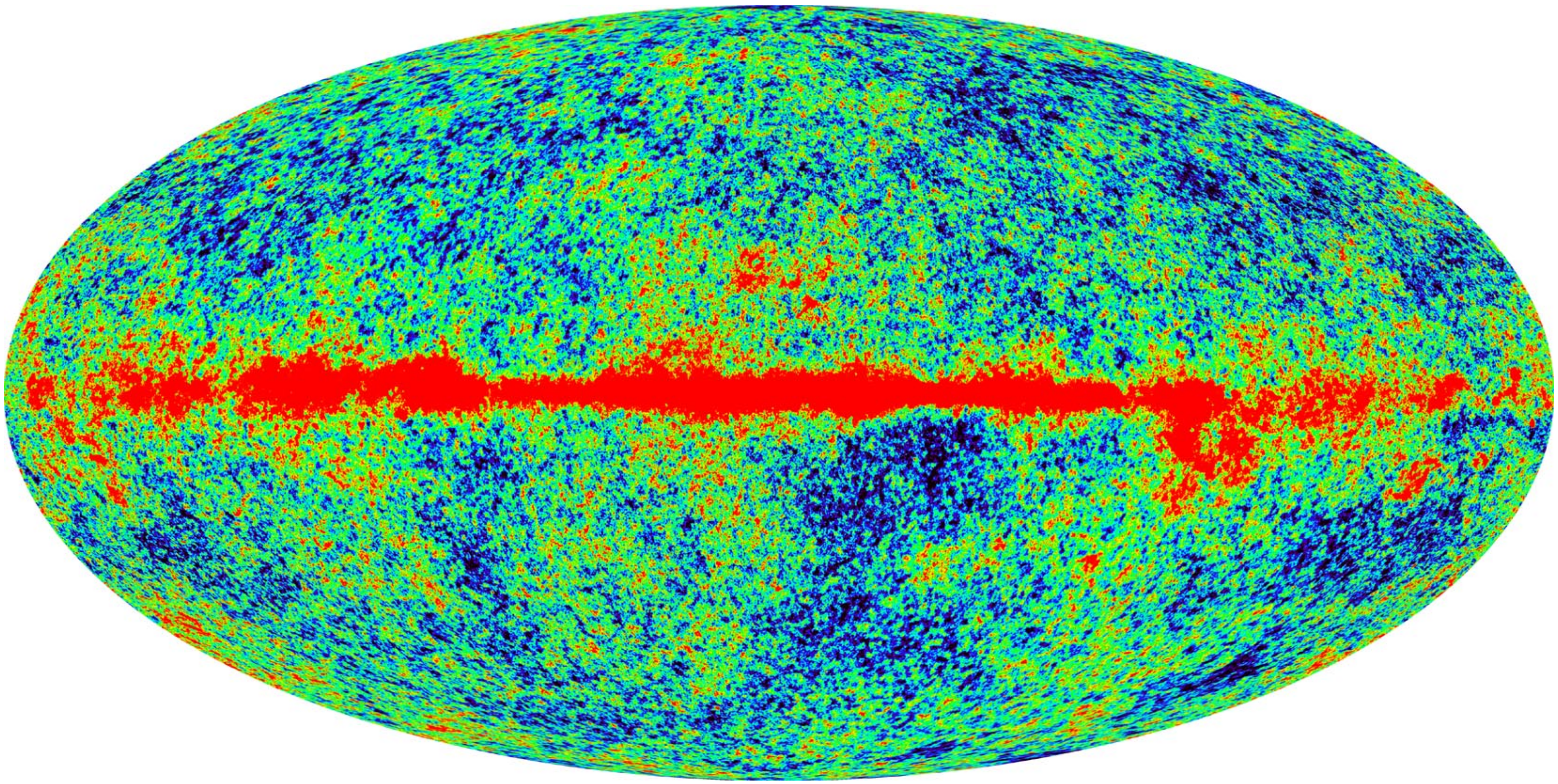
# Set 9:

