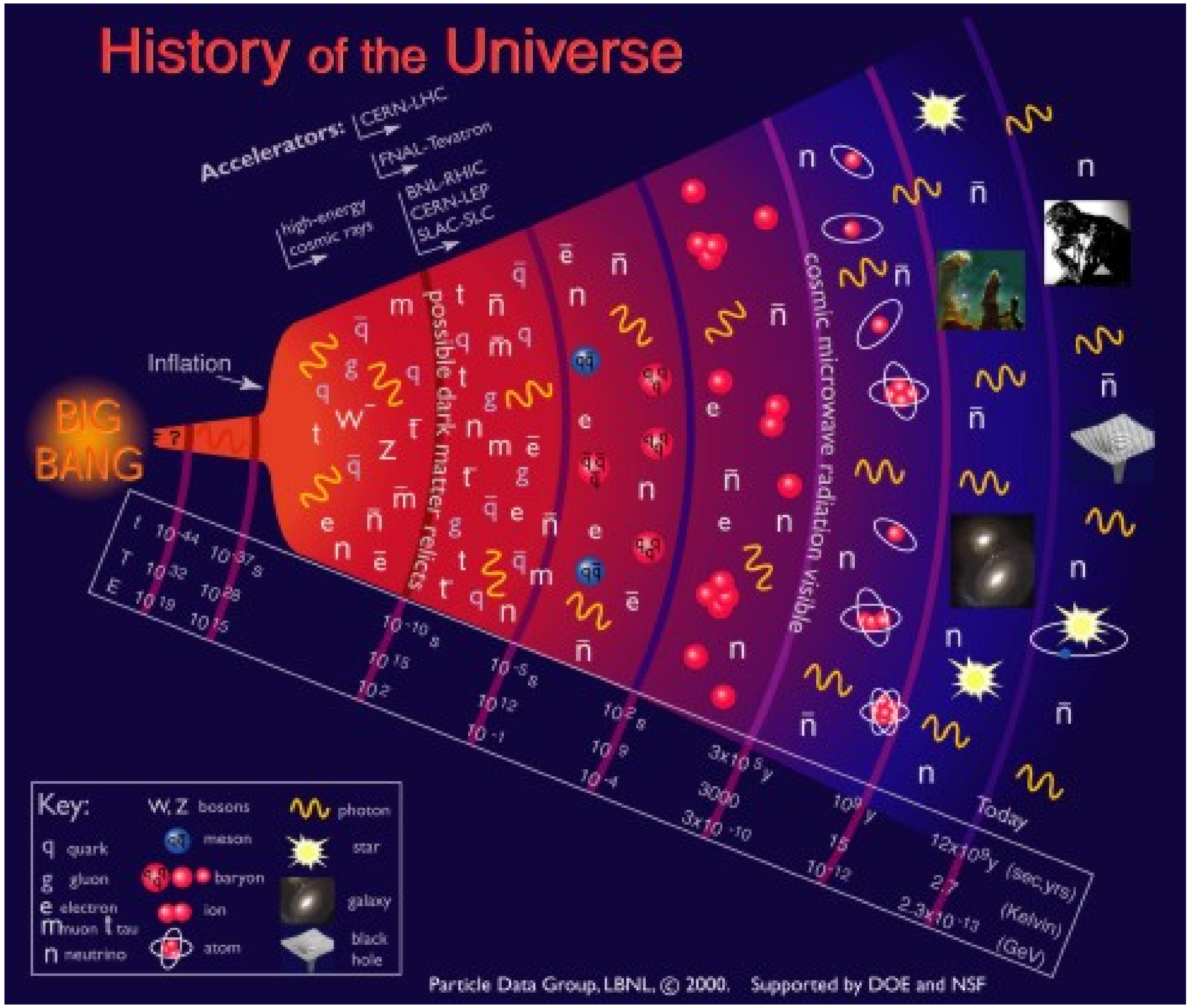
Set 7:

