Set 8:
Inflationary Origins
Horizon Problem

- The horizon in a decelerating universe scales as $\eta \propto a^{(1+3w)/2}$, $w > -1/3$. For example in a matter dominated universe

  $$\eta \propto a^{1/2}$$

- CMB decoupled at $a_* = 10^{-3}$ so subtends an angle on the sky

  $$\frac{\eta_*}{\eta_0} = a_*^{1/2} \approx 0.03 \approx 2^\circ$$

- So why is the CMB sky isotropic to $10^{-5}$ in temperature if it is composed of $\sim 10^4$ causally disconnected regions

- If smooth by fiat, why are there $10^{-5}$ fluctuations correlated on superhorizon scales
Flatness & Relic Problems

- Flatness problem: why is the radius of curvature larger than the observable universe. (Before the CMB determinations, why is it at least comparable to observable universe $|\Omega_K| \lesssim \Omega_m$)

- Also phrased as a coincidence problem: since $\rho_K \propto a^{-2}$ and $\rho_m \propto a^{-3}$, why would they be comparable today – modern version is dark energy coincidence $\rho_\Lambda = \text{const}$.

- Relic problem – why don’t relics like monopoles dominate the energy density

- Inflation is a theory that solves all three problems at once and also supplies a source for density perturbations
Accelerating Expansion

- In a matter or radiation dominated universe, the horizon grows as a power law in $a$ so that there is no way to establish causal contact on a scale longer than the inverse Hubble length ($1/aH$, comoving coordinates) at any given time: general for a decelerating universe

$$
\eta = \int d \ln a \frac{1}{aH(a)}
$$

- $H^2 \propto \rho \propto a^{-3(1+w)}$, $aH \propto a^{-(1+3w)/2}$, critical value of $w = -1/3$, the division between acceleration and deceleration

- In an accelerating universe, the Hubble length shrinks in comoving coordinates and so the horizon gets its contribution at the earliest times, e.g. in a cosmological constant universe, the horizon saturates to a constant value
Causal Contact

- Note confusion in nomenclature: the true horizon always grows meaning that one always sees more and more of the universe. The Hubble length decreases: the difference in conformal time, the distance a photon can travel between two epochs denoted by the scale factor decreases. Regions that were in causal contact, leave causal contact.

- Horizon problem solved if the universe was in an acceleration phase up to $\eta_i$ and the conformal time since then is shorter than the total conformal age

\[ \eta_0 \gg \eta_0 - \eta_i \]

total distance $\gg$ distance traveled since inflation

apparent horizon
Flatness & Relic

- Comoving radius of curvature is constant and can even be small compared to the full horizon $R \ll \eta_0$ yet still $\eta_0 \gg R \gg \eta_0 - \eta_i$

- In physical coordinates, the rapid expansion of the universe makes the current observable universe much smaller than the curvature scale

- Likewise, the number density of relics formed before the accelerating (or inflationary) epoch is diluted to make them rare in the current observable volume

- Common to reference time to the end of inflation $\tilde{\eta} \equiv \eta - \eta_i$. Here conformal time is negative during inflation and its value (as a difference in conformal time) reflects the comoving Hubble length - defines leaving the horizon as $k|\tilde{\eta}| = 1$
Exponential Expansion

- If the accelerating component has equation of state $w = -1$, $\rho = \text{const.}$, $H = H_i \text{ const.}$ so that $a \propto \exp(Ht)$

\[ \tilde{\eta} = -\int_{a_i}^{a} d\ln a \frac{1}{aH} = \frac{1}{aH_i} \Bigg|^{a_i}_{a} \]

\[ \approx -\frac{1}{aH_i} \quad (a_i \gg a) \]

- In particular, the current horizon scale $H_0 \tilde{\eta}_0 \approx 1$ exited the horizon during inflation at

\[ \tilde{\eta}_0 \approx H_0^{-1} = \frac{1}{a_H H_i} \]

\[ a_H = \frac{H_0}{H_i} \]
Sufficient Inflation

• Current horizon scale must have exited the horizon during inflation so that the start of inflation could not be after \( a_H \). How long before the end of inflation must it have began?

