7=\frac{9c_2^2}{c_1}(c_7c_{11}-c_{12}^2),\\
&iv) \quad c_3=\frac{c_2^2}{c_1}, \quad c_{10}=0, \quad c_6=0,\ c_9=\frac{c_4^2}{c_1}, && 9c_5^2c_{11}+6c_2c_5c_{12}+c_2^2c_7=\frac{9c_2^2}{c_1}(c_7c_{11}-c_{12}^2).
\end{aligned}
\tag{A17}
$$

In this case, corrections beyond the static approximation can change the coefficients of the dispersion relation at $c_sk/aH\gg 1$ but not the form and so these provide the complete conditions.

Note that the above branch is not included in DHOST theories. While (A14) and (A15) should satisfy $c_3=c_2^2/c_1$, $c_5=c_7=c_{12}=0$, $c_{10}=(2c_1c_2c_8-c_2^2c_6)/c_1^2$, the above branch satisfies the degeneracy without requiring c_5,c_7,c_{12} to be vanishing. By definition c_7 and c_{12} are nonvanishing in the presence of quadratic curvature terms, whereas c_5 is nonvanishing if Lagrangian includes terms such as $\left({}^{(4)}R+\Box\phi\right)^2$. Also, i) and ii) does not require any condition on c_{10}, and iii) and iv) requires a different condition $c_{10}=0$ which can be satisfied with $2c_1c_2c_8-c_2^2c_6\neq 0$ as c_8 does not appear in the degeneracy conditions and hence is a free parameter.

c. Finally, we consider $f_2f_5\neq 0$ and all other f's zero. This leads to six possible cases

$$
\begin{aligned}
&i) \quad c_3=\frac{c_2^2}{c_1}, \quad c_4=c_9=0, && c_{11}=\frac{c_1}{9}, && c_6c_{10}=c_8^2,\\
&ii) \quad c_3=\frac{c_2^2}{c_1}, \quad c_4=c_9=0, && c_6=\frac{c_1c_8}{c_2}, && c_{10}=\frac{c_2c_8}{c_1},\\
&iii) \quad c_3=\frac{c_2^2}{c_1}, \quad c_6=0,\ c_9=\frac{c_4^2}{c_1}, && c_{11}=\frac{c_1}{9}, && c_8=\frac{3c_4c_{12}}{c_1},\\
&iv) \quad c_3=\frac{c_2^2}{c_1}, \quad c_6=0,\ c_9=\frac{c_4^2}{c_1}, && c_{10}=\frac{2(c_2c_8+c_4c_5)}{c_1}, && c_8^2c_{11}=\frac{c_4^2}{c_1}(c_7c_{11}-c_{12}^2)+c_4\left(\frac{2}{3}c_8c_{12}-\frac{1}{9}c_4c_7\right),\\
&v) \quad c_3=\frac{c_2^2}{c_1}, \quad c_6=0,\ c_{11}=\frac{c_1}{9}, && c_{11}c_8^2=c_9\left(\frac{1}{9}c_1c_7-c_{12}^2\right)+c_4\left(\frac{2}{3}c_8c_{12}-\frac{1}{9}c_4c_7\right)=0,\\
&vi) \quad c_3=\frac{c_2^2}{c_1}, \quad c_9=\frac{c_4^2}{c_1},\ c_{11}=\frac{c_1}{9}, && c_6c_{10}=\frac{(c_1c_8-3c_4c_{12})^2}{c_1^2}.
\end{aligned}
\tag{A18}
$$

Again, note that this branch is not included in DHOST theories as the DHOST conditions $c_3=c_2^2/c_1$, $c_5=c_7=c_{12}=0$, $c_{10}=(2c_1c_2c_8-c_2^2c_6)/c_1^2$ are not satisfied in general. Clearly, the conditions i), ii) do not include c_5, c_7, c_{12}; the condition iii) does not include c_5, c_7, c_{10}; the condition v) does not include c_5, c_{10}; and the condition vi) does not include c_5, c_7. Also, the condition iv) as well as i), ii), vi) require different conditions on c_{10} which do not coincide with the DHOST condition in general.