Thermal History

Brief Thermal History



Distribution Function

- The distribution function f gives the number of particles per unit phase space d^3xd^3q where q is the momentum (conventional to work in physical coordinates)
- Consider a box of volume $V = L^3$. Periodicity implies that the allowed momentum states are given by $q_i = n_i 2\pi/L$ so that the density of states is

$$dN_s = g\frac{V}{(2\pi)^3}d^3q$$

  where g is the degeneracy factor (spin/polarization states)
- The distribution function $f(\mathbf{x}, \mathbf{q}, t)$ describes the particle occupancy of these states, i.e.

$$N = \int dN_s f = gV \int \frac{d^3q}{(2\pi)^3} f$$

Bulk Properties

- Integrals over the distribution function define the bulk properties of the collection of particles
- Number density

$$n(\mathbf{x}, t) \equiv N/V = g \int \frac{d^3q}{(2\pi)^3} f$$

- Energy density

$$\rho(\mathbf{x}, t) = g \int \frac{d^3q}{(2\pi)^3} E(q) f$$

where $E^2 = q^2 + m^2$

Bulk Properties

- Pressure: particles bouncing off a surface of area A in a volume spanned by L_x: per momentum state

$$p_q = \frac{F}{A} = \frac{N_{\rm part}}{A}\frac{\Delta q}{\Delta t}$$

$$(\Delta q = 2|q_x|, \quad \Delta t = 2L_x/v_x)$$

$$= \frac{N_{\rm part}}{V}|q_x||v_x| = f\frac{|q||v|}{3} = f\frac{q^2}{3E}$$

so that summed over states

$$p(\mathbf{x},t) = g\int \frac{d^3q}{(2\pi)^3}\frac{|q|^2}{3E(q)}f$$

Liouville Equation

- Liouville theorem states that the phase space distribution function is conserved along a trajectory in the absence of particle interactions

$$\frac{Df}{Dt} = \left[\frac{\partial}{\partial t} + \frac{d\mathbf{q}}{dt}\frac{\partial}{\partial \mathbf{q}} + \frac{d\mathbf{x}}{dt}\frac{\partial}{\partial \mathbf{x}}\right] f = 0$$

subtlety in expanding universe is that the de Broglie wavelength of particles changes with the expansion so that

$$q \propto a^{-1}$$

- Homogeneous and isotropic limit

$$\frac{\partial f}{\partial t} + \frac{dq}{dt}\frac{\partial f}{\partial q} = \frac{\partial f}{\partial t} - H(a)\frac{\partial f}{\partial \ln q} = 0$$

Energy Density Evolution

- Integrate Liouville equation over $g\int d^3q/(2\pi)^3 E$ to form

$$\frac{\partial\rho}{\partial t} = H(a)g\int \frac{d^3q}{(2\pi)^3} Eq\frac{\partial}{\partial q}f$$
$$= H(a)g\int \frac{d\Omega}{(2\pi)^3}\int dq q^3 E\frac{\partial}{\partial q}f$$
$$= -H(a)g\int \frac{d\Omega}{(2\pi)^3}\int dq\frac{d(q^3E)}{dq}f$$
$$= -H(a)g\int \frac{d\Omega}{(2\pi)^3}\int dq(3q^2E + q^3\frac{dE}{dq})f$$
$$(\frac{dE}{dq} = \frac{d(q^2+m^2)^{1/2}}{dq} = \frac{1}{2}\frac{2q}{E} = \frac{q}{E})$$
$$= -3H(a)g\int \frac{d^3q}{(2\pi)^3}(E+\frac{q^2}{3E})f = -3H(a)(\rho+p)$$

as derived previously from energy conservation

Boltzmann Equation

- Boltzmann equation says that Liouville theorem must be modified to account for collisions

$$\frac{Df}{Dt} = C[f]$$

- If collisions are sufficiently rapid, distribution will tend to thermal equilibrium form

Poor Man's Boltzmann Equation

- Non expanding medium

$$\frac{\partial f}{\partial t} = \Gamma\,(f - f_{\rm eq})$$

where Γ is some rate for collisions

- Add in expansion in a homogeneous medium

$$\frac{\partial f}{\partial t} + \frac{dq}{dt}\frac{\partial f}{\partial q} = \Gamma\,(f - f_{\rm eq})$$

$$(q \propto a^{-1} \rightarrow \frac{1}{q}\frac{dq}{dt} = -\frac{1}{a}\frac{da}{dt} = H)$$

$$\frac{\partial f}{\partial t} - H\frac{\partial f}{\partial \ln q} = \Gamma\,(f - f_{\rm eq})$$

