

Set 8:

Inflationary Origins

Horizon Problem

- The horizon in a decelerating universe scales as $\eta \propto a^{(1+3w)/2}$, $w > -1/3$. For example in a matter dominated universe

$$\eta \propto a^{1/2}$$

- CMB decoupled at $a_* = 10^{-3}$ so subtends an angle on the sky

$$\frac{\eta_*}{\eta_0} = a_*^{1/2} \approx 0.03 \approx 2^\circ$$

- So why is the CMB sky isotropic to 10^{-5} in temperature if it is composed of $\sim 10^4$ causally disconnected regions
- If smooth by fiat, why are there 10^{-5} fluctuations correlated on superhorizon scales

Flatness & Relic Problems

- Flatness problem: why is the radius of curvature larger than the observable universe. (Before the CMB determinations, why is it at least comparable to observable universe $|\Omega_K| \lesssim \Omega_m$)
- Also phrased as a coincidence problem: since $\rho_K \propto a^{-2}$ and $\rho_m \propto a^{-3}$, why would they be comparable today – modern version is dark energy coincidence $\rho_\Lambda = \text{const.}$
- Relic problem – why don't relics like monopoles dominate the energy density
- Inflation is a theory that solves all three problems at once and also supplies a source for density perturbations

Accelerating Expansion

- In a matter or radiation dominated universe, the horizon grows as a power law in a so that there is no way to establish causal contact on a scale longer than the inverse Hubble length ($1/aH$, comoving coordinates) at any given time: general for a decelerating universe

$$\eta = \int d \ln a \frac{1}{aH(a)}$$

- $H^2 \propto \rho \propto a^{-3(1+w)}$, $aH \propto a^{-(1+3w)/2}$, critical value of $w = -1/3$, the division between acceleration and deceleration
- In an accelerating universe, the Hubble length shrinks in comoving coordinates and so the horizon gets its contribution at the earliest times, e.g. in a cosmological constant universe, the horizon saturates to a constant value

Causal Contact

- Note confusion in nomenclature: the true horizon always grows meaning that one always sees more and more of the universe. The Hubble length decreases: the difference in conformal time, the distance a photon can travel between two epochs denoted by the scale factor decreases. Regions that were in causal contact, leave causal contact.
- Horizon problem solved if the universe was in an acceleration phase up to η_i and the conformal time since then is shorter than the total conformal age

$$\eta_0 \gg \eta_0 - \eta_i$$

total distance \gg distance traveled since inflation
apparent horizon

Flatness & Relic

- Comoving radius of curvature is constant and can even be small compared to the full horizon $R \ll \eta_0$ yet still $\eta_0 \gg R \gg \eta_0 - \eta_i$
- In physical coordinates, the rapid expansion of the universe makes the current observable universe much smaller than the curvature scale
- Likewise, the number density of relics formed before the accelerating (or inflationary) epoch is diluted to make them rare in the current observable volume
- Common to reference time to the end of inflation $\tilde{\eta} \equiv \eta - \eta_i$. Here conformal time is negative during inflation and its value (as a difference in conformal time) reflects the comoving Hubble length - defines leaving the horizon as $k|\tilde{\eta}| = 1$

Exponential Expansion

- If the accelerating component has equation of state $w = -1$, $\rho = \text{const.}$, $H = H_i \text{ const.}$ so that $a \propto \exp(Ht)$

$$\begin{aligned}\tilde{\eta} &= - \int_a^{a_i} d \ln a \frac{1}{aH} = \frac{1}{aH_i} \Big|_a^{a_i} \\ &\approx -\frac{1}{aH_i} \quad (a_i \gg a)\end{aligned}$$

- In particular, the current horizon scale $H_0 \tilde{\eta}_0 \approx 1$ exited the horizon during inflation at

$$\begin{aligned}\tilde{\eta}_0 &\approx H_0^{-1} = \frac{1}{a_H H_i} \\ a_H &= \frac{H_0}{H_i}\end{aligned}$$

Sufficient Inflation

- Current horizon scale must have exited the horizon during inflation so that the start of inflation could not be after a_H . How long before the end of inflation must it have begun?