\[
\frac{a_H}{a_i} = \frac{H_0}{H_i a_i}
\]

\[
H_0 \frac{H}{H_i} = \sqrt{\frac{\rho_c}{\rho_i}}, \quad a_i = \frac{T_{\text{CMB}}}{T_i}
\]

• \( \rho_c^{1/4} = 3 \times 10^{-12} \text{ GeV} \), \( T_{\text{CMB}} = 3 \times 10^{-13} \text{ GeV} \)

\[
\frac{a_H}{a_i} = 3 \times 10^{-29} \left( \frac{\rho_i^{1/4}}{10^{14}\text{GeV}} \right)^{-2} \left( \frac{T_i}{10^{10}\text{GeV}} \right)
\]

\[
\ln \frac{a_i}{a_H} = 65 + 2 \ln \left( \frac{\rho_i^{1/4}}{10^{14}\text{GeV}} \right) - \ln \left( \frac{T_i}{10^{10}\text{GeV}} \right)
\]
Perturbation Generation

- Horizon scale $\tilde{\eta}$ during inflation acts like an even horizon - things leaving causal contact
- Particle creation similar to Hawking radiation from a black hole with hubble length replacing the BH horizon

$$T_H \approx H_i$$

- Because $H_i$ remains roughly constant during inflation the result is a scale invariant spectrum of fluctuations due to zero-point fluctuations becoming classical
- Fluctuations in the field driving inflation (inflaton) carry the energy density of the universe and so their zero point fluctuations are net energy density or curvature fluctuations
- Any other light field (gravitational waves, etc...) will also carry scale invariant perturbations but are iso-curvature fluctuations
Scalar Fields

- Stress-energy tensor of a scalar field

\[ T^{\mu}_{\nu} = \nabla^{\mu}\phi \nabla_{\nu}\phi - \frac{1}{2}(\nabla^{\alpha}\phi \nabla_{\alpha}\phi + 2V)\delta^{\mu}_{\nu}. \]

- For the background \( \langle \phi \rangle \equiv \phi_0 \) (\( a^{-2} \) from conformal time)

\[
\rho_\phi = \frac{1}{2} a^{-2} \dot{\phi}_0^2 + V, \quad p_\phi = \frac{1}{2} a^{-2} \dot{\phi}_0^2 - V
\]

- So for kinetic dominated \( w_\phi = p_\phi/\rho_\phi \rightarrow 1 \)

- And potential dominated \( w_\phi = p_\phi/\rho_\phi \rightarrow -1 \)

- A slowly rolling (potential dominated) scalar field can accelerate the expansion and so solve the horizon problem or act as a dark energy candidate
Equation of Motion

- Can use general equations of motion of dictated by stress energy conservation

\[ \dot{\rho}_\phi = -3(\rho_\phi + p_\phi) \frac{\dot{a}}{a}, \]

to obtain the equation of motion of the background field \( \phi \)

\[ \ddot{\phi}_0 + 2\frac{\dot{a}}{a} \dot{\phi}_0 + a^2 V' = 0, \]

- In terms of time instead of conformal time

\[ \frac{d^2 \phi_0}{dt^2} + 3H \frac{d\phi_0}{dt} + V' = 0 \]

- Field rolls down potential hill but experiences “Hubble friction” to create slow roll. In slow roll \( 3H \frac{d\phi_0}{dt} \approx -V' \) and so kinetic energy is determined by field position \( \rightarrow \) adiabatic – both kinetic and potential energy determined by single degree of freedom \( \phi_0 \)
Slow Roll Inflation

- Alternately can derive directly from the Klein-Gordon equation for scalar field
- Scalar field equation of motion $V' \equiv dV/d\phi$

$$
\nabla_\mu \nabla^\mu \phi + V'(\phi) = 0
$$

so that in the background $\phi = \phi_0$ and

$$
\ddot{\phi}_0 + 2 \frac{\dot{a}}{a} \dot{\phi}_0 + a^2 V' = 0
$$

$$
\frac{d^2 \phi_0}{dt^2} + 3H \frac{d\phi_0}{dt} + V' = 0
$$

- Simply the continuity equation with the associations

$$
\rho_\phi = \frac{1}{2} a^{-2} \dot{\phi}_0^2 + V \quad p_\phi = \frac{1}{2} a^{-2} \dot{\phi}_0^2 - V
$$
Slow Roll Parameters