- So equilibrium will be maintained if collision rate exceeds expansion rate $\Gamma = n\langle\sigma v\rangle > H$

Thermal & Diffusive Equilibrium

- A gas in thermal & diffusive contact with a reservoir at temperature T
- Probability of system being in state of energy E_i and number N_i (Gibbs Factor)

$$P(E_i, N_i) \propto \exp[-(E_i - \mu N_i)/T]$$

  where μ is the chemical potential (defines the free energy "cost" for adding a particle at fixed temperature and volume)

- Chemical potential appears when particles are conserved
- CMB photons can carry chemical potential if creation and annihilation processes inefficient, as they are after $t \sim 1$yr.

Distribution Function

- Mean occupation of the state in thermal equilibrium

$$f \equiv \frac{\sum N_i P(E_i, N_i)}{\sum P(E_i, N_i)}$$

  where the total energy is related to the particle energy as $E_i = N_i E$ (ignoring zero pt)

- Density of (energy) states in phase space makes the net spatial density of particles

$$n = g \int \frac{d^3 p}{(2\pi)^3} f$$

  where g is the number of spin states

Fermi-Dirac Distribution

- For fermions, the occupancy can only be $N_i = 0, 1$

$$f = \frac{P(E,1)}{P(0,0) + P(E,1)}$$
$$= \frac{e^{-(E-\mu)/T}}{1 + e^{-(E-\mu)/T}}$$
$$= \frac{1}{e^{(E-\mu)/T} + 1}$$

- In the non-relativistic limit

$$E = (p^2 + m^2)^{1/2} \approx m + \frac{1}{2}\frac{p^2}{m}$$

and $m \gg T$ so the distribution is Maxwell-Boltzmann

$$f = e^{-(m-\mu)/T} e^{-p^2/2mT} = e^{-(m-\mu)/T} e^{-mv^2/2T}$$

Bose-Einstein Distribution

- For bosons each state can have multiple occupation,

$$f = \frac{\frac{d}{d\mu/T}\sum_{N=0}^{\infty}(e^{-(E-\mu)/T})^N}{\sum_{N=0}^{\infty}(e^{-(E-\mu)/T})^N} \qquad \text{with} \sum_{N=0}^{\infty} x^N = \frac{1}{1-x}$$

$$= \frac{1}{e^{(E-\mu)/T}-1}$$

- Again, non relativistic distribution is Maxwell-Boltzmann

$$f = e^{-(m-\mu)/T} e^{-p^2/2mT} = e^{-(m-\mu)/T} e^{-mv^2/2T}$$

with a spatial number density

$$n = g e^{-(m-\mu)/T} \int \frac{d^3p}{(2\pi)^3} e^{-p^2/2mT}$$

$$= g e^{-(m-\mu)/T} \left(\frac{mT}{2\pi}\right)^{3/2}$$

Ultra-Relativistic Bulk Properties

- Chemical potential $\mu = 0$, $\zeta(3) \approx 1.202$
- Number density

$$n_{\rm boson} = gT^3 \frac{\zeta(3)}{\pi^2} \qquad \zeta(n+1) \equiv \frac{1}{n!} \int_0^\infty dx \frac{x^n}{e^x - 1}$$

$$n_{\rm fermion} = \frac{3}{4} gT^3 \frac{\zeta(3)}{\pi^2}$$

- Energy density

$$\rho_{\rm boson} = gT^4 \frac{3}{\pi^2} \zeta(4) = gT^4 \frac{\pi^2}{30}$$

$$\rho_{\rm fermion} = \frac{7}{8} gT^4 \frac{3}{\pi^2} \zeta(4) = \frac{7}{8} gT^4 \frac{\pi^2}{30}$$

- Pressure $q^2/3E = E/3 \rightarrow p = \rho/3$, $w_r = 1/3$

Entropy Density

- Second law of thermodynamics

$$dS = \frac{1}{T}(d\rho(T)V + p(T)dV)$$

so that

$$\left.\frac{\partial S}{\partial V}\right|_T = \frac{1}{T}[\rho(T) + p(T)]$$
$$\left.\frac{\partial S}{\partial T}\right|_V = \frac{V}{T}\frac{d\rho}{dT}$$