2. $c_1 = c_2 = 0$ Case

In case 2, $c_1 = c_2 = 0$ and the kinetic term for ζ vanishes leaving $f_1 = 0$ and δN as the propagating degree of freedom. Since models where δN and not ζ is propagating are unlikely to recover Newtonian gravity, we include this case for completeness and pedagogical interest.

a. Let us begin with the case $f_4 f_7 \neq 0$ and all other f's zero. In this case we must again check for corrections to the dispersion relation beyond the static limit. Using only the condition $c_1 = c_2 = 0$, the quadratic Lagrangian (26) reduces to

$$\begin{aligned}\mathcal{L}_2 = &\frac{1}{2}c_3\dot{\delta N}^2 + \left(c_4 + c_5\frac{k^2}{a^2}\right)\dot{\zeta}\delta N \\ &+ \frac{1}{2}\left(c_6\frac{k^2}{a^2} + c_7\frac{k^4}{a^4}\right)\zeta^2 + c_8\frac{k^2}{a^2}\zeta\delta N + \frac{1}{2}\left(c_9 + c_{10}\frac{k^2}{a^2}\right)\delta N^2 \\ &+ \frac{1}{2}c_{11}\frac{k^4}{a^4}\psi^2 + \psi\left(\frac{c_4}{3}\frac{k^2}{a^2}\delta N + c_{12}\frac{k^4}{a^4}\zeta\right).\end{aligned} \tag{A19}$$

Here, again, the problematic term is $c_5(k/a)^2\dot{\zeta}\delta N$, and hence we impose $c_5 = 0$ as a supplemental degeneracy condition.

Requiring $f_2 = f_3 = f_5 = f_6 = 0$ leads to four possible cases

$$\begin{aligned}
&i) \quad c_1 = c_2 = 0, \quad c_5 = c_{10} = 0, && c_3 = 0, \quad 9c_9(c_{12}^2 - c_7c_{11}) + 9c_8^2c_{11} + c_4^2c_7 - 6c_4c_8c_{12} = 0, \\
&ii) \quad c_1 = c_2 = 0, \quad c_5 = c_{10} = c_{12}^2 - c_7c_{11} = 0, && c_6 = 0, \quad 9c_8^2c_{11} + c_4^2c_7 - 6c_4c_8c_{12} = 0, \\
&iii) \quad c_1 = c_2 = 0, \quad c_5 = c_{12}^2 - c_7c_{11} = 0, && c_3 = 0, \quad 9c_6c_{10}c_{11} - 9c_8^2c_{11} - c_4^2c_7 + 6c_4c_8c_{12} = 0, \\
&iv) \quad c_1 = c_2 = 0, \quad c_5 = c_{12}^2 - c_7c_{11} = 0, && c_6 = 0, \quad 9c_8^2c_{11} + c_4^2c_7 - 6c_4c_8c_{12} = 0.
\end{aligned} \tag{A20}$$

b. Next we consider the case $f_3 f_6 \neq 0$ and all other f's zero. Requiring $f_4 = f_7 = 0$ under $f_3 = c_{11}(c_3c_6 - 2c_4c_5) \neq 0$ allows $c_4 = c_9 = 0$ only. By further requiring $f_2 = f_5 = 0$, we obtain three possible cases

$$\begin{aligned}
&i) \quad c_1 = c_2 = 0, \quad c_4 = c_9 = 0, \quad c_{10} = c_3(c_7c_{11} - c_{12}^2) - c_5^2c_{11} = 0, \\
&ii) \quad c_1 = c_2 = 0, \quad c_4 = c_9 = 0, \quad c_5 = c_{12}^2 - c_7c_{11} = 0, \\
&iii) \quad c_1 = c_2 = 0, \quad c_4 = c_9 = 0, \quad c_{11} = c_{12} = 0.
\end{aligned} \tag{A21}$$