$$\frac{a_H}{a_i} = \frac{H_0}{H_i a_i}$$
$$\frac{H_0}{H_i} = \sqrt{\frac{\rho_c}{\rho_i}}, \quad a_i = \frac{T_{\text{CMB}}}{T_i}$$

- $\rho_c^{1/4} = 3 \times 10^{-12} \text{ GeV}$, $T_{\text{CMB}} = 3 \times 10^{-13} \text{ GeV}$

$$\frac{a_H}{a_i} = 3 \times 10^{-29} \left(\frac{\rho_i^{1/4}}{10^{14} \text{ GeV}} \right)^{-2} \left(\frac{T_i}{10^{10} \text{ GeV}} \right)$$
$$\ln \frac{a_i}{a_H} = 65 + 2 \ln \left(\frac{\rho_i^{1/4}}{10^{14} \text{ GeV}} \right) - \ln \left(\frac{T_i}{10^{10} \text{ GeV}} \right)$$

Perturbation Generation

- Horizon scale $\tilde{\eta}$ during inflation acts like an event horizon - things leaving causal contact
- Particle creation similar to Hawking radiation from a black hole with hubble length replacing the BH horizon

$$T_H \approx H_i$$

- Because H_i remains roughly constant during inflation the result is a scale invariant spectrum of fluctuations due to zero-point fluctuations becoming classical
- Fluctuations in the field driving inflation (inflaton) carry the energy density of the universe and so their zero point fluctuations are net energy density or curvature fluctuations
- Any other light field (gravitational waves, etc...) will also carry scale invariant perturbations but are iso-curvature fluctuations

Scalar Fields

- Stress-energy tensor of a scalar field

$$T^{\mu}_{\nu} = \nabla^{\mu}\varphi \nabla_{\nu}\varphi - \frac{1}{2}(\nabla^{\alpha}\varphi \nabla_{\alpha}\varphi + 2V)\delta^{\mu}_{\nu}.$$

- For the background $\langle\phi\rangle \equiv \phi_0$ (a^{-2} from conformal time)

$$\rho_{\phi} = \frac{1}{2}a^{-2}\dot{\phi}_0^2 + V, \quad p_{\phi} = \frac{1}{2}a^{-2}\dot{\phi}_0^2 - V$$

- So for kinetic dominated $w_{\phi} = p_{\phi}/\rho_{\phi} \rightarrow 1$
- And potential dominated $w_{\phi} = p_{\phi}/\rho_{\phi} \rightarrow -1$
- A slowly rolling (potential dominated) scalar field can accelerate the expansion and so solve the horizon problem or act as a dark energy candidate

Equation of Motion

- Can use general equations of motion dictated by stress energy conservation

$$\dot{\rho}_\phi = -3(\rho_\phi + p_\phi)\frac{\dot{a}}{a},$$

to obtain the equation of motion of the background field ϕ

$$\ddot{\phi}_0 + 2\frac{\dot{a}}{a}\dot{\phi}_0 + a^2V' = 0,$$

- In terms of time instead of conformal time

$$\frac{d^2\phi_0}{dt^2} + 3H\frac{d\phi_0}{dt} + V' = 0$$

- Field rolls down potential hill but experiences “Hubble friction” to create slow roll. In slow roll $3Hd\phi_0/dt \approx -V'$ and so kinetic energy is determined by field position \rightarrow adiabatic – both kinetic and potential energy determined by single degree of freedom ϕ_0

Slow Roll Inflation

- Alternately can derive directly from the Klein-Gordon equation for scalar field
- Scalar field equation of motion $V' \equiv dV/d\phi$

$$\nabla_{\mu} \nabla^{\mu} \phi + V'(\phi) = 0$$

so that in the background $\phi = \phi_0$ and

$$\ddot{\phi}_0 + 2\frac{\dot{a}}{a}\dot{\phi}_0 + a^2 V' = 0$$
$$\frac{d^2 \phi_0}{dt^2} + 3H \frac{d\phi_0}{dt} + V' = 0$$

- Simply the continuity equation with the associations

$$\rho_{\phi} = \frac{1}{2} a^{-2} \dot{\phi}_0^2 + V \quad p_{\phi} = \frac{1}{2} a^{-2} \dot{\phi}_0^2 - V$$