- Net energy is dominated by potential energy and so acts like a cosmological constant $w \to -1$

- First slow roll parameter

$$\epsilon = \frac{3}{2} (1 + w) = \frac{1}{16\pi G} \left( \frac{V'}{V} \right)^2$$

- Second slow roll parameter $d^2\phi_0/dt^2 \approx 0$, or $\ddot{\phi}_0 \approx (\dot{a}/a)\dot{\phi}_0$

$$\delta = \frac{\ddot{\phi}_0}{\dot{\phi}_0} \left( \frac{\dot{a}}{a} \right)^{-1} - 1 = \epsilon - \frac{1}{8\pi G} \frac{V''}{V}$$

- Slow roll condition $\epsilon, \delta \ll 1$ corresponds to a very flat potential
Perturbations

- Linearize perturbation $\phi = \phi_0 + \phi_1$ then

$$\ddot{\phi}_1 + 2\frac{\dot{a}}{a}\dot{\phi}_1 + k^2 \phi_1 + a^2 V'' \phi_1 = 0$$

in slow roll inflation $V''$ term negligible

- Implicitly assume that the spatial metric fluctuations (curvature $R$) vanishes, otherwise covariant derivatives pick these up

- GR: work in the spatially flat slicing and transform back to comoving slicing once done.

- Curvature is local scale factor $a \rightarrow (1 + R)a$ or $\delta a/a = R$

$$R = \frac{\delta a}{a} = \frac{\dot{a}}{a} \delta \eta = \frac{\dot{a}}{a} \frac{\phi_1}{\phi_0}$$

a change in the field value $\phi_1$ defines a change in the epoch that inflation ends, imprinting a curvature fluctuation
Slow-Roll Evolution

- Rewrite in $u \equiv a\phi$ to remove expansion damping

$$\ddot{u} + \left[ k^2 - 2 \left( \frac{\dot{a}}{a} \right)^2 \right] u = 0$$

- or for conformal time measured from the end of inflation

$$\tilde{\eta} = \eta - \eta_{\text{end}}$$

$$\tilde{\eta} = \int_{a_{\text{end}}}^{a} \frac{da}{H a^2} \approx - \frac{1}{aH}$$

- Compact, slow-roll equation:

$$\ddot{u} + \left[ k^2 - \frac{2}{\tilde{\eta}^2} \right] u = 0$$
Slow Roll Limit

- Slow roll equation has the exact solution:

\[ u = A(k \pm \frac{i}{\eta})e^{\pm ik\eta} \]

- For \(|k\eta| \gg 1\) (early times, inside Hubble length) behaves as free oscillator

\[ \lim_{|k\eta| \to \infty} u = Ak e^{\pm ik\eta} \]

- Normalization \(A\) will be set by origin in quantum fluctuations of free field
Slow Roll Limit

- For $|k\eta| \ll 1$ (late times, $\gg$ Hubble length) fluctuation freezes in

$$\lim_{|k\eta| \rightarrow 0} u = \pm \frac{i}{\eta} A = \pm iHaA$$

$$\phi_1 = \pm iHA$$

$$\mathcal{R} = \mp iHA \left( \frac{\dot{a}}{a} \right) \frac{1}{\dot{\phi}_0}$$

- Slow roll replacement

$$\left( \frac{\dot{a}}{a} \right)^2 \frac{1}{\dot{\phi}_0^2} = \frac{8\pi Ga^2V}{3} \frac{3}{2a^2V}\epsilon = \frac{4\pi G}{\epsilon} = \frac{1}{2\epsilon M^2_{\text{pl}}}$$

- Comoving curvature power spectrum

$$\Delta^2_\mathcal{R} \equiv \frac{k^3|\mathcal{R}|^2}{2\pi^2} = \frac{k^3}{4\pi^2 \epsilon M^2_{\text{pl}}} \frac{H^2}{A^2}$$
Quantum Fluctuations