- Since $S(V,T) \propto V$ is extensive

$$S = \frac{V}{T}[\rho(T) + p(T)] \quad \sigma = S/V = \frac{1}{T}[\rho(T) + p(T)]$$

Entropy Density

- Integrability condition $dS/dVdT = dS/dTdV$ relates the evolution of entropy density

$$\frac{d\sigma}{dT} = \frac{1}{T}\frac{d\rho}{dT}$$
$$\frac{d\sigma}{dt} = \frac{1}{T}\frac{d\rho}{dt} = \frac{1}{T}[-3(\rho+p)]\frac{d\ln a}{dt}$$
$$\frac{d\ln\sigma}{dt} = -3\frac{d\ln a}{dt} \qquad \sigma \propto a^{-3}$$

comoving entropy density is conserved in thermal equilibrium

- For ultra relativisitic bosons $\sigma_{\rm boson} = 3.602 n_{\rm boson}$; for fermions factor of $7/8$ from energy density.

$$g_* = \sum_{\rm bosons} g_b + \frac{7}{8}\sum g_f$$

Neutrino Freezeout

- Neutrino equilibrium maintained by weak interactions, e.g. $e^+ + e^- \leftrightarrow \nu + \bar{\nu}$
- Weak interaction cross section $T_{10} = T/10^{10}K \sim T/1\text{MeV}$

$$\sigma_w \sim G_F^2 E_\nu^2 \approx 4 \times 10^{-44}\, T_{10}^2 \text{cm}^2$$

- Rate $\Gamma = n_\nu \sigma_w = H$ at $T_{10} \sim 3$ or $t \sim 0.2$s
- After neutrino freezeout, electrons and positrons annihilate dumping their entropy into the photons
- Before g_*: $\gamma, e^+, e^- = 2 + \frac{7}{8}(2+2) = \frac{11}{2}$
- After g_*: γ = 2; so conservation of entropy gives

$$g_* T^3\Big|_{\text{initial}} = g_* T^3\Big|_{\text{final}} \qquad T_\nu = \left(\frac{4}{11}\right)^{1/3} T_\gamma$$

Relic Neutrinos

- Relic number density (zero chemical potential; now required by oscillations & BBN)

$$n_\nu = n_\gamma \frac{3}{4}\frac{4}{11} = 112\mathrm{cm}^{-3}$$

- Relic energy density assuming one species with finite m_ν:

$\rho_\nu = m_\nu n_\nu$

$$\rho_\nu = 112\frac{m_\nu}{\mathrm{eV}}\,\mathrm{eV\,cm}^{-3} \qquad \rho_c = 1.05 \times 10^4 h^2\,\mathrm{eVcm}^{-3}$$

$$\Omega_\nu h^2 = \frac{m_\nu}{93.7\mathrm{eV}}$$

- Candidate for dark matter? an eV mass neutrino goes non relativistic around $z \sim 1000$ and retains a substantial velocity dispersion σ_ν.

Hot Dark Matter

- Momenta for a nonrelativistic species redshifts like temperature for a relativistic one, so average momentum is still given by

