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WEAK LENSING: PROSPECTS FOR MEASURING COSMOLOGICAL PARAMETERS

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ABSTRACT

Weak lensing of galaxies by large-scale structure can potentially measure cosmological quantities as accurately as the cosmic microwave background (CMB). However, the relation between observables and fundamental parameters is more complex and degenerate, especially in the full space of adiabatic cold dark matter models considered here. We introduce a Fisher matrix analysis of the information contained in weak-lensing surveys to address these issues and provide a simple means of estimating how survey properties and source redshift uncertainties affect parameter measurement. We find that surveys on degree scales and above can improve the accuracy on parameters that affect the growth rate of structure by up to an order of magnitude compared to using the CMB alone, even if the characteristic redshift of the sources must be determined from the data itself. Surprisingly, both sparse sampling and increasing the source redshift can weaken the cosmological constraints.

Subject headings: cosmic microwave background — gravitational lensing

1. INTRODUCTION

Weak lensing of faint galaxies by large-scale structure can in principle provide precise constraints on the spectrum and evolution of mass fluctuations in the universe (Miralda-Escude 1991; Blandford et al. 1991; Kaiser 1992). Given the same sky coverage, the statistical errors on these measurements should be as small as those from the cosmic microwave background (CMB). The main systematic errors are instrumental rather than astrophysical; although difficult, detector problems are in principle surmountable (Kaiser, Squires, & Broadhurst 1995; Fischer & Tyson 1997; Kamionkowski et al. 1997; Schneider et al. 1998).

Because lensing convolves aspects of the spectrum of present-day mass fluctuations, their evolution, and the distribution of source galaxies, it is not obvious how to translate precision in the observables into precision in the cosmological parameters. Previous work has focussed on a relatively small number of parameters such as the matter density and its present-day fluctuation amplitude assuming a fixed functional form and a fixed distribution of sources (e.g., Jain & Seljak 1997; Bernardeau, Waerbeke, & Mellier 1997; Kaiser 1998). Even so, predictions depend strongly on prior assumptions for these parameters.

In this Letter, we use a Fisher matrix approach to assess the information contained in the weak-lensing power spectrum. This quantifies how assumptions about survey properties, parameter space, fiducial model, and prior knowledge from other cosmological measurements affect parameter estimation. We study an 11-dimensional parameter space based on the adiabatic cold dark matter (CDM) model and show that information from CMB anisotropy measurements can be used in lieu of large sky coverage to isolate several key cosmological parameters and measure the redshift distribution of the sources.

We begin in § 2 with the Fisher matrix formalism. In § 3, we describe the parameterization of the cosmological model to which we apply this formalism in § 4. We study the effect of the source sampling and distribution in § 5 and summarize our conclusions in § 6.

2. FISHER MATRIX

By measuring the distortion of the shapes of galaxies due to the tidal deflection of light by large-scale structure, one can determine the power spectrum of the convergence as a function of multipole or angular frequency on the sky ℓ (Kaiser 1992, 1998):

$$P_\ell^\kappa = \ell^4 \int d\chi \, \frac{g^2(\chi)}{\sinh^6 \chi} P_\Phi(\ell/\sinh\chi, \chi), \qquad (1)$$

where P_Φ is the three-dimensional power spectrum of the gravitational potential, $\chi \equiv R^{-1}\int_0^z H^{-1} dz$ is the distance to z in units of the radius of curvature $R \equiv 1/H_0\,(1-\Omega_{\rm tot})^{1/2}$ with H as the Hubble parameter, and $g(\chi)$ weights the galaxy source distribution by the lensing probability

$$g(\chi) = \sinh\chi \int_\chi^\infty d\chi' n(\chi') \frac{\sinh(\chi'-\chi)}{\sinh(\chi')}, \qquad (2)$$

where $n(\chi)$ is the distribution of sources normalized to $\int d\chi n(\chi) = 1$. We use the Peacock & Dodds (1994) scaling relation to obtain the nonlinear density and hence the potential power spectrum. For models with massive neutrinos, we replace their growth rates with the scale-dependent rates from Hu & Eisenstein (1998).