Slow Roll Parameters

- Net energy is dominated by potential energy and so acts like a cosmological constant $w \rightarrow -1$
- First slow roll parameter

$$\epsilon = \frac{3}{2}(1 + w) = \frac{1}{16\pi G} \left(\frac{V'}{V} \right)^2$$

- Second slow roll parameter $d^2\phi_0/dt^2 \approx 0$, or $\ddot{\phi}_0 \approx (\dot{a}/a)\dot{\phi}_0$

$$\delta = \frac{\ddot{\phi}_0}{\dot{\phi}_0} \left(\frac{\dot{a}}{a} \right)^{-1} - 1 = \epsilon - \frac{1}{8\pi G} \frac{V''}{V}$$

- Slow roll condition $\epsilon, \delta \ll 1$ corresponds to a very flat potential

Perturbations

- Linearize perturbation $\phi = \phi_0 + \phi_1$ then

$$\ddot{\phi}_1 + 2\frac{\dot{a}}{a}\dot{\phi}_1 + k^2\phi_1 + a^2V''\phi_1 = 0$$

in slow roll inflation V'' term negligible

- Implicitly assume that the spatial metric fluctuations (curvature \mathcal{R}) vanishes, otherwise covariant derivatives pick these up
- GR: work in the spatially flat slicing and transform back to comoving slicing once done.
- Curvature is local scale factor $a \rightarrow (1 + \mathcal{R})a$ or $\delta a/a = \mathcal{R}$

$$\mathcal{R} = \frac{\delta a}{a} = \frac{\dot{a}}{a}\delta\eta = \frac{\dot{a}}{a}\frac{\phi_1}{\dot{\phi}_0}$$

a change in the field value ϕ_1 defines a change in the epoch that inflation ends, imprinting a curvature fluctuation

Slow-Roll Evolution

- Rewrite in $u \equiv a\phi$ to remove expansion damping

$$\ddot{u} + \left[k^2 - 2 \left(\frac{\dot{a}}{a} \right)^2 \right] u = 0$$

- or for conformal time measured from the end of inflation

$$\tilde{\eta} = \eta - \eta_{\text{end}}$$

$$\tilde{\eta} = \int_{a_{\text{end}}}^a \frac{da}{Ha^2} \approx -\frac{1}{aH}$$

- Compact, slow-roll equation:

$$\ddot{u} + \left[k^2 - \frac{2}{\tilde{\eta}^2} \right] u = 0$$

Slow Roll Limit

- Slow roll equation has the exact solution:

$$u = A\left(k \pm \frac{i}{\tilde{\eta}}\right)e^{\mp ik\tilde{\eta}}$$

- For $|k\tilde{\eta}| \gg 1$ (early times, inside Hubble length) behaves as free oscillator

$$\lim_{|k\tilde{\eta}| \rightarrow \infty} u = Ake^{\mp ik\tilde{\eta}}$$

- Normalization A will be set by origin in quantum fluctuations of free field

Slow Roll Limit

- For $|k\tilde{\eta}| \ll 1$ (late times, \gg Hubble length) fluctuation freezes in

$$\lim_{|k\tilde{\eta}| \rightarrow 0} u = \pm \frac{i}{\tilde{\eta}} A = \pm i H a A$$

$$\phi_1 = \pm i H A$$

$$\mathcal{R} = \mp i H A \left(\frac{\dot{a}}{a} \right) \frac{1}{\dot{\phi}_0}$$

- Slow roll replacement

$$\left(\frac{\dot{a}}{a} \right)^2 \frac{1}{\dot{\phi}_0^2} = \frac{8\pi G a^2 V}{3} \frac{3}{2a^2 V \epsilon} = \frac{4\pi G}{\epsilon} = \frac{1}{2\epsilon M_{\text{pl}}^2}$$

- Comoving curvature power spectrum

$$\Delta_{\mathcal{R}}^2 \equiv \frac{k^3 |\mathcal{R}|^2}{2\pi^2} = \frac{k^3}{4\pi^2} \frac{H^2}{\epsilon M_{\text{pl}}^2} A^2$$