$$\langle q \rangle = 3T_\nu = m\sigma_\nu$$

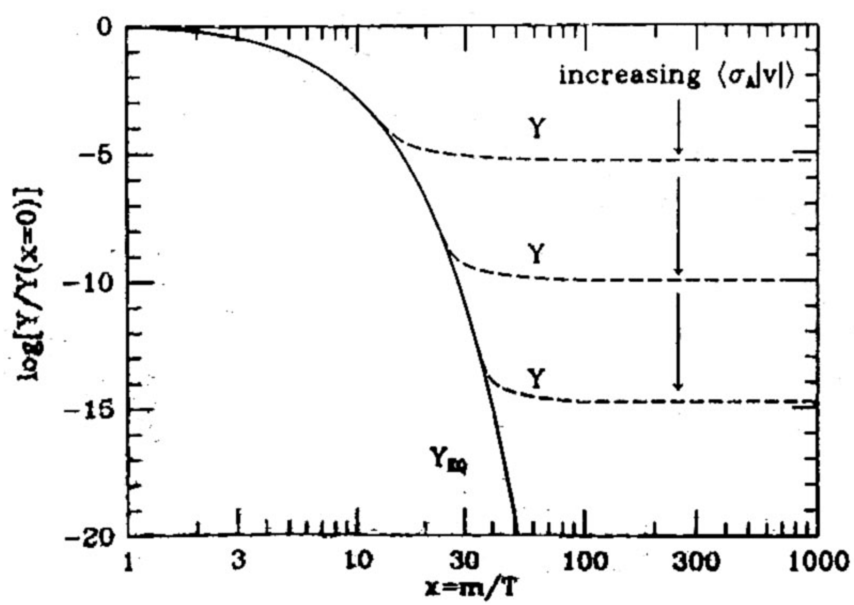
$$\sigma_\nu = 3\left(\frac{m_\nu}{1\text{eV}}\right)^{-1}\left(\frac{T_\nu}{1\text{eV}}\right) = 3\left(\frac{m_\nu}{1\text{eV}}\right)^{-1}\left(\frac{T_\nu}{10^4\text{K}}\right)$$

$$= 6\times 10^{-4}\left(\frac{m_\nu}{1\text{eV}}\right)^{-1} = 200\text{km/s}\left(\frac{m_\nu}{1\text{eV}}\right)^{-1}$$

on order the rotation velocity of galactic halos and higher at higher redshift - small objects can't form: top down structure formation – not observed – must not constitute the bulk of the dark matter

Cold Dark Matter

- Problem with neutrinos is they decouple while relativistic and hence have a comparable number density to photons - for a reasonable energy density, the mass must be small



- The equilibrium distribution for a non-relativistic species declines exponentially beyond the mass threshold

$$n = g(\frac{mT}{2\pi})^{3/2} e^{-m/T}$$

Cold Dark Matter

- Freezeout when annihilation rate equal expansion rate $\Gamma \propto \sigma_A$, increasing annihilation cross section decreases abundance
- Appropriate candidates supplied by supersymmetry
- Alternate solution: keep light particle but not created in thermal equilibrium, axion dark matter

Big Bang Nucleosynthesis

- Most of light element synthesis can be understood through nuclear statistical equilibrium and reaction rates
- Equilibrium abundance of species with mass number A and charge Z (Z protons and $A - Z$ neutrons)

$$n_A = g_A(\frac{m_A T}{2\pi})^{3/2} e^{(\mu_A - m_A)/T}$$

- In chemical equilibrium with protons and neutrons

$$\mu_A = Z\mu_p + (A - Z)\mu_n$$

$$n_A = g_A(\frac{m_A T}{2\pi})^{3/2} e^{-m_A/T} e^{(Z\mu_p + (A-Z)\mu_n)/T}$$

Big Bang Nucleosynthesis

- Eliminate chemical potentials with n_p, n_n

$$e^{\mu_p/T} = \frac{n_p}{g_p}\left(\frac{2\pi}{m_p T}\right)^{3/2} e^{m_p/T}$$

$$e^{\mu_n/T} = \frac{n_n}{g_n}\left(\frac{2\pi}{m_n T}\right)^{3/2} e^{m_n/T}$$

$$n_A = g_A g_p^{-Z} g_n^{Z-A} (\frac{m_A T}{2\pi})^{3/2} \left(\frac{2\pi}{m_p T}\right)^{3Z/2} \left(\frac{2\pi}{m_n T}\right)^{3(A-Z)/2}$$

$$\times\, e^{-m_A/T} e^{(Z\mu_p + (A-Z)\mu_n)/T} n_p^Z n_n^{A-Z}$$

$$(g_p = g_n = 2; m_p \approx m_n = m_b = m_A/A)$$

$$(B_A = Z m_p + (A-Z) m_n - m_A)$$

$$= g_A 2^{-A} \left(\frac{2\pi}{m_b T}\right)^{3(A-1)/2} A^{3/2} n_p^Z n_n^{A-Z} e^{B_A/T}$$