# CMB and Large Scale Structure

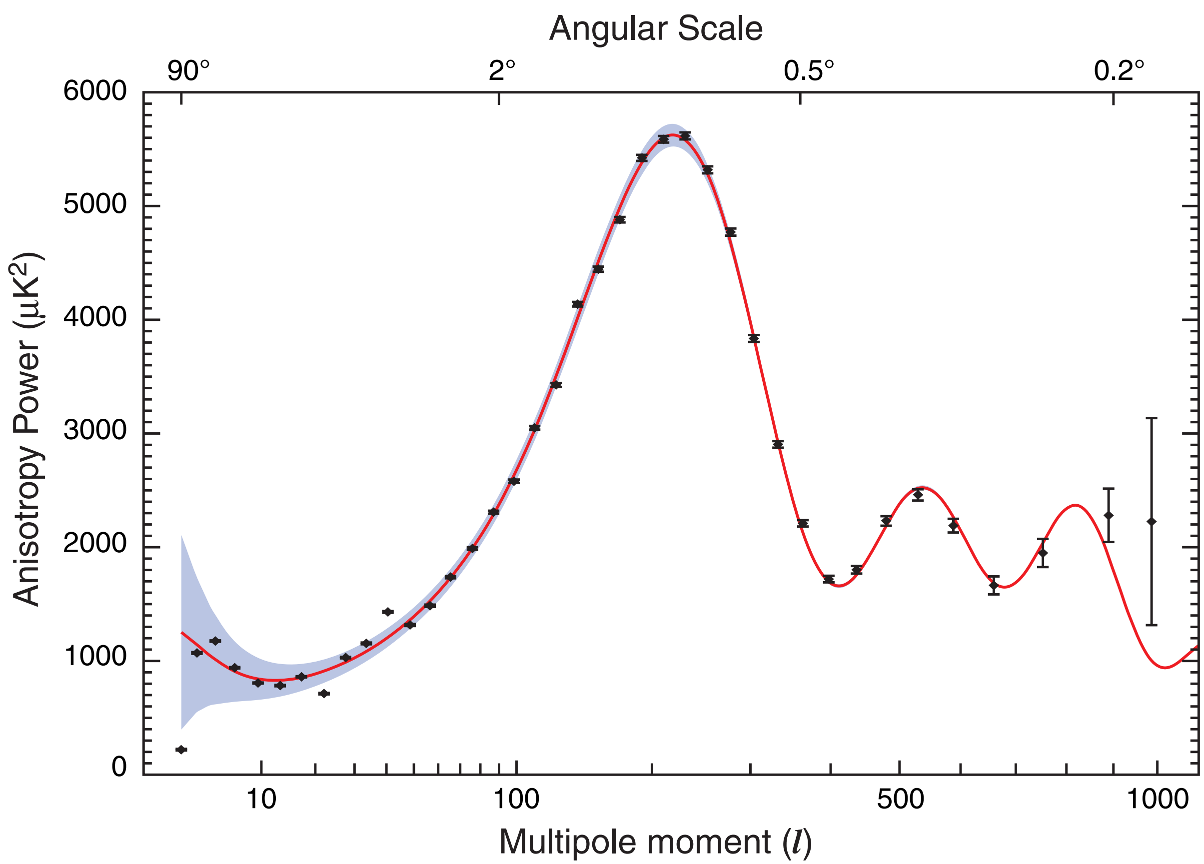
# CMB Temperature Anisotropy

- WMAP measured the temperature anisotropy (first discovered by COBE) from recombination:



# CMB Temperature Anisotropy

- Power spectrum shows characteristic scales where the intensity of variations peak - reveals geometry and contents of the universe:



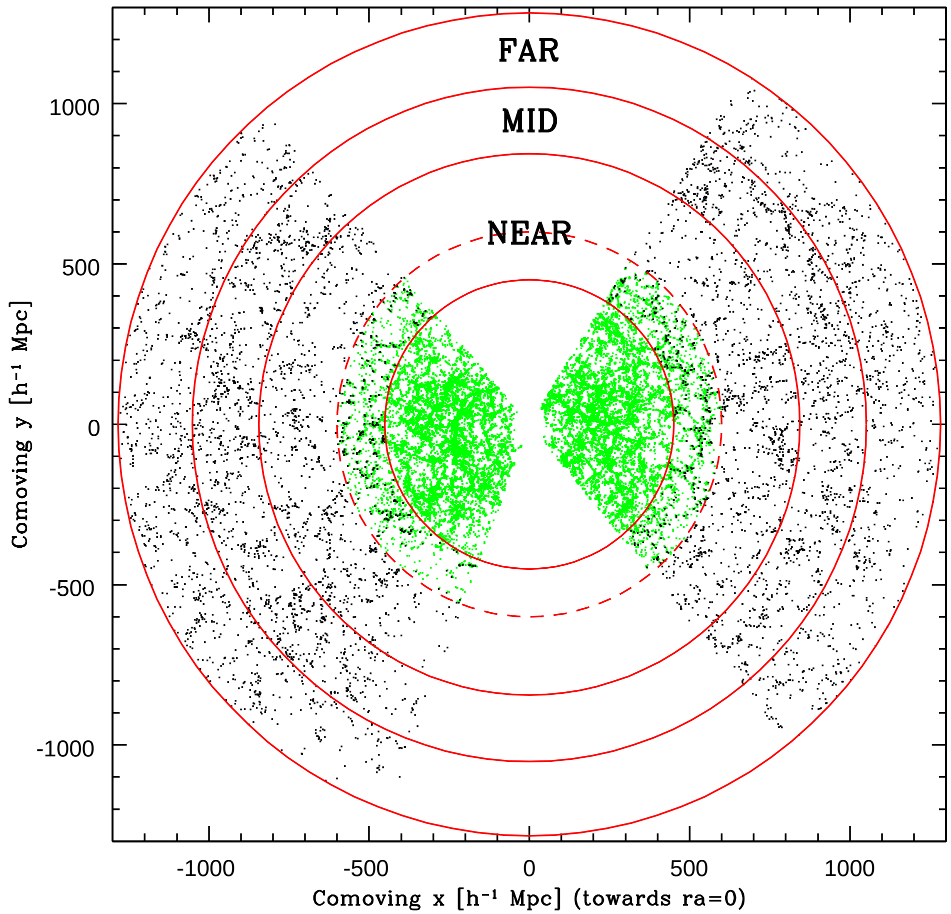
# CMB Parameter Inferences

- Spectrum constrains the matter-energy contents of the universe

| Parameter | First Year Mean | WMAPext Mean | Three Year Mean (No SZ) | Three Year Mean | Three Year+ALL Mean |
|---|---|---|---|---|---|
| $100\Omega_b h^2$ | $2.38^{+0.13}_{-0.12}$ | $2.32^{+0.12}_{-0.11}$ | $2.23 \pm 0.08$ | $2.229 \pm 0.073$ | $2.186 \pm 0.068$ |
| $\Omega_m h^2$ | $0.144^{+0.016}_{-0.016}$ | $0.134^{+0.006}_{-0.006}$ | $0.126 \pm 0.009$ | $0.1277^{+0.0080}_{-0.0079}$ | $0.1324^{+0.0042}_{-0.0041}$ |
| $H_0$ | $72^{+5}_{-5}$ | $73^{+3}_{-3}$ | $73.5 \pm 3.2$ | $73.2^{+3.1}_{-3.2}$ | $70.4^{+1.5}_{-1.6}$ |
| $\tau$ | $0.17^{+0.08}_{-0.07}$ | $0.15^{+0.07}_{-0.07}$ | $0.088^{+0.029}_{-0.030}$ | $0.089 \pm 0.030$ | $0.073^{+0.027}_{-0.028}$ |
| $n_s$ | $0.99^{+0.04}_{-0.04}$ | $0.98^{+0.03}_{-0.03}$ | $0.961 \pm 0.017$ | $0.958 \pm 0.016$ | $0.947 \pm 0.015$ |
| $\Omega_m$ | $0.29^{+0.07}_{-0.07}$ | $0.25^{+0.03}_{-0.03}$ | $0.234 \pm 0.035$ | $0.241 \pm 0.034$ | $0.268 \pm 0.018$ |
| $\sigma_8$ | $0.92^{+0.1}_{-0.1}$ | $0.84^{+0.06}_{-0.06}$ | $0.76 \pm 0.05$ | $0.761^{+0.049}_{-0.048}$ | $0.776^{+0.031}_{-0.032}$ |
| Parameter | First Year ML | WMAPext ML | Three Year ML (No SZ) | Three Year ML | Three Year + ALL ML |
| $100\Omega_b h^2$ | 2.30 | 2.21 | 2.23 | 2.22 | 2.19 |
| $\Omega_m h^2$ | 0.145 | 0.138 | 0.125 | 0.127 | 0.131 |
| $H_0$ | 68 | 71 | 73.4 | 73.2 | 73.2 |
| $\tau$ | 0.10 | 0.10 | 0.0904 | 0.091 | 0.0867 |
| $n_s$ | 0.97 | 0.96 | 0.95 | 0.954 | 0.949 |
| $\Omega_m$ | 0.32 | 0.27 | 0.232 | 0.236 | 0.259 |
| $\sigma_8$ | 0.88 | 0.82 | 0.737 | 0.756 | 0.783 |