Quantum Fluctuations

- Simple harmonic oscillator \ll Hubble length

$$\ddot{u} + k^2 u = 0$$

- Quantize the simple harmonic oscillator

$$\hat{u} = u(k, \eta) \hat{a} + u^*(k, \eta) \hat{a}^\dagger$$

where $u(k, \eta)$ satisfies classical equation of motion and the creation and annihilation operators satisfy

$$[a, a^\dagger] = 1, \quad a|0\rangle = 0$$

- Normalize wavefunction $[\hat{u}, d\hat{u}/d\eta] = i$

$$u(k, \eta) = \frac{1}{\sqrt{2k}} e^{-ik\tilde{\eta}}$$

Quantum Fluctuations

- Zero point fluctuations of ground state

$$\begin{aligned}\langle u^2 \rangle &= \langle 0 | u^\dagger u | 0 \rangle \\ &= \langle 0 | (u^* \hat{a}^\dagger + u \hat{a}) (u \hat{a} + u^* \hat{a}^\dagger) | 0 \rangle \\ &= \langle 0 | \hat{a} \hat{a}^\dagger | 0 \rangle |u(k, \tilde{\eta})|^2 \\ &= \langle 0 | [\hat{a}, \hat{a}^\dagger] + \hat{a}^\dagger \hat{a} | 0 \rangle |u(k, \tilde{\eta})|^2 \\ &= |u(k, \tilde{\eta})|^2 = \frac{1}{2k}\end{aligned}$$

- Classical equation of motion take this quantum fluctuation outside horizon where it freezes in. Slow roll equation
- So $A = (2k^3)^{-1/2}$ and curvature power spectrum

$$\Delta_{\mathcal{R}}^2 \equiv \frac{1}{8\pi^2} \frac{H^2}{\epsilon M_{\text{pl}}^2}$$

Tilt

- Curvature power spectrum is scale invariant to the extent that H is constant
- Scalar spectral index

$$\begin{aligned}\frac{d \ln \Delta_{\mathcal{R}}^2}{d \ln k} &\equiv n_S - 1 \\ &= 2 \frac{d \ln H}{d \ln k} - \frac{d \ln \epsilon}{d \ln k}\end{aligned}$$

- Evaluate at horizon crossing where fluctuation freezes

$$\begin{aligned}\frac{d \ln H}{d \ln k} \Big|_{-k\tilde{\eta}=1} &= \frac{k}{H} \frac{dH}{d\tilde{\eta}} \Big|_{-k\tilde{\eta}=1} \frac{d\tilde{\eta}}{dk} \Big|_{-k\tilde{\eta}=1} \\ &= \frac{k}{H} (-aH^2\epsilon) \Big|_{-k\tilde{\eta}=1} \frac{1}{k^2} = -\epsilon\end{aligned}$$

where $aH = -1/\tilde{\eta} = k$

Tilt

- Evolution of ϵ

$$\frac{d \ln \epsilon}{d \ln k} = -\frac{d \ln \epsilon}{d \ln \tilde{\eta}} = -2(\delta + \epsilon) \frac{\dot{a}}{a} \tilde{\eta} = 2(\delta + \epsilon)$$

- Tilt in the slow-roll approximation

$$n_S = 1 - 4\epsilon - 2\delta$$

Gravitational Waves

- Gravitational wave amplitude satisfies Klein-Gordon equation ($K = 0$), same as scalar field

$$\ddot{h}_{+,\times} + 2\frac{\dot{a}}{a}\dot{h}_{+,\times} + k^2 h_{+,\times} = 0.$$

- Acquires quantum fluctuations in same manner as ϕ . Lagrangian sets the normalization
- Scale-invariant gravitational wave amplitude

$$\Delta_{+,\times}^2 = 16\pi G \Delta_{\phi_1}^2 = 16\pi G \frac{H^2}{(2\pi)^2} = \frac{H^2}{2\pi^2 M_{\text{pl}}^2}$$