Big Bang Nucleosynthesis

- Convenient to define abundance fraction

$$X_A \equiv A\frac{n_A}{n_b} = A g_A 2^{-A}\left(\frac{2\pi}{m_b T}\right)^{3(A-1)/2} A^{3/2} n_p^Z n_n^{A-Z} n_b^{-1} e^{B_A/T}$$

$$= A g_A 2^{-A}\left(\frac{2\pi n_b^{2/3}}{m_b T}\right)^{3(A-1)/2} A^{3/2} e^{B_A/T} X_p^Z X_n^{A-Z}$$

$$(n_\gamma = \frac{2}{\pi^2} T^3 \zeta(3) \qquad \eta_{b\gamma} \equiv n_b/n_\gamma)$$

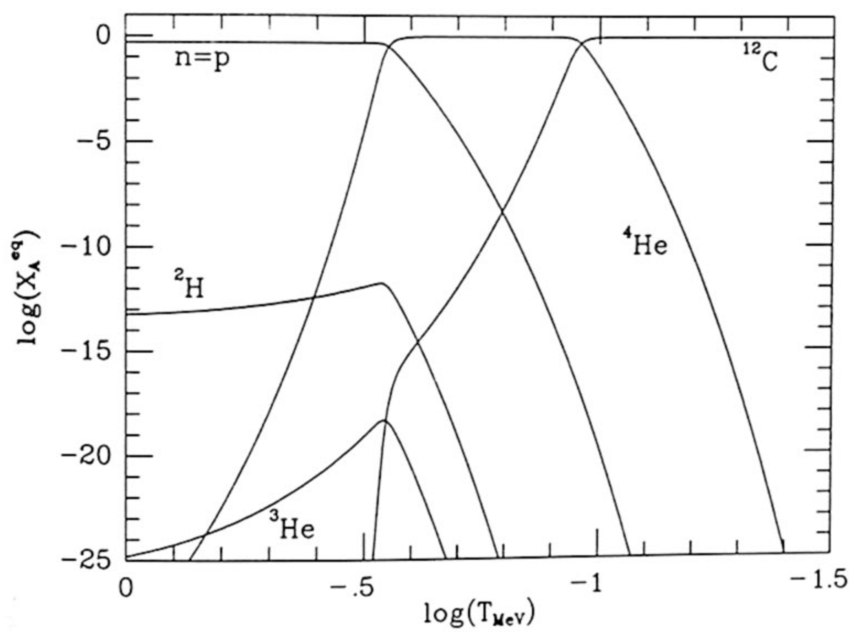
$$= A^{5/2} g_A 2^{-A}\left[\left(\frac{2\pi T}{m_b}\right)^{3/2} \frac{2\zeta(3)\eta_{b\gamma}}{\pi^2}\right]^{A-1} e^{B_A/T} X_p^Z X_n^{A-Z}$$

Deuterium

- Deuterium $A = 2$, $Z = 1$, $g_2 = 3$, $B_2 = 2.225$ MeV

$$X_2 = \frac{3}{\pi^2}\left(\frac{4\pi T}{m_b}\right)^{3/2}\eta_{b\gamma}\zeta(3)e^{B_2/T}X_pX_n$$

- Deuterium "bottleneck" is mainly due to the low baryon-photon number of the universe $\eta_{b\gamma} \sim 10^{-9}$, secondarily due to the low binding energy B_2