- Simple harmonic oscillator $\ll$ Hubble length

$$\ddot{u} + k^2 u = 0$$

- Quantize the simple harmonic oscillator

$$\hat{u} = u(k, \eta) \hat{a} + u^*(k, \eta) \hat{a}^\dagger$$

where $u(k, \eta)$ satisfies classical equation of motion and the creation and annihilation operators satisfy

$$[a, a^\dagger] = 1, \quad a|0\rangle = 0$$

- Normalize wavefunction $[\hat{u}, d\hat{u}/d\eta] = i$

$$u(k, \eta) = \frac{1}{\sqrt{2k}} e^{-ik\tilde{\eta}}$$
Quantum Fluctuations

- Zero point fluctuations of ground state

\[ \langle u^2 \rangle = \langle 0 | u^{\dagger} u | 0 \rangle = \langle 0 | (u^* \hat{a}^{\dagger} + u \hat{a}) (u \hat{a} + u^* \hat{a}^{\dagger}) | 0 \rangle = \langle 0 | \hat{a} \hat{a}^{\dagger} | 0 \rangle |u(k, \tilde{\eta})|^2 = \langle 0 | [\hat{a}, \hat{a}^{\dagger}] + \hat{a}^{\dagger} \hat{a} | 0 \rangle |u(k, \tilde{\eta})|^2 = |u(k, \tilde{\eta})|^2 = \frac{1}{2k} \]

- Classical equation of motion take this quantum fluctuation outside horizon where it freezes in. Slow roll equation

- So \( A = (2k^3)^{-1/2} \) and curvature power spectrum

\[ \Delta_{\mathcal{R}}^2 \equiv \frac{1}{8\pi^2} \frac{H^2}{\epsilon M_{\text{pl}}^2} \]
Tilt

- Curvature power spectrum is scale invariant to the extent that $H$ is constant
- Scalar spectral index

\[
\frac{d \ln \Delta^2_R}{d \ln k} \equiv n_S - 1
\]

\[
= 2 \frac{d \ln H}{d \ln k} - \frac{d \ln \epsilon}{d \ln k}
\]

- Evaluate at horizon crossing where fluctuation freezes

\[
\frac{d \ln H}{d \ln k} \bigg|_{-k\tilde{\eta}=1} = \frac{k}{H} \frac{dH}{d\tilde{\eta}} \bigg|_{-k\tilde{\eta}=1} \frac{d\tilde{\eta}}{dk} \bigg|_{-k\tilde{\eta}=1}
\]

\[
= \frac{k}{H} (-aH^2\epsilon) \bigg|_{-k\tilde{\eta}=1} \frac{1}{k^2} = -\epsilon
\]

where $aH = -1/\tilde{\eta} = k$
Tilt

- Evolution of $\epsilon$

\[
\frac{d \ln \epsilon}{d \ln k} = -\frac{d \ln \epsilon}{d \ln \tilde{\eta}} = -2(\delta + \epsilon) \frac{\dot{a}}{a} \tilde{\eta} = 2(\delta + \epsilon)
\]

- Tilt in the slow-roll approximation

\[
n_s = 1 - 4\epsilon - 2\delta
\]
Gravitational Waves

- Gravitational wave amplitude satisfies Klein-Gordon equation ($K = 0$), same as scalar field
  \[ \ddot{h}_{+,\times} + 2\frac{a}{\dot{a}}\dot{h}_{+,\times} + k^2 h_{+,\times} = 0. \]

- Acquires quantum fluctuations in same manner as $\phi$. Lagrangian sets the normalization

- Scale-invariant gravitational wave amplitude
  \[ \Delta_{+,\times}^2 = 16\pi G \Delta_{\phi_1}^2 = 16\pi G \frac{H^2}{(2\pi)^2} = \frac{H^2}{2\pi^2 M_{\text{pl}}^2}. \]
Gravitational Waves

- Gravitational wave power $\propto H^2 \propto V \propto E_i^4$ where $E_i$ is the energy scale of inflation