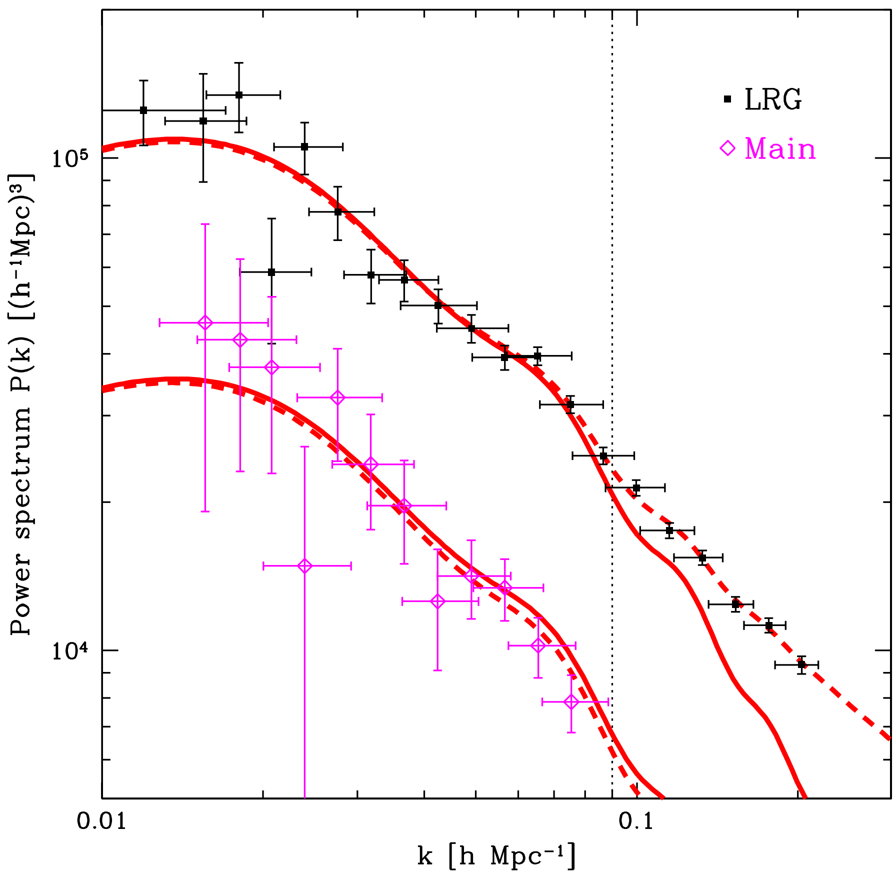
# Galaxy Redshift Surveys

- Galaxy redshift surveys (e.g. 2dF and SDSS) measure the three dimensional distribution of galaxies today:



# Galaxy Power Spectrum

- SDSS LRG and Main power spectrum:



# Structure Formation

- Small perturbations from inflation over the course of the 14Gyr life of the universe are gravitationally enhanced into all of the structure seen today
- Cosmic microwave background shows a snapshot at a few hundred thousand years old at recombination
- Discovery in 1992 of cosmic microwave background anisotropy provided the observational breakthrough - convincing support for adiabatic initial density fluctuations of amplitude $10^{-5}$
- Combine with galaxy clustering - large scale structure seen in galaxy surveys - right amplitude given cold dark matter
- Following notes are at a slightly more advanced level than the book and are provided here for completeness

# Angular Power Spectrum

- Angular distribution of radiation is essentially the 3D temperature field projected onto a shell at the distance from the observer to recombination: called the last scattering surface
- Take the radiation distribution at last scattering to also be described by an isotropic temperature fluctuation field $\Theta(\mathbf{x})$ and recombination to be instantaneous

$$\Theta(\hat{\mathbf{n}}) = \int dD\, \Theta(\mathbf{x})\delta(D - D_*)$$

where $D$ is the comoving distance and $D_*$ denotes recombination.

- Describe the temperature field by its Fourier moments

$$\Theta(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3}\Theta(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}$$

# Angular Power Spectrum

- Power spectrum

$$\langle\Theta(\mathbf{k})^*\Theta(\mathbf{k}')\rangle = (2\pi)^3\delta(\mathbf{k}-\mathbf{k}')P_T(k)$$

$$\Delta_T^2 = k^3 P_T/2\pi^2$$

- Temperature field

$$\Theta(\hat{\mathbf{n}}) = \int \frac{d^3k}{(2\pi)^3}\Theta(\mathbf{k})e^{i\mathbf{k}\cdot D_*\hat{\mathbf{n}}}$$

- Multipole moments $\Theta(\hat{\mathbf{n}}) = \sum_{\ell m}\Theta_{\ell m}Y_{\ell m}$
- Expand out plane wave in spherical coordinates

$$e^{i\mathbf{k}D_*\cdot\hat{\mathbf{n}}} = 4\pi\sum_{\ell m} i^\ell j_\ell(kD_*)Y^*_{\ell m}(\mathbf{k})Y_{\ell m}(\hat{\mathbf{n}})$$