c. Finally, we consider $f_2 f_5 \neq 0$ and all other f's zero. This leads to five possible cases

$$\begin{aligned}
&i) \quad c_1 = c_2 = 0, \quad c_4 = c_9 = 0, \quad c_3 = 0, \quad c_6c_{10} - c_8^2 = 0, \\
&ii) \quad c_1 = c_2 = 0, \quad c_4 = c_9 = 0, \quad c_6 = 0, \quad c_{11}(c_6c_{10} - c_8^2) = 0, \\
&iii) \quad c_1 = c_2 = 0, \quad c_4 = c_9 = 0, \quad c_{11} = 0, \\
&iv) \quad c_1 = c_2 = 0, \quad c_4 = c_6 = 0, \quad c_9(c_7c_{11} - c_{12}^2) - c_8^2c_{11} = 0, \\
&v) \quad c_1 = c_2 = 0, \quad c_6 = c_{11} = 0, \quad 9c_9c_{12}^2 + c_4^2c_7 - 6c_4c_8c_{12} = 0.
\end{aligned} \tag{A22}$$

3. $c_{11} = c_1/9$ Case

Finally we consider the case 3 where $c_{11} = c_1/9$. Here the Euler-Lagrange equation for ψ provides a contribution that cancels the kinetic term for ζ, and δN is again the propagating degree of freedom. As in case 2, this case is unlikely to provide viable theories of gravity. While generally we expect 5 static degeneracy conditions, in this case there are fewer since some of the f terms are identically zero once other degeneracy conditions are applied.

a. The $f_4 f_7 \neq 0$ branch does not exist since $c_{11} = c_1/9$ implies $f_4 = 0$.

b. Next, for $f_3 f_6 \neq 0$, requiring that additionally $f_2 = f_5 = f_7 = 0$ leads to two possible cases

$$\begin{aligned}
&i) \quad c_{11} = \frac{c_1}{9}, \quad c_4^2 = c_1c_9, \quad c_7 = \frac{c_1c_5^2 + 6c_2c_5c_{12} + 9c_3c_{12}^2}{c_1c_3 - c_2^2}, \quad c_{10} = 0, \\
&ii) \quad c_{11} = \frac{c_1}{9}, \quad c_4^2 = c_1c_9, \quad c_7 = \frac{c_1c_5^2 + 6c_2c_5c_{12} + 9c_3c_{12}^2}{c_1c_3 - c_2^2}, \quad c_1c_5 + 3c_2c_{12} = 0.
\end{aligned} \tag{A23}$$

c. Finally $f_2 f_5 \neq 0$ leads to two possible cases

$$\begin{aligned} &i) \qquad c_{11} = \frac{c_1}{9}, \qquad c_4^2 = c_1 c_9, \qquad c_2^2 = c_1 c_3, \qquad c_1(c_6 c_{10} - c_8^2) + 6c_4 c_8 c_{12} - 9c_9 c_{12}^2 = 0, \\ &ii) \qquad c_{11} = \frac{c_1}{9}, \qquad c_6 = 0, \qquad c_9 = -\frac{6c_4 c_8 c_{12} - c_1 c_8^2 - c_4^2 c_7}{c_1 c_7 - 9c_{12}^2}. \end{aligned} \tag{A24}$$