Kaiser (1992, 1998) showed that the errors on a galaxy ellipticity based estimator of P_ℓ^κ are described by

$$\Delta P_\ell^\kappa = \sqrt{\frac{2}{(2\ell+1)f_{\rm sky}}}\left(P_\ell^\kappa + \frac{\langle\gamma_{\rm int}^2\rangle}{\bar{n}}\right) \qquad (3)$$

(see also Seljak 1998), where $f_{\rm sky} = \Theta^2\pi/129{,}600$ deg^2 is the fraction of the sky covered by a survey of area Θ^2 and $\langle\gamma_{\rm int}^2\rangle^{1/2} \approx 0.4$ is the galaxy-intrinsic rms shear in one component. We assume throughout that $\bar{n} \approx 2\times10^5$ deg^{-2}= 6.6 × 10^8 sr^{-1}, which corresponds roughly to a magnitude limit of $R \sim 25$ (e.g., Smail et al. 1995b). The first term is simply the sampling error assuming Gaussian statistics for the underlying field. Equating the two terms gives the ℓ above which shot noise from the finite number of source galaxies dominates. We plot an example of the band power $\ell(\ell+1)P_\ell^\kappa/2\pi$ and its errors averaged over bands in ℓ in Figure 1.

Equation (3) tells us that weak lensing can, in principle, provide measurements as precise as the CMB. Unlike the CMB,

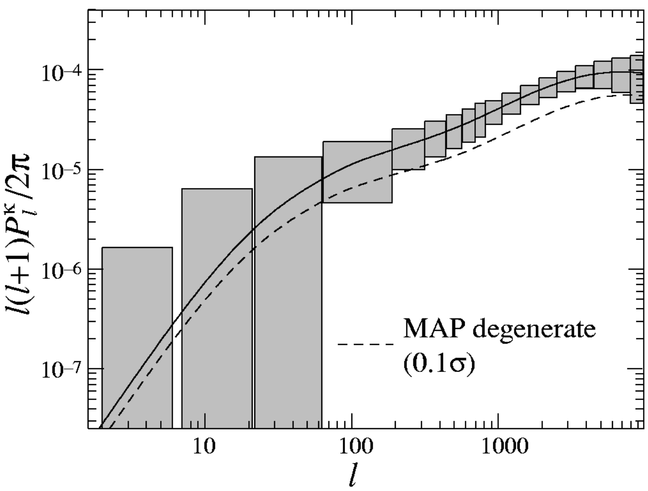

FIG. 1.—Weak-lensing power spectrum for the fiducial ΛCDM model and errors for a $\Theta = 3°$ survey. These are compared with a model that is degenerate with respect to CMB measurements from *MAP*. With the fixed $z_s = 1$ assumed here for illustration purposes, the 0.1 σ *MAP* separation increases to many σ in the lensing survey.

the angular power spectrum of weak lensing is rather featureless because of the radial projection in equation (1). Thus, the translation of these measurements into cosmological parameters will suffer from more severe parameter degeneracies.

We estimate the accuracy with which these cosmological parameters p_i can be jointly measured by computing the so-called Fisher information matrix $\mathbf{F}$ (see § 17 of Kendall & Stuart 1969). $\mathbf{F}^{-1}$ can be thought of as the covariance matrix of the best possible unbiased estimator of the parameter vector $\boldsymbol{p}$, i.e., p_i cannot be measured with a variance less than $(\mathbf{F}^{-1})_{ii}$. If all other parameters were known a priori, this minimum variance drops to $(\mathbf{F}_{ii})^{-1}$. In the approximation that the power spectrum measurements P_ℓ^κ have uncorrelated and Gaussian errors $\Delta P_\ell^\kappa \ll P_\ell^\kappa$, the Fisher matrix is given by $\mathbf{F}_{ij} = \sum_\ell (\Delta P_\ell^\kappa)^{-2}(\partial P_\ell^\kappa/\partial p_i)(\partial P_\ell^\kappa/\partial p_j)$ (Tegmark, Taylor, & Heavens 1997). Equation (3) therefore gives

$$\mathbf{F}_{ij} = \sum_{\ell=\ell_{\min}}^{\ell_{\max}} \frac{\ell + 1/2}{f_{\rm sky}(P_\ell^\kappa + \langle\gamma_{\rm int}^2/\bar{n}\rangle)^2} \frac{\partial P_\ell^\kappa}{\partial p_i}\frac{\partial P_\ell^\kappa}{\partial p_j}. \tag{4}$$