# Angular Power Spectrum

- Power spectrum

$$\Theta_{\ell m} = \int \frac{d^3k}{(2\pi)^3} \Theta(\mathbf{k}) 4\pi i^\ell j_\ell(kD_*) Y_{\ell m}(\mathbf{k})$$

$$\begin{aligned}\langle \Theta^*_{\ell m} \Theta_{\ell' m'} \rangle &= \int \frac{d^3k}{(2\pi)^3} (4\pi)^2 (i)^{\ell-\ell'} j_\ell(kD_*) j_{\ell'}(kD_*) Y^*_{\ell m}(\mathbf{k}) Y_{\ell' m'}(\mathbf{k}) P_T(k) \\ &= \delta_{\ell\ell'} \delta_{mm'} 4\pi \int d\ln k \, j_\ell^2(kD_*) \Delta_T^2(k)\end{aligned}$$

with $\int_0^\infty j_\ell^2(x) d\ln x = 1/(2\ell(\ell+1))$, slowly varying $\Delta_T^2$

- Angular power spectrum:

$$C_\ell = \frac{4\pi \Delta_T^2(\ell/D_*)}{2\ell(\ell+1)} = \frac{2\pi}{\ell(\ell+1)} \Delta_T^2(\ell/D_*)$$

# Thomson Scattering

- Thomson scattering of photons off of free electrons is the most important CMB process with a cross section (averaged over polarization states) of

$$\sigma_T = \frac{8\pi\alpha^2}{3m_e^2} = 6.65 \times 10^{-25}\mathrm{cm}^2$$

- Density of free electrons in a fully ionized $x_e = 1$ universe

$$n_e = (1 - Y_p/2)x_e n_b \approx 10^{-5}\Omega_b h^2 (1+z)^3 \mathrm{cm}^{-3}\,,$$

where $Y_p \approx 0.24$ is the Helium mass fraction, creates a high (comoving) Thomson opacity

$$\dot\tau \equiv n_e \sigma_T a$$

where dots are conformal time $\eta \equiv \int dt/a$ derivatives and $\tau$ is the optical depth.

# Tight Coupling Approximation

- Near recombination $z \approx 10^3$ and $\Omega_b h^2 \approx 0.02$, the (comoving) mean free path of a photon

$$\lambda_C \equiv \frac{1}{\dot{\tau}} \sim 2.5\text{Mpc}$$

  small by cosmological standards!

- On scales $\lambda \gg \lambda_C$ photons are tightly coupled to the electrons by Thomson scattering which in turn are tightly coupled to the baryons by Coulomb interactions

- Specifically, their bulk velocities are defined by a single fluid velocity $v_\gamma = v_b$ and the photons carry no anisotropy in the rest frame of the baryons

- $\rightarrow$ No heat conduction or viscosity (anisotropic stress) in fluid

# Zeroth Order Approximation

- Momentum density of a fluid is $(\rho + p)v$, where $p$ is the pressure
- Neglect the momentum density of the baryons

$$R \equiv \frac{(\rho_b + p_b)v_b}{(\rho_\gamma + p_\gamma)v_\gamma} = \frac{\rho_b + p_b}{\rho_\gamma + p_\gamma} = \frac{3\rho_b}{4\rho_\gamma}$$
$$\approx 0.6 \left(\frac{\Omega_b h^2}{0.02}\right)\left(\frac{a}{10^{-3}}\right)$$

since $\rho_\gamma \propto T^4$ is fixed by the CMB temperature $T = 2.73(1+z)$K – OK substantially before recombination

- Neglect radiation in the expansion

$$\frac{\rho_m}{\rho_r} = 3.6 \left(\frac{\Omega_m h^2}{0.15}\right)\left(\frac{a}{10^{-3}}\right)$$

- Neglect gravity

# Fluid Equations

- Density $\rho_\gamma \propto T^4$ so define temperature fluctuation $\Theta$

$$\delta_\gamma = 4\frac{\delta T}{T} \equiv 4\Theta$$

- Real space continuity equation transformed to Fourier space $\nabla \to i\mathbf{k}$

$$\dot{\delta}_\gamma = -(1+w_\gamma)kv_\gamma$$
$$\dot{\Theta} = -\frac{1}{3}kv_\gamma$$

- Euler equation (neglecting gravity)

$$\dot{v}_\gamma = -(1 - 3w_\gamma)\frac{\dot{a}}{a}v + \frac{kc_s^2}{1 + w_\gamma}\delta_\gamma$$

$$\dot{v}_\gamma = kc_s^2\frac{3}{4}\delta_\gamma = 3c_s^2 k\Theta$$

# Oscillator: Take One

- Combine these to form the simple harmonic oscillator equation

$$\ddot{\Theta} + c_s^2 k^2 \Theta = 0$$

where the sound speed is adiabatic

$$c_s^2 = \frac{\delta p}{\delta \rho} = \frac{\dot{p}_\gamma}{\dot{\rho}_\gamma}$$

here $c_s^2 = 1/3$ since we are photon-dominated

- General solution:

$$\Theta(\eta) = \Theta(0)\cos(ks) + \frac{\dot{\Theta}(0)}{kc_s}\sin(ks)$$

where the sound horizon is defined as $s \equiv \int c_s d\eta$

# Harmonic Extrema

- All modes are frozen in at recombination (denoted with a subscript $*$) yielding temperature perturbations of different amplitude for different modes. For the adiabatic (curvature mode) $\dot{\Theta}(0) = 0$

$$\Theta(\eta_*) = \Theta(0)\cos(ks_*)$$

- Modes caught in the extrema of their oscillation will have enhanced fluctuations

$$k_n s_* = n\pi$$

yielding a fundamental scale or frequency, related to the inverse sound horizon

$$k_A = \pi/s_*$$

and a harmonic relationship to the other extrema as $1:2:3\ldots$

# Peak Location

- The fundmental physical scale is translated into a fundamental angular scale by simple projection according to the angular diameter distance $D_A$

$$
\begin{aligned}
\theta_A &= \lambda_A / D_A \\
\ell_A &= k_A D_A
\end{aligned}
$$