Appendix B: Relationship to literature

Our approach is most similar to Ref. [29] and in this Appendix we make the explicit connection to that work and discuss the differences. First some of the terms in Ref. [29] take a superficially different form that is related to ours through integration by parts. Up to a total derivative

$$N\sqrt{h}\tilde{\mathcal{C}}_{KR}\delta K^i_j \delta R^j_i \sim \frac{a^3}{2}\left[(\dot{\tilde{\mathcal{C}}}_{KR} + H\tilde{\mathcal{C}}_{KR})\left(\frac{\delta\sqrt{h}}{a^3}\delta R + \delta_2 R\right) + \tilde{\mathcal{C}}_{KR}\delta R\delta K + H\tilde{\mathcal{C}}_{KR}\delta N \delta R\right], \tag{B1}$$

and hence we can rewrite our quadratic Lagrangian (23) in the form of Eq. (1.2) of [29] expose the difference between the two

$$\begin{aligned} \delta\mathcal{L}_2 \sim\ & -a^3\mathcal{C}_\beta\left[\delta N\left(\frac{\delta_1\sqrt{h}}{a^3}\right)^{\cdot} + \delta N^i \delta N_{,i}\right] \\ & + a^3\left[\mathcal{C}_{\beta R}\delta_1 R \delta\dot{N} + \frac{1}{2}\mathcal{C}_{RR}\delta_1 R^2 + \frac{1}{2}\tilde{\mathcal{C}}_{RR}\delta_1 R^i_j \delta_1 R^j_i + \left(\mathcal{C}_{KR} + \frac{1}{2}\tilde{\mathcal{C}}_{KR}\right)\delta K \delta_1 R\right]. \end{aligned} \tag{B2}$$

The quadratic Lagrangian (23) thus contains terms that differ from Eq. (1.2) of [29]. Since the first term in (B2) has δN or $\delta N_{,i}$, it is nonvanishing only for scalar perturbations. For scalar perturbation, it can be expressed up to a total derivative as

$$\begin{aligned} -a^3\mathcal{C}_\beta\left[\delta N\left(\frac{\delta_1\sqrt{h}}{a^3}\right)^{\cdot} + \delta N^i\delta N_{,i}\right] &\sim -a^3\mathcal{C}_\beta\left(3\dot{\zeta} - \frac{\partial^2\psi}{a^2}\right)\delta N \\ &= -a^3\mathcal{C}_\beta(\delta K + 3H\delta N)\delta N, \end{aligned} \tag{B3}$$

which can be absorbed into the α_B and α_K terms in Eq. (1.2) of [29]. On the other hand, the third line of (B2) is not considered in [29] as these terms have derivatives higher than second order in total. If we assume these terms are vanishing by imposing

$$\mathcal{C}_{\beta R} = \mathcal{C}_{RR} = \tilde{\mathcal{C}}_{RR} = \mathcal{C}_{KR} + \frac{1}{2}\tilde{\mathcal{C}}_{KR} = 0, \tag{B4}$$

we have $c_5 = c_7 = c_{12} = 0$ in (27). These conditions hold in the 1a degeneracy subclasses defined by Eq. (A14) and (A15). Ref. [29] considered only these cases. They furthermore assume $\phi \propto t$ and so their

$$V = -\frac{\dot{\phi}}{N}\beta \tag{B5}$$

vanishes in the background $\bar{V} = 0$ or $\bar{\beta} = 0$ in our notation. Generalizing this does not change the functional form of their Lagrangian, just the mapping between the scalar field and ADM representations and so we retain $\bar{\beta} \neq 0$ in the correspondences below. Note that if a field redefinition $\varphi \propto t(\phi)$ is performed instead after solving for the background $\phi(t)$, which alternately reestablishes the generality of their expressions, then the DHOST coefficients must correspondingly be redefined (cf. [30] v2).