We choose $\ell_{\min} = 100°/\Theta$ when evaluating equation (4), since it corresponds roughly to the survey size. The precise value does not matter for parameter estimation because of the increase in sample variance on the survey scale. We choose a maximum value of $\ell_{\max} = 3000$, since here nonlinear effects can produce non-Gaussianity in the angular distribution, which increases the errors on the power spectrum estimator (Jain & Seljak 1997; Jain, Seljak, & White 1999). Since the convergence arises from many independent density fluctuations along the line of sight, it remains Gaussian deeper into the nonlinear regime. Nonetheless, determining a more precise value for $\ell_{\max}$ is an important issue that will be addressed by future simulations. We explore variations in $\ell_{\max}$ in § 4.

Although information in the power spectrum is degraded by non-Gaussianity, it can be recovered from the non-Gaussian measures such as the skewness of the convergence. We neglect such information here to be conservative (see Jain & Seljak 1997; Bernardeau et al. 1997).

TABLE 1
FULL-SKY WEAK LENSING COMPARED WITH CMB

$\sigma(p_i)$	Weak Lensing	*MAP*	*Planck*
$\sigma(\Omega_m h^2)$	0.024 (430)	0.029	0.0027
$\sigma(\Omega_b h^2)$	0.0092 (310)	0.0029	0.0002
$\sigma(m_\nu)$	0.29 (230)	0.77	0.25
$\sigma(\Omega_\Lambda)$	0.079 (180)	1.0	0.11
$\sigma(\Omega_K)$	0.048 (200)	0.29	0.030
$\sigma(n_S)$	0.066 (470)	0.1	0.009
$\sigma(\ln A)$	0.28 (310)	1.21	0.045
$\sigma(z_s)$	0.047 (56)	[1]	[1]
$\sigma(\tau)$	…	0.63	0.004
σ(T/S)	…	0.45	0.012
$\sigma(Y_p)$	[0.02]	[0.02]	0.01

NOTE.—*MAP* assumes temperature information; *Planck* assumes additional polarization information. These span the range of predicted CMB parameter estimation prospects. Square brackets denote priors of $\sigma(Y_p) = 0.02$ and $\sigma(z_s) = 1$.

3. PARAMETERIZED MODEL

Projections for how well weak lensing can measure cosmological parameters depend crucially on the extent of the parameter space considered as well as the location in this space (or "fiducial model") around which we quote our errors. Previous works have focused on models with essentially two parameters, the matter density Ω_m and the amplitude of mass fluctuations on the 8 h^{-1} Mpc scale today, σ_8 (e.g., Bernardeau et al. 1997; Jain & Seljak 1997; Jain et al. 1999). Since all cosmological parameters that affect the amplitude of power across a wide range of physical ($H_0 \lesssim k \lesssim 10\ h$ Mpc^{-1}) and temporal scales ($z \lesssim 1$) are accessible to weak lensing, it seems prudent to consider a wider parameter space and then impose any external constraints as prior information.

We consider the adiabatic cold dark matter model space and include 11 free parameters. Weak lensing is only sensitive to eight of the parameters: the matter density $\Omega_m h^2$, the baryon density $\Omega_b h^2$, the mass of the neutrinos m_ν, the cosmological constant Ω_Λ, the curvature $\Omega_K = 1 - \Omega_m - \Omega_\Lambda$, the scalar tilt n_S, the power normalization P_Φ(3000 Mpc^{-1}) initially (A), and the characteristic redshift of the sources z_s. When considering prior information provided by the CMB, the optical depth to reionization τ, the primordial helium abundance Y_p, and the scalar-tensor ratio T/S must be considered because of their covariance with the eight lensing parameters.