Gravitational Waves

- Gravitational wave power $\propto H^2 \propto V \propto E_i^4$ where E_i is the energy scale of inflation
- Tensor-scalar ratio - various definitions - WMAP standard is

$$r \equiv 4 \frac{\Delta_+^2}{\Delta_{\mathcal{R}}^2} = 16\epsilon$$

- Tensor tilt:

$$\frac{d \ln \Delta_+^2}{d \ln k} \equiv n_T = 2 \frac{d \ln H}{d \ln k} = -2\epsilon$$

Gravitational Waves

- Consistency relation between tensor-scalar ratio and tensor tilt

$$r = 16\epsilon = -8n_T$$

- Measurement of scalar tilt and gravitational wave amplitude constrains inflationary model in the slow roll context
- Comparison of tensor-scalar ratio and tensor tilt tests the idea of slow roll itself

Gravitational Wave Phenomenology

- A gravitational wave makes a quadrupolar (transverse-traceless) distortion to metric
- Just like the scale factor or spatial curvature, a temporal variation in its amplitude leaves a residual temperature variation in CMB photons – here anisotropic
- Before recombination, anisotropic variation is eliminated by scattering
- Gravitational wave temperature effect drops sharply at the horizon scale at recombination

Large Field Models

- For detectable gravitational waves, scalar field must roll by order

$$M_{\text{pl}} = (8\pi G)^{-1/2}$$

$$\frac{d\phi_0}{dN} = \frac{d\phi_0}{d \ln a} = \frac{d\phi_0}{dt} \frac{1}{H}$$

- The larger ϵ is the more the field rolls in an e-fold

$$\epsilon = \frac{r}{16} = \frac{3}{2V} \left(H \frac{d\phi_0}{dN} \right)^2 = \frac{8\pi G}{2} \left(\frac{d\phi_0}{dN} \right)^2$$

- Observable scales span $\Delta N \sim 5$ so

$$\Delta\phi_0 \approx 5 \frac{d\phi}{dN} = 5(r/8)^{1/2} M_{\text{pl}} \approx 0.6(r/0.1)^{1/2} M_{\text{pl}}$$

- Does this make sense as an effective field theory? [Lyth \(1997\)](#)

Large Field Models

- Large field models include monomial potentials $V(\phi) = A\phi^n$

$$\epsilon \approx \frac{n^2}{16\pi G\phi^2}$$

$$\delta \approx \epsilon - \frac{n(n-1)}{8\pi G\phi^2}$$

- Slow roll requires large field values of $\phi > M_{\text{pl}}$
- Thus $\epsilon \sim |\delta|$ and a finite tilt indicates finite ϵ
- Given WMAP tilt, potentially observable gravitational waves

Small Field Models

- If the field is near an maximum of the potential

$$V(\phi) = V_0 - \frac{1}{2}\mu^2\phi^2$$

- Inflation occurs if the V_0 term dominates

$$\epsilon \approx \frac{1}{16\pi G} \frac{\mu^4\phi^2}{V_0^2}$$

$$\delta \approx \epsilon + \frac{1}{8\pi G} \frac{\mu^2}{V_0} \rightarrow \frac{\delta}{\epsilon} = \frac{V_0}{\mu^2\phi^2} \gg 1$$

- Tilt reflects δ : $n_S \approx 1 - 2\delta$ and ϵ is much smaller
- The field does not roll significantly during inflation and gravitational waves are negligible

Hybrid Models

- If the field is rolling toward a minimum of the potential

$$V(\phi) = V_0 + \frac{1}{2}m^2\phi^2$$

- Slow roll parameters similar to small field but a real m^2

$$\epsilon \approx \frac{1}{16\pi G} \frac{m^4\phi^2}{V_0^2}$$

$$\delta \approx \epsilon - \frac{1}{8\pi G} \frac{m^2}{V_0}$$

- Then V_0 domination $\epsilon, \delta < 0$ and $n_S > 1$ - blue tilt
- For m^2 domination, monomial-like.
- Intermediate cases with intermediate predictions - can have observable gravity waves but does not require it.

Hybrid Models

- But how does inflation end? V_0 remains as field settles to minimum
- Implemented as multiple field model with V_0 supplied by second field
- Inflation ends when rolling triggers motion in the second field to the true joint minimum