- In a flat universe, the distance is simply $D_A = D \equiv \eta_0 - \eta_* \approx \eta_0$, the horizon distance, and $k_A = \pi / s_* = \sqrt{3}\pi/\eta_*$ so

$$
\theta_A \approx \frac{\eta_*}{\eta_0}
$$

- In a matter-dominated universe $\eta \propto a^{1/2}$ so $\theta_A \approx 1/30 \approx 2^\circ$ or

$$
\ell_A \approx 200
$$

# Curvature

- In a curved universe, the apparent or angular diameter distance is no longer the conformal distance $D_A = R\sin(D/R) \neq D$
- Objects in a closed universe are further than they appear! gravitational lensing of the background...
- Curvature scale of the universe must be substantially larger than current horizon
- Flat universe indicates critical density and implies missing energy given local measures of the matter density "dark energy"
- $D$ also depends on dark energy density $\Omega_{\rm DE}$ and equation of state $w = p_{\rm DE}/\rho_{\rm DE}$.
- Expansion rate at recombination or matter-radiation ratio enters into calculation of $k_A$.

# Doppler Effect

- Bulk motion of fluid changes the observed temperature via Doppler shifts

$$\left(\frac{\Delta T}{T}\right)_{\rm dop} = \hat{\mathbf{n}} \cdot \mathbf{v}_\gamma$$

- Averaged over directions

$$\left(\frac{\Delta T}{T}\right)_{\rm rms} = \frac{v_\gamma}{\sqrt{3}}$$

- Acoustic solution

$$\begin{aligned}\frac{v_\gamma}{\sqrt{3}} &= -\frac{\sqrt{3}}{k}\dot{\Theta} = \frac{\sqrt{3}}{k} k c_s\, \Theta(0) \sin(ks) \\ &= \Theta(0) \sin(ks)\end{aligned}$$

# Doppler Peaks?

- Doppler effect for the photon dominated system is of equal amplitude and $\pi/2$ out of phase: extrema of temperature are turning points of velocity
- Effects add in quadrature:

$$\left(\frac{\Delta T}{T}\right)^2 = \Theta^2(0)[\cos^2(ks) + \sin^2(ks)] = \Theta^2(0)$$

- No peaks in $k$ spectrum! However the Doppler effect carries an angular dependence that changes its projection on the sky $\hat{\mathbf{n}} \cdot \mathbf{v}_\gamma \propto \hat{\mathbf{n}} \cdot \hat{\mathbf{k}}$
- Coordinates where $\hat{\mathbf{z}} \parallel \hat{\mathbf{k}}$

$$Y_{10} Y_{\ell 0} \to Y_{\ell \pm 1\, 0}$$

recoupling $j_\ell' Y_{\ell 0}$: no peaks in Doppler effect

# Restoring Gravity

- Take a simple photon dominated system with gravity
- Continuity altered since a gravitational potential represents a stretching of the spatial fabric that dilutes number densities – formally a spatial curvature perturbation
- Think of this as a perturbation to the scale factor $a \rightarrow a(1+\Phi)$ so that the cosmogical redshift is generalized to

$$\frac{\dot{a}}{a} \rightarrow \frac{\dot{a}}{a} + \dot{\Phi}$$

so that the continuity equation becomes

$$\dot{\Theta} = -\frac{1}{3} k v_\gamma - \dot{\Phi}$$

# Restoring Gravity

- Gravitational force in momentum conservation $\mathbf{F} = -m\nabla\Psi$ generalized to momentum density modifies the Euler equation to

$$\dot{v}_\gamma = k(\Theta + \Psi)$$

- General relativity says that $\Phi$ and $\Psi$ are the relativistic analogues of the Newtonian potential and that $\Phi \approx -\Psi$.
- In our matter-dominated approximation, $\Phi$ represents matter density fluctuations through the cosmological Poisson equation

$$k^2\Phi = 4\pi G a^2 \rho_m \Delta_m$$

where the difference comes from the use of comoving coordinates for $k$ ($a^2$ factor), the removal of the background density into the background expansion $(\rho\Delta_m)$ and finally a coordinate subtlety that enters into the definition of $\Delta_m$

# Constant Potentials

- In the matter dominated epoch potentials are constant because infall generates velocities as $v_m \sim k\eta\Psi$
- Velocity divergence generates density perturbations as $\Delta_m \sim -k\eta v_m \sim -(k\eta)^2\Psi$
- And density perturbations generate potential fluctuations as $\Phi \sim \Delta_m/(k\eta)^2 \sim -\Psi$, keeping them constant. Note that because of the expansion, density perturbations must grow to keep potentials constant.
- Here we have used the Friedman equation $H^2 = 8\pi G\rho_m/3$ and $\eta = \int d\ln a/(aH) \sim 1/(aH)$
- More generally, if stress perturbations are negligible compared with density perturbations ( $\delta p \ll \delta\rho$ ) then potential will remain roughly constant – more specifically a variant called the Bardeen or comoving curvature $\zeta$ is constant

# Oscillator: Take Two

- Combine these to form the simple harmonic oscillator equation

$$\ddot{\Theta} + c_s^2 k^2 \Theta = -\frac{k^2}{3}\Psi - \ddot{\Phi}$$

- In a CDM dominated expansion $\dot{\Phi} = \dot{\Psi} = 0$. Also for photon domination $c_s^2 = 1/3$ so the oscillator equation becomes

$$\ddot{\Theta} + \ddot{\Psi} + c_s^2 k^2 (\Theta + \Psi) = 0$$

- Solution is just an offset version of the original

$$[\Theta + \Psi](\eta) = [\Theta + \Psi](0)\,\cos(ks)$$

- $\Theta + \Psi$ is also the observed temperature fluctuation since photons lose energy climbing out of gravitational potentials at recombination