For our source redshift distribution, we assume a common redshift given by z_s since the errors on cosmological parameters are insensitive to the shape of the distribution as long as it is considered known. We return to this point in § 6. Our fiducial model is the same ΛCDM model as chosen in Eisenstein, Hu, & Tegmark (1998): $\Omega_m = 0.35$, $h = 0.65$, $\Omega_b = 0.05$, $\Omega_\Lambda = 0.65$, $m_\nu = 0.7$ eV, $\tau = 0.05$, $n_S = 1$, T/S = 0, and A given by the *COBE* normalization.

4. PARAMETER ESTIMATION RESULTS

In Table 1, we present the Fisher estimates of errors $(\mathbf{F}^{-1})_{ii}$ on cosmological parameters from weak lensing assuming full sky coverage $f_{\rm sky} = 1$ and $\ell_{\max} = 3000$. Reducing $\ell_{\max}$ to 1000 typically degrades errors by no more than a factor 1.5. Errors for more realistic survey sizes Θ scale roughly as $f_{\rm sky}^{-1/2}$ until $\Theta \sim 100°/\ell_{\max}$, at which the errors diverge. Although these errors (per $f_{\rm sky}^{-1/2}$) are comparable in precision to CMB estimates projected for the *Microwave Anisotropy Probe* (*MAP*) and *Planck* satellites from Eisenstein et al. (1999), they fail to achieve their ultimate potential because of parameter degeneracies. We have included in parenthesis the degradation factors due to degeneracies $(\mathbf{F}_{ii})^{-1/2}/(\mathbf{F}^{-1})_{ii}^{1/2}$. These are on the order of hundreds and reflect that lowering $\Omega_m h^2$ or n_S or raising $\Omega_b h^2$ all reduce the primordial small-scale power in mass fluctuations, whereas raising Ω_Λ, Ω_K, or m_ν all slow the growth of structure. This mimics changes in the amplitude A and source redshift z_s at the well-sampled high ℓ's. On the other hand, the features in $\ell(\ell+1)P_\ell^\kappa$ can basically be characterized by four parameters: an amplitude, a slope, the nonlinear scale ($\ell \sim 10^3$), and the turnover scale ($\ell \sim 10^2$).

Surveys in the near future will be limited to several degrees on the side at best ($f_{\rm sky} \sim 10^{-3}$), and the precision lost to parameter degeneracies is crucial. The best constrained combinations of the parameters can be determined from the eigenvectors of $\mathbf{F}$. The best combination involves ($\Omega_m h^2$, $\Omega_b h^2$, $\Omega_\nu h^2 = m_\nu/94$ eV); variation in the direction $(-0.24, 0.45, 1)$ is constrained to have 10^{-5} amplitude for $f_{\rm sky} = 1$. Moving in this direction rapidly reduces the small-scale power in mass fluctuations to which weak lensing is most sensitive. From analytic treatments of growth rates, we also expect that neutrinos are twice as effective as baryons in reducing small-scale power (Hu & Eisenstein 1998).

These considerations imply that external constraints can help weak-lensing measurements regain their precision. CMB satellite missions provide the ideal source of such information since the CMB angular power spectrum they measure is sensitive to the same cosmological parameters but in different combinations. The CMB is particularly useful in the example above, since it can provide precise measurements of $\Omega_b h^2$ and $\Omega_m h^2$, leaving weak lensing free to constrain the neutrino mass. Furthermore, it is well known that CMB temperature measurements suffer from degeneracies themselves, especially between Ω_Λ and Ω_K along the direction that keeps the angular diameter distance to last scattering fixed. Because Ω_Λ must be raised to compensate Ω_K in the CMB angular diameter distance but must be lowered to compensate Ω_K in the growth rate of structure, one expects that weak lensing will be particularly useful in breaking the degeneracy.

Figure 2 quantifies these expectations. The top panel shows the improvement over projected *MAP* satellite errors on cosmological parameters (Eisenstein et al. 1999) when adding the weak-lensing information with different survey sizes by summing Fisher matrices. Dividing the CMB numbers of Table 1 with these gain factors gives the absolute errors on cosmological parameters. As expected, even a rather modest survey size of $\Theta = 0\overset{\circ}{.}3$ is sufficient to improve *MAP* errors on Ω_Λ and Ω_K by a factor of 3 (see also Fig. 1). Ultimately, weak lensing can improve *MAP*'s measurement of these quantities by over an order of magnitude. Amusingly, it also improves the measurement of τ by a comparable factor, since the angular diameter distance degeneracy in the CMB requires τ-variations to offset the amplitude changes from Ω_Λ and Ω_K. Once the degeneracy is broken by weak lensing, τ becomes better measured. With survey sizes of several degrees and beyond, constraints on m_ν improve to reach the ultimate limit of $\sigma(m_\nu) = 0.1$ eV.