In summary, in the subclass of (B4), the quadratic Lagrangian (23) for scalar perturbation takes the same functional

form as Eq. (1.2) of [29] with the correspondence

$$
\begin{aligned}
M^2 &= \tilde{\mathcal{C}}_{KK},\\
\alpha_K &= \frac{1}{H^2\tilde{\mathcal{C}}_{KK}}\left(2\mathcal{C}_N+\mathcal{C}_{NN}-2\bar{\beta}\mathcal{C}_{\beta N}+\bar{\beta}^2\mathcal{C}_{\beta\beta}+\frac{(a^3\mathcal{C}_\beta)^\cdot}{a^3}-\frac{(a^3\mathcal{C}_{\beta N})^\cdot}{a^3}+\frac{(a^3\bar{\beta}\mathcal{C}_{\beta\beta})^\cdot}{a^3}-6H\mathcal{C}_\beta\right),\\
\alpha_B &= \frac{\mathcal{C}_{NK}-\bar{\beta}\mathcal{C}_{\beta K}-\mathcal{C}_\beta}{2H\tilde{\mathcal{C}}_{KK}},\\
\alpha_T &= \frac{2\mathcal{C}_R+\dot{\tilde{\mathcal{C}}}_{KR}+H\tilde{\mathcal{C}}_{KR}}{\tilde{\mathcal{C}}_{KK}}-1,\\
\alpha_H &= \frac{2(\mathcal{C}_{NR}+\mathcal{C}_R)+H\tilde{\mathcal{C}}_{KR}}{\tilde{\mathcal{C}}_{KK}}-1,\\
\alpha_L &= -\frac{3}{2}\left(\frac{\mathcal{C}_{KK}}{\tilde{\mathcal{C}}_{KK}}+1\right),\\
\beta_1 &= \frac{\mathcal{C}_{\beta K}}{2\tilde{\mathcal{C}}_{KK}},\\
\beta_2 &= \frac{\mathcal{C}_{\beta\beta}}{\tilde{\mathcal{C}}_{KK}},\\
\beta_3 &= \frac{2\mathcal{C}_\alpha}{\tilde{\mathcal{C}}_{KK}}.
\end{aligned}
\tag{B6}
$$

Equivalently, the inverse correspondence between notations for the subclass (B4) is given by

$$
a^{-3}c_1=-6M^2(1+\alpha_L),\quad a^{-3}c_2=6M^2\beta_1,\quad a^{-3}c_3=M^2\beta_2,\quad a^{-3}c_{10}=M^2\beta_3,\quad a^{-3}c_{11}=-\frac{2}{3}M^2\alpha_L,
\tag{B7}
$$

and

$$
\begin{aligned}
\Theta &\equiv -\frac{1}{3}a^{-3}c_4=-2HM^2(1+\alpha_B+\alpha_L),\\
\Psi &\equiv \frac{1}{4}a^{-3}c_6=\frac{1}{2}M^2(1+\alpha_T),\\
\Xi &\equiv \frac{1}{4}a^{-3}c_8=\frac{1}{2}M^2(1+\alpha_H),\\
\Phi &\equiv a^{-3}c_9=H^2M^2[\alpha_K-6(1+\alpha_L)-12\alpha_B]+6a^{-3}\left(a^3M^2H\beta_1\right)^\cdot.
\end{aligned}
\tag{B8}
$$

with c_5, c_7, c_{12} vanishing in this class.

With these relations we can also translate the degeneracy conditions Eqs. (2.15), (2.16) of [29]:

$$
\mathrm{C_I}:\quad \alpha_L=0,\quad \beta_2=-6\beta_1^2,\quad \beta_3=-2\beta_1[2(1+\alpha_H)+\beta_1(1+\alpha_T)],
\tag{B9}
$$

$$
\mathrm{C_{II}}:\quad \beta_1=-(1+\alpha_L)\frac{1+\alpha_H}{1+\alpha_T},\quad \beta_2=-6(1+\alpha_L)\frac{(1+\alpha_H)^2}{(1+\alpha_T)^2},\quad \beta_3=2\frac{(1+\alpha_H)^2}{1+\alpha_T},
\tag{B10}
$$

into our notation to confirm that their $\mathrm{C_I}$ and $\mathrm{C_{II}}$ correspond to (A14) and (A15), respectively. The Lagrangian for $\tilde{\zeta}$ in (36) is equivalent to Eq. (4.8) of [29] for $\mathrm{C_I}$.

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