# Effective Temperature

- Photons climb out of potential wells at last scattering
- Lose energy to gravitational redshifts
- Observed or effective temperature

$$\Theta + \Psi$$

- Effective temperature oscillates around zero with amplitude given by the initial conditions
- Note: initial conditions are set when the perturbation is outside of horizon, need inflation or other modification to matter-radiation FRW universe.
- GR says that initial temperature is given by initial potential

# Sachs-Wolfe Effect and the Magic 1/3

- A gravitational potential is a perturbation to the temporal coordinate [formally a gauge transformation]

$$\frac{\delta t}{t} = \Psi$$

- Convert this to a perturbation in the scale factor,

$$t = \int \frac{da}{aH} \propto \int \frac{da}{a\rho^{1/2}} \propto a^{3(1+w)/2}$$

where $w \equiv p/\rho$ so that during matter domination

$$\frac{\delta a}{a} = \frac{2}{3}\frac{\delta t}{t}$$

- CMB temperature is cooling as $T \propto a^{-1}$ so

$$\Theta + \Psi \equiv \frac{\delta T}{T} + \Psi = -\frac{\delta a}{a} + \Psi = \frac{1}{3}\Psi$$

# Baryon Loading

- Baryons add extra mass to the photon-baryon fluid
- Controlling parameter is the momentum density ratio:

$$R \equiv \frac{p_b + \rho_b}{p_\gamma + \rho_\gamma} \approx 30\Omega_b h^2 \left(\frac{a}{10^{-3}}\right)$$

of order unity at recombination

- Momentum density of the joint system is conserved

$$\begin{aligned}(\rho_\gamma + p_\gamma)v_\gamma + (\rho_b + p_b)v_b &\approx (p_\gamma + p_\gamma + \rho_b + \rho_\gamma)v_\gamma \\ &= (1 + R)(\rho_\gamma + p_\gamma)v_{\gamma b}\end{aligned}$$

where the controlling parameter is the momentum density ratio:


$$R \equiv \frac{p_b + \rho_b}{p_\gamma + \rho_\gamma} \approx 30\Omega_b h^2 \left(\frac{a}{10^{-3}}\right)$$

of order unity at recombination

# New Euler Equation

- Momentum density ratio enters as

$$[(1+R)v_{\gamma b}]^{\cdot} = k\Theta + (1+R)k\Psi$$

- Photon continuity remains the same

$$\dot{\Theta} = -\frac{k}{3}v_{\gamma b} - \dot{\Phi}$$

- Modification of oscillator equation

$$[(1+R)\dot{\Theta}]^{\cdot} + \frac{1}{3}k^2\Theta = -\frac{1}{3}k^2(1+R)\Psi - [(1+R)\dot{\Phi}]^{\cdot}$$

# Oscillator: Take Three

- Combine these to form the not-quite-so simple harmonic oscillator equation

$$c_s^2 \frac{d}{d\eta}(c_s^{-2}\dot{\Theta}) + c_s^2 k^2 \Theta = -\frac{k^2}{3}\Psi - c_s^2 \frac{d}{d\eta}(c_s^{-2}\dot{\Phi})$$

where $c_s^2 \equiv \dot{p}_{\gamma b}/\dot{\rho}_{\gamma b}$

$$c_s^2 = \frac{1}{3}\frac{1}{1+R}$$

- In a CDM dominated expansion $\dot{\Phi} = \dot{\Psi} = 0$ and the adiabatic approximation $\dot{R}/R \ll \omega = kc_s$

$$[\Theta + (1+R)\Psi](\eta) = [\Theta + (1+R)\Psi](0)\cos(ks)$$

# Baryon Peak Phenomenology

- Photon-baryon ratio enters in three ways
- Overall larger amplitude:

$$[\Theta + (1+R)\Psi](0) = \frac{1}{3}(1+3R)\Psi(0)$$

- Even-odd peak modulation of effective temperature

$$[\Theta + \Psi]_{\rm peaks} = [\pm(1+3R) - 3R]\,\frac{1}{3}\Psi(0)$$

$$[\Theta + \Psi]_1 - [\Theta + \Psi]_2 = [-6R]\frac{1}{3}\Psi(0)$$

- Shifting of the sound horizon down or $\ell_A$ up

$$\ell_A \propto \sqrt{1+R}$$

- Actual effects smaller since $R$ evolves

# Photon Baryon Ratio Evolution

- Oscillator equation has time evolving mass

$$c_s^2 \frac{d}{d\eta}(c_s^{-2}\dot{\Theta}) + c_s^2 k^2 \Theta = 0$$

- Effective mass is is $m_{\rm eff} = 3c_s^{-2} = (1+R)$
- Adiabatic invariant

$$\frac{E}{\omega} = \frac{1}{2} m_{\rm eff} \omega A^2 = \frac{1}{2} 3 c_s^{-2} k c_s A^2 \propto A^2 (1+R)^{1/2} = const.$$

- Amplitude of oscillation $A \propto (1+R)^{-1/4}$ decays adiabatically as the photon-baryon ratio changes

# Oscillator: Take Three and a Half

- The not-quite-so simple harmonic oscillator equation is a forced harmonic oscillator

$$c_s^2 \frac{d}{d\eta}(c_s^{-2}\dot{\Theta}) + c_s^2 k^2 \Theta = -\frac{k^2}{3}\Psi - c_s^2 \frac{d}{d\eta}(c_s^{-2}\Phi)$$

  changes in the gravitational potentials alter the form of the acoustic oscillations

- If the forcing term has a temporal structure that is related to the frequency of the oscillation, this becomes a driven harmonic oscillator

- Term involving $\Psi$ is the ordinary gravitational force

- Term involving $\Phi$ involves the $\dot{\Phi}$ term in the continuity equation as a (curvature) perturbation to the scale factor

# Potential Decay

- Matter-to-radiation ratio

$$\frac{\rho_m}{\rho_r} \approx 24\Omega_m h^2 \left(\frac{a}{10^{-3}}\right)$$

  of order unity at recombination in a low $\Omega_m$ universe

- Radiation is not stress free and so impedes the growth of structure

$$k^2\Phi = 4\pi G a^2 \rho_r \Delta_r$$

  $\Delta_r \sim 4\Theta$ oscillates around a constant value, $\rho_r \propto a^{-4}$ so the Netwonian curvature decays.