Weak lensing can improve on cosmological parameter estimation even if the CMB reaches its full potential with precision temperature and polarization measurements from the *Planck* satellite (see Fig. 2*b*). In this case, gains will mainly come from survey sizes $\Theta \gtrsim 10°$. Again there is the potential to improve measurements of Ω_K, Ω_Λ, and m_ν by nearly an order of magnitude, e.g., $\sigma(m_\nu) = 0.04$ eV. This number is of par-

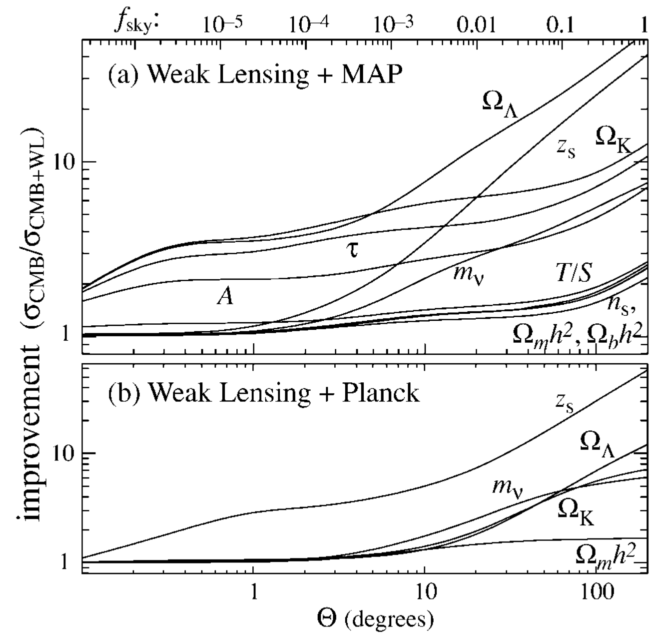

FIG. 2.—Improving CMB parameter estimation with weak lensing. Here and throughout, we have assumed a prior of $\sigma(z_s) = 1$.

ticular interest since the atmospheric neutrino anomaly is currently suggesting mass squared separations of $\Delta m_\nu^2 \sim 10^{-3}$. More generally, this result suggests that weak lensing and CMB measurements can be combined to study the clustering properties of the dark matter beyond the CDM paradigm. For example, lensing can potentially test whether the cosmological constant or scalar fields drives the acceleration in the expansion rate.

5. EFFECT OF GALAXY SAMPLING AND DISTRIBUTION

How does the sampling of galaxies and their redshift distribution affect parameter estimation? Kaiser (1998) noted that at degree scales, the large ratio of sample variance to noise variance in equation (3) implies that one can obtain better constraints on P_ℓ^κ here by sparse sampling, i.e., tiling a large field with small fields at the same depth. One can estimate the effect on cosmological parameters, under the optimistic assumption that aliasing of power from small scales is negligible, by replacing $\bar{n}$ with $\bar{n}_{\rm eff} = \bar{n} N_{\rm tile}(\Theta_{\rm tile}/\Theta)^2$ in equation (3) (see Kaiser 1998, eq. [44]). Unfortunately, we find that going from a filled $\Theta = 3°$ survey at $\bar{n} = 2 \times 10^5$ deg^{-2} to $\Theta = 10°$ at $\bar{n}_{\rm eff} = 2 \times 10^4$ deg^{-2} not only fails to improve the errors, but can actually degrade them. This is because the main source of cosmological information if $\Theta < 10°$ comes from the translinear regime near $\ell = 1000$. Accordingly, parameter errors start improving rapidly in Figure 2 only if $\Theta > 10°$ by resolving the power spectrum bend below $\ell \sim 10^2$. Aliasing problems may unfortunately preclude such aggressive sparse sampling.