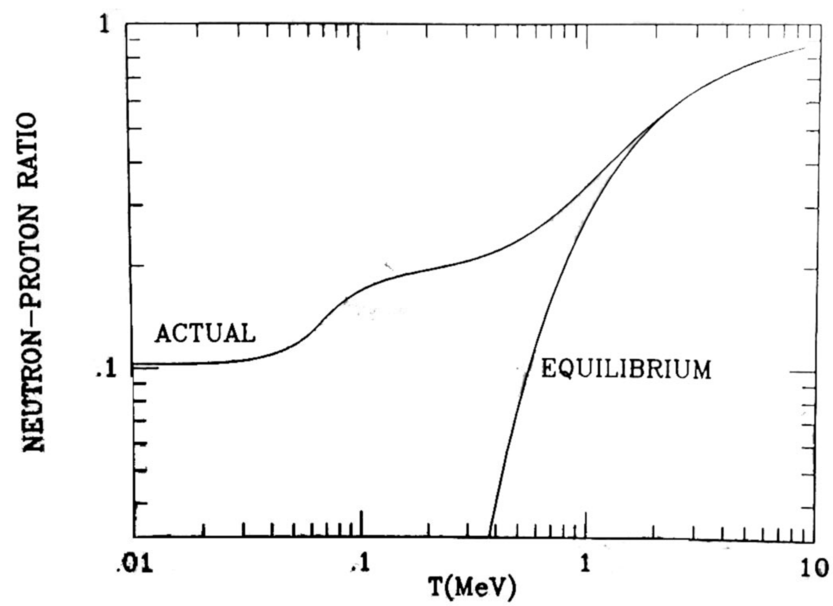
Deuterium

- $X_2/X_pX_n \approx \mathcal{O}(1)$ at $T \approx 100\text{keV}$ or 10^9 K, much lower than the binding energy B_2
- Most of the deuterium formed then goes through to helium via $\mathrm{D} + \mathrm{D} \to {}^3\mathrm{He} + p$, ${}^3\mathrm{He} + \mathrm{D} \to {}^4\mathrm{He} + n$
- Deuterium freezes out as number abundance becomes too small to maintain reactions $n_D = \text{const}$. The deuterium freezeout fraction $n_D/n_b \propto \eta_{b\gamma}^{-1} \propto (\Omega_b h^2)^{-1}$ and so is fairly sensitive to the baryon density.
- Observations of the ratio in quasar absorption systems give $\Omega_b h^2 \approx 0.02$

Helium

- Essentially all neutrons around during nucleosynthesis end up in Helium
- In equilibrium, the neutron-to-proton ratio is determined by the mass difference $Q = m_n - m_p = 1.293$ MeV

$$\frac{n_n}{n_p} = \exp[-Q/T]$$



Helium

- Equilibrium is maintained through weak interactions, e.g. $n \leftrightarrow p + e^- + \bar{\nu}$, $\nu + n \leftrightarrow p + e^-$, $e^+ + n \leftrightarrow p + \bar{\nu}$ with rate

$$\frac{\Gamma}{H} \approx \frac{T}{0.8\mathrm{MeV}}$$

- Freezeout fraction

$$\frac{n_n}{n_p} = \exp[-1.293/0.8] \approx 0.2$$

- Finite lifetime of neutrons brings this to $\sim 1/7$ by 10^9K
- Helium mass fraction