- General rule: potential decays if the dominant energy component has substantial stress fluctuations, i.e. below the generalized sound horizon or Jeans scale

# Radiation Driving

- Decay is timed precisely to drive the oscillator - close to fully coherent

$$
\begin{aligned}
[\Theta + \Psi](\eta) &= [\Theta + \Psi](0) + \Delta\Psi - \Delta\Phi \\
&= \frac{1}{3}\Psi(0) - 2\Psi(0) = \frac{5}{3}\Psi(0)
\end{aligned}
$$

- $5\times$ the amplitude of the Sachs-Wolfe effect!
- Coherent approximation is exact for a photon-baryon fluid but reality is reduced to $\sim 4\times$ because of neutrino contribution to radiation
- Actual initial conditions are $\Theta + \Psi = \Psi/2$ for radiation domination but comparison to matter dominated SW correct

# Damping

- Tight coupling equations assume a perfect fluid: no viscosity, no heat conduction
- Fluid imperfections are related to the mean free path of the photons in the baryons

$$\lambda_C = \dot{\tau}^{-1} \quad \text{where} \quad \dot{\tau} = n_e \sigma_T a$$

  is the conformal opacity to Thomson scattering
- Dissipation is related to the diffusion length: random walk approximation

$$\lambda_D = \sqrt{N} \lambda_C = \sqrt{\eta/\lambda_C}\, \lambda_C = \sqrt{\eta \lambda_C}$$

  the geometric mean between the horizon and mean free path
- $\lambda_D/\eta_* \sim$ few %, so expect the peaks :> 3 to be affected by dissipation

# Equations of Motion

- Continuity

$$\dot\Theta = -\frac{k}{3}v_\gamma - \dot\Phi\,, \quad \dot\delta_b = -kv_b - 3\dot\Phi$$

where the photon equation remains unchanged and the baryons follow number conservation with $\rho_b = m_b n_b$

- Euler

$$\begin{aligned}
\dot v_\gamma &= k(\Theta + \Psi) - \frac{k}{6}\pi_\gamma - \dot\tau(v_\gamma - v_b) \\
\dot v_b &= -\frac{\dot a}{a}v_b + k\Psi + \dot\tau(v_\gamma - v_b)/R
\end{aligned}$$

where the photons gain an anisotropic stress term $\pi_\gamma$ from radiation viscosity and a momentum exchange term with the baryons and are compensated by the opposite term in the baryon Euler equation

# Viscosity

- Viscosity is generated from radiation streaming from hot to cold regions
- Expect

$$\pi_\gamma \sim v_\gamma \frac{k}{\dot{\tau}}$$

generated by streaming, suppressed by scattering in a wavelength of the fluctuation. Radiative transfer says

$$\pi_\gamma \approx 2 A_v v_\gamma \frac{k}{\dot{\tau}}$$

where $A_v = 16/15$

$$\dot{v}_\gamma = k(\Theta + \Psi) - \frac{k}{3} A_v \frac{k}{\dot{\tau}} v_\gamma$$

# Oscillator: Penultimate Take

- Adiabatic approximation ( $\omega \gg \dot{a}/a$)

$$\dot{\Theta} \approx -\frac{k}{3} v_\gamma$$

- Oscillator equation contains a $\dot{\Theta}$ damping term

$$c_s^2 \frac{d}{d\eta}(c_s^{-2}\dot{\Theta}) + \frac{k^2 c_s^2}{\dot{\tau}} A_v \dot{\Theta} + k^2 c_s^2 \Theta = -\frac{k^2}{3}\Psi - c_s^2 \frac{d}{d\eta}(c_s^{-2}\dot{\Phi})$$

- Heat conduction term similar in that it is proportional to $v_\gamma$ and is suppressed by scattering $k/\dot{\tau}$. Expansion of Euler equations to leading order in $k\dot{\tau}$ gives

$$A_h = \frac{R^2}{1+R}$$

since the effects are only significant if the baryons are dynamically important

# Oscillator: Final Take

- Final oscillator equation

$$c_s^2 \frac{d}{d\eta}(c_s^{-2}\dot{\Theta}) + \frac{k^2 c_s^2}{\dot{\tau}}[A_v + A_h]\dot{\Theta} + k^2 c_s^2 \Theta = -\frac{k^2}{3}\Psi - c_s^2 \frac{d}{d\eta}(c_s^{-2}\dot{\Phi})$$

- Solve in the adiabatic approximation

$$\Theta \propto \exp(i \int \omega d\eta)$$

$$-\omega^2 + \frac{k^2 c_s^2}{\dot{\tau}}(A_v + A_h)i\omega + k^2 c_s^2 = 0 \qquad (1)$$

# Dispersion Relation

- Solve

$$\omega^2 = k^2 c_s^2 \left[1 + i\frac{\omega}{\dot\tau}(A_v + A_h)\right]$$
$$\omega = \pm k c_s \left[1 + \frac{i}{2}\frac{\omega}{\dot\tau}(A_v + A_h)\right]$$
$$= \pm k c_s \left[1 \pm \frac{i}{2}\frac{k c_s}{\dot\tau}(A_v + A_h)\right]$$

- Exponentiate

$$\exp(i\int \omega d\eta) = e^{\pm iks}\exp[-k^2\int d\eta \frac{1}{2}\frac{c_s^2}{\dot\tau}(A_v + A_h)]$$
$$= e^{\pm iks}\exp[-(k/k_D)^2] \quad (2)$$

- Damping is exponential under the scale $k_D$

# Diffusion Scale

- Diffusion wavenumber

$$k_D^{-2} = \int d\eta \frac{1}{\dot\tau} \frac{1}{6(1+R)} \left( \frac{16}{15} + \frac{R^2}{(1+R)} \right)$$