External knowledge of the redshift distribution of the sources can aid parameter estimation especially for survey sizes with $\Theta < 10°$ where z_s is not well measured internally. Redshifts on a fair sample of 100 galaxies would be sufficient to pin down the characteristic redshift to $\sigma(z_s) = 0.1$, improving errors on Ω_Λ and Ω_K by up to a factor of 2 for $\Theta < 10°$. Unfortunately, spectroscopy on a fair sample of these faint galaxies may be prohibitive. Alternatively, one can use photometric redshifts to select a subsample of galaxies whose individual redshifts are known to ∼10%. For example, even separating the 1.3% of galaxies that are around $z = 3$ (Steidel et al. 1996) improve errors on Ω_Λ, m_ν, and z_s by factors of 1.7, 1.3, and 2.1, respectively, for $\Theta = 6°$.

The actual value of z_s affects the sensitivity of weak lensing to cosmological parameters, but in a counterintuitive manner. As the characteristic redshift of the source galaxies rise, the lensing effect increases because of the increased amount of intervening large-scale structure. Although this makes the signal easier to detect, it does not necessarily improve errors on cosmological parameters. In fact, errors on Ω_Λ and Ω_K worsen as z_s increases! The reason is that Ω_Λ and Ω_K only affect low-redshift structure. The intervening high-redshift structure is insensitive to these parameters, and the sample variance on this larger signal swamps that of Ω_Λ and Ω_K. With $z_s = 3$, errors on Ω_Λ and Ω_K are a factor of 6 and 2 larger for $f_{\rm sky} = 1$.

Finally, we have assumed that the galaxy redshift distribution is parameterized by a single number, the characteristic redshift. While this is indeed the main effect (Smail, Ellis, & Fitchet 1995a; Fort, Mellier, & Dantel-Fort 1995; Luppino & Kaiser 1997), the fact that weak lensing has the statistical power to measure the characteristic redshift to better than 10% for survey sizes $\Theta \gtrsim 10°$ implies that more detailed aspects of the distribution, e.g., its width and skewness, can in principle be measured from large surveys. Allowing the data itself to determine the form of the distribution will of course introduce more uncertainty in the cosmological parameter determinations, but this would be a small price to pay given the statistical power of such large surveys.

6. DISCUSSION

The Fisher matrix analysis introduced here allows one to explore with ease how assumptions about the survey properties, the fiducial model, and any prior knowledge from other cosmological measurements affect parameter estimation. Weak-lensing surveys are in principle sensitive to all cosmological parameters that affect the shape of the matter power spectrum, the growth rate of fluctuations, and the source redshift distribution. Here we have included the effects of a cosmological constant, spatial curvature, cold dark matter, baryonic dark matter, hot (neutrino) dark matter, power spectrum tilt and amplitude, and the characteristic redshift of sources. We find that even a relatively modest sample size of 0°.3 would suffice to improve our knowledge of cosmological parameters, such as the cosmological constant and the curvature, over those provided by *MAP* satellite measurements of the CMB temperature power spectrum. Order-of-magnitude improvements in many cosmological parameters are available with survey sizes $\gtrsim 3°$.

We have also explored how properties of the sample affect parameter estimation. Sparse sampling can help extend power spectrum determinations to larger angles but can degrade parameter estimation because of the rather featureless nature of the lensing power spectrum in the range $100 < \ell < 1000$. The cosmological constant and curvature can be best measured with a moderate-redshift ($z_s \sim 1$) population of sources, since the larger signal at high redshifts is insensitive to these parameters and therefore acts like noise. On the other hand, even separating out the ∼1% of galaxies at $z_s \sim 3$ by photometric redshifts (Steidel et al. 1996) can improve errors in the absence of redshifts for the bulk of the galaxies.

The potential of weak lensing for cosmology will only be realized once systematic errors are reduced below the statistical errors considered here. Anisotropies in the point-spread function of telescopes can mask the percent-level cosmological signal and pose a daunting challenge for the current generation of weak-lensing surveys. Our analysis reinforces the conclusion that the returns for cosmology justify this great expenditure of effort.

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