- Tensor-scalar ratio - various definitions - WMAP standard is

$$r \equiv 4 \frac{\Delta^2_+}{\Delta^2_R} = 16\epsilon$$

- Tensor tilt:

$$\frac{d \ln \Delta^2_+}{d \ln k} \equiv n_T = 2\frac{d \ln H}{d \ln k} = -2\epsilon$$
Gravitational Waves

- Consistency relation between tensor-scalar ratio and tensor tilt
  \[ r = 16\epsilon = -8n_T \]

- Measurement of scalar tilt and gravitational wave amplitude constrains inflationary model in the slow roll context

- Comparison of tensor-scalar ratio and tensor tilt tests the idea of slow roll itself
Gravitational Wave Phenomenology

- A gravitational wave makes a quadrupolar (transverse-traceless) distortion to metric.
- Just like the scale factor or spatial curvature, a temporal variation in its amplitude leaves a residual temperature variation in CMB photons – here anisotropic.
- Before recombination, anisotropic variation is eliminated by scattering.
- Gravitational wave temperature effect drops sharply at the horizon scale at recombination.
Large Field Models

- For detectable gravitational waves, scalar field must roll by order $M_{pl} = \left(8\pi G\right)^{-1/2}$

  $$\frac{d\phi_0}{dN} = \frac{d\phi_0}{d\ln a} = \frac{d\phi_0}{dt} \frac{1}{H}$$

- The larger $\epsilon$ is the more the field rolls in an e-fold

  $$\epsilon = \frac{r}{16} = \frac{3}{2V} \left(H \frac{d\phi_0}{dN}\right)^2 = \frac{8\pi G}{2} \left(\frac{d\phi_0}{dN}\right)^2$$

- Observable scales span $\Delta N \sim 5$ so

  $$\Delta\phi_0 \approx 5 \frac{d\phi}{dN} = 5\left(r/8\right)^{1/2} M_{pl} \approx 0.6 \left(r/0.1\right)^{1/2} M_{pl}$$

- Does this make sense as an effective field theory? Lyth (1997)
Large Field Models

- Large field models include monomial potentials $V(\phi) = A\phi^n$

\[
\epsilon \approx \frac{n^2}{16\pi G\phi^2}
\]

\[
\delta \approx \epsilon - \frac{n(n - 1)}{8\pi G\phi^2}
\]

- Slow roll requires large field values of $\phi > M_{\text{pl}}$

- Thus $\epsilon \sim |\delta|$ and a finite tilt indicates finite $\epsilon$

- Given WMAP tilt, potentially observable gravitational waves
Small Field Models

• If the field is near an maximum of the potential

\[ V(\phi) = V_0 - \frac{1}{2} \mu^2 \phi^2 \]

• Inflation occurs if the \( V_0 \) term dominates

\[ \epsilon \approx \frac{1}{16\pi G} \frac{\mu^4 \phi^2}{V_0^2} \]

\[ \delta \approx \epsilon + \frac{1}{8\pi G} \frac{\mu^2}{V_0} \rightarrow \frac{\delta}{\epsilon} = \frac{V_0}{\mu^2 \phi^2} \gg 1 \]

• Tilt reflects \( \delta \): \( n_S \approx 1 - 2\delta \) and \( \epsilon \) is much smaller

• The field does not roll significantly during inflation and gravitational waves are negligible
Hybrid Models

- If the field is rolling toward a minimum of the potential

\[ V(\phi) = V_0 + \frac{1}{2} m^2 \phi^2 \]

- Slow roll parameters similar to small field but a real \( m^2 \)

\[ \epsilon \approx \frac{1}{16\pi G} \frac{m^4 \phi^2}{V_0^2} \]

\[ \delta \approx \epsilon - \frac{1}{8\pi G} \frac{m^2}{V_0} \]

- Then \( V_0 \) domination \( \epsilon, \delta < 0 \) and \( n_S > 1 \) - blue tilt

- For \( m^2 \) domination, monomial-like.

- Intermediate cases with intermediate predictions - can have observable gravity waves but does not require it.
Hybrid Models

- But how does inflation end? $V_0$ remains as field settles to minimum
- Implemented as multiple field model with $V_0$ supplied by second field
- Inflation ends when rolling triggers motion in the second field to the true joint minimum