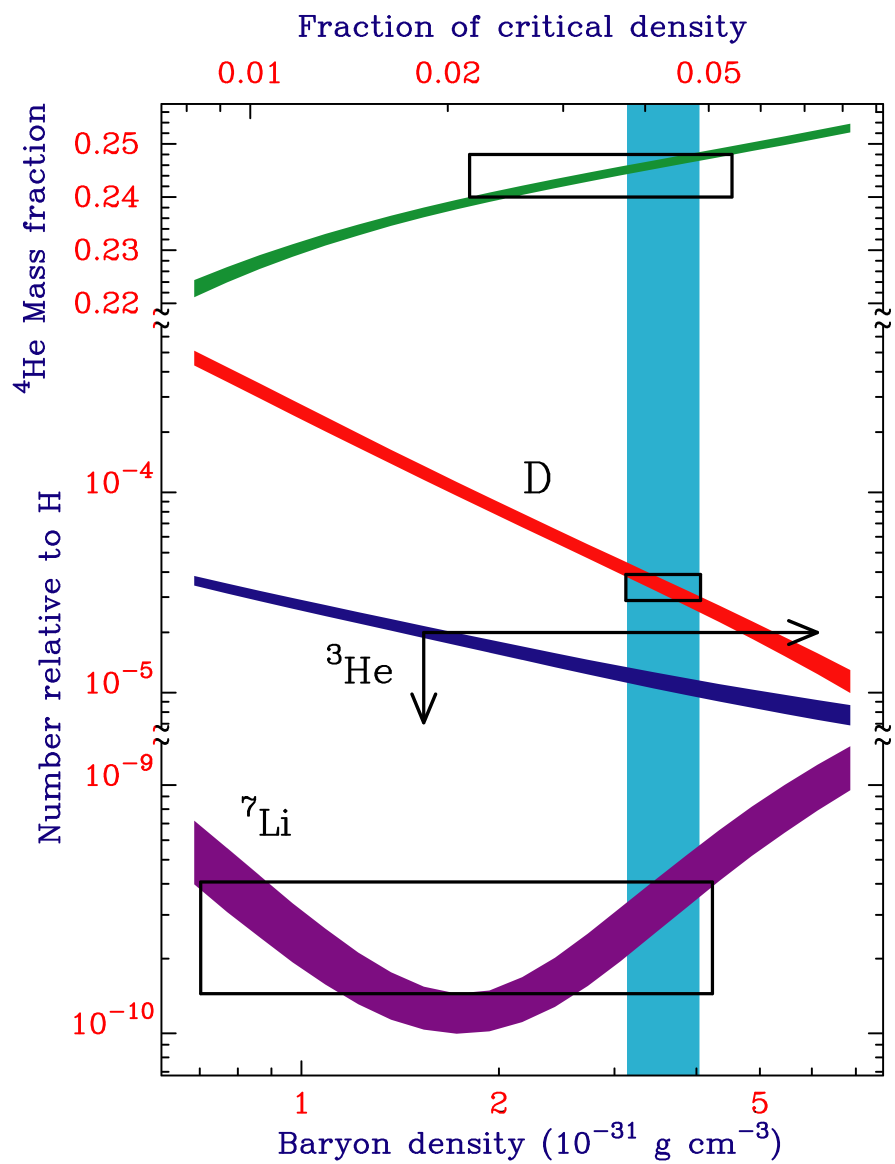
$$Y_{\mathrm{He}} = \frac{4n_{He}}{n_b} = \frac{4(n_n/2)}{n_n + n_p}$$

$$= \frac{2n_n/n_p}{1 + n_n/n_p} \approx \frac{2/7}{8/7} \approx \frac{1}{4}$$

Helium

- Depends mainly on the expansion rate during BBN - measure number of relativistic species
- Traces of ^{7}Li as well. Measured abundances in reasonable agreement with deuterium measure $\Omega_b h^2 = 0.02$

Light Elements



Burles, Nollett, Turner (1999)

Recombination

- Maxwell-Boltzmann distribution determines the equilibrium distribution for reactions, e.g. big-bang nucleosynthesis, recombination:

$$p + e^- \leftrightarrow H + \gamma$$

$$\frac{n_p n_e}{n_H} \approx e^{-B/T} \left(\frac{m_e T}{2\pi} \right)^{3/2} e^{(\mu_p + \mu_e - \mu_H)/T}$$

where $B = m_p + m_e - m_H = 13.6$eV is the binding energy, $g_p = g_e = \frac{1}{2} g_H = 2$, and $\mu_p + \mu_e = \mu_H$ in equilibrium

- Define ionization fraction

$$n_p = n_e = x_e n_b$$

$$n_H = n_{\rm tot} - n_b = (1 - x_e) n_b$$

Recombination

- Saha Equation

$$\frac{n_e n_p}{n_H n_b} = \frac{x_e^2}{1 - x_e}$$
$$= \frac{1}{n_b}\left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-B/T}$$

- Naive guess of $T_* = B$ wrong due to the low baryon-photon ratio
  – $T_* \approx 0.3$eV so recombination at $z_* \approx 1000$
- But the photon-baryon ratio is very low

$$\eta_{b\gamma} \equiv n_b/n_\gamma \approx 3 \times 10^{-8}\Omega_b h^2$$

Recombination

- Eliminate in favor of $\eta_{b\gamma}$ and B/T through

$$n_\gamma = 0.244 T^3 \,, \quad \frac{m_e}{B} = 3.76 \times 10^4$$

- Big coefficient

$$\frac{{x_e}^2}{1 - x_e} = 3.16 \times 10^{15} \left(\frac{B}{T}\right)^{3/2} e^{-B/T}$$

$T = 1/3\text{eV} \to x_e = 0.7$, $T = 0.3\text{eV} \to x_e = 0.2$

- Further delayed by inability to maintain equilibrium since net is through 2γ process and redshifting out of line

Recombination