- Limiting forms

$$\lim_{R \to 0} k_D^{-2} = \frac{1}{6}\frac{16}{15} \int d\eta \frac{1}{\dot\tau}$$

$$\lim_{R \to \infty} k_D^{-2} = \frac{1}{6} \int d\eta \frac{1}{\dot\tau}$$

- Geometric mean between horizon and mean free path as expected from a random walk

$$\lambda_D = \frac{2\pi}{k_D} \sim \frac{2\pi}{\sqrt{6}} (\eta \dot\tau^{-1})^{1/2}$$

# Thomson Scattering

- Polarization state of radiation in direction $\hat{\mathbf{n}}$ described by the intensity matrix $\langle E_i(\hat{\mathbf{n}})E_j^*(\hat{\mathbf{n}})\rangle$, where $\mathbf{E}$ is the electric field vector and the brackets denote time averaging.
- Differential cross section

$$\frac{d\sigma}{d\Omega} = \frac{3}{8\pi}|\hat{\mathbf{E}}' \cdot \hat{\mathbf{E}}|^2 \sigma_T \,,$$

where $\sigma_T = 8\pi\alpha^2/3m_e$ is the Thomson cross section, $\hat{\mathbf{E}}'$ and $\hat{\mathbf{E}}$ denote the incoming and outgoing directions of the electric field or polarization vector.

- Summed over angle and incoming polarization

$$\sum_{i=1,2} \int d\hat{\mathbf{n}}' \frac{d\sigma}{d\Omega} = \sigma_T$$

# Polarization Generation

- Heuristic: incoming radiation shakes an electron in direction of electric field vector $\hat{\mathbf{E}}'$
- Radiates photon with polarization also in direction $\hat{\mathbf{E}}'$
- But photon cannot be longitudinally polarized so that scattering into $90^\circ$ can only pass one polarization
- Linearly polarized radiation like polarization by reflection
- Unlike reflection of sunlight, incoming radiation is nearly isotropic
- Missing linear polarization supplied by scattering from direction orthogonal to original incoming direction
- Only quadrupole anisotropy generates polarization by Thomson scattering

# Acoustic Polarization

- Break down of tight-coupling leads to quadrupole anisotropy of

$$\pi_\gamma \approx \frac{k}{\dot\tau} v_\gamma$$

- Scaling $k_D = (\dot\tau/\eta_*)^{1/2} \rightarrow \dot\tau = k_D^2 \eta_*$
- Know: $k_D s_* \approx k_D \eta_* \approx 10$
- So:

$$\pi_\gamma \approx \frac{k}{k_D} \frac{1}{10} v_\gamma$$

$$\Delta_P \approx \frac{\ell}{\ell_D} \frac{1}{10} \Delta_T$$

# Acoustic Polarization

- Gradient of velocity is along direction of wavevector, so polarization is pure $E$-mode
- Velocity is $90^\circ$ out of phase with temperature – turning points of oscillator are zero points of velocity:

$$\Theta + \Psi \propto \cos(ks); \quad v_\gamma \propto \sin(ks)$$

- Polarization peaks are at troughs of temperature power

# Cross Correlation

- Cross correlation of temperature and polarization

$$(\Theta + \Psi)(v_\gamma) \propto \cos(ks)\sin(ks) \propto \sin(2ks)$$

- Oscillation at twice the frequency
- Correlation: radial or tangential around hot spots
- Partial correlation: easier to measure if polarization data is noisy, harder to measure if polarization data is high $S/N$ or if bands do not resolve oscillations
- Good check for systematics and foregrounds
- Comparison of temperature and polarization is proof against features in initial conditions mimicking acoustic features

# CMB Normalization

- Normalization of potential, hence inflationary power spectrum, set by CMB observations, aka COBE or WMAP normalization
- Angular power spectrum:

$$C_\ell = \frac{4\pi\Delta_T^2(\ell/D_*)}{2\ell(\ell+1)} = \frac{2\pi}{\ell(\ell+1)}\Delta_T^2(\ell/D_*)$$

- $\ell(\ell+1)C_\ell/2\pi = \Delta_T^2$ is commonly used log power
- Sachs-Wolfe effect says $\Delta_T^2 = \Delta_\Phi^2/9$, $\Phi = \frac{3}{5}\zeta$ initial
- Observed number at recombination

$$\Delta_T^2 = \left(\frac{28\mu\mathrm{K}}{2.725\times 10^6\mu\mathrm{K}}\right)^2$$

$$\Delta_\Phi^2 \approx (3\times 10^{-5})^2$$

$$\Delta_\zeta^2 \approx (5\times 10^{-5})^2$$

# COBE vs WMAP Normalization

- Given that the temperature response to an inflationary initial perturbation is known for all $k$ through the Boltzmann solution of the acoustic physics, one can translate $\Delta_T^2$ to $\Delta_\zeta^2$ at the best measured $k \approx \ell/D_*$.
- The CMB normalization was first extracted from COBE at $\ell \sim 10$ or $k \sim H_0$. A low $\ell$ normalization point suffers from cosmic variance: only $2\ell + 1$ samples of a given $\ell$ mode.
- WMAP measures very precisely the first acoustic peak at $\ell \approx 200$. This is the current best place to normalize the spectrum ($k \sim 0.02$ Mpc$^{-1}$).
- To account for future improvements, WMAP chose $k = 0.05$ Mpc$^{-1}$ as the normalization point. Taking out the CMB transfer function $\Delta_\zeta^2(k = 0.05) = (5.07 \times 10^{-5})^2$ consistent with a scale invariant spectrum from $0.0002 - 0.05$ Mpc$^{-1}$

# Transfer Function

- Transfer function transfers the initial Newtonian curvature to its value today (linear response theory)

$$T(k) = \frac{\Phi(k, a=1)}{\Phi(k, a_{\rm init})} \frac{\Phi(k_{\rm norm}, a_{\rm init})}{\Phi(k_{\rm norm}, a=1)}$$

- Conservation of Bardeen curvature: Newtonian curvature is a constant when stress perturbations are negligible: above the horizon during radiation and dark energy domination, on all scales during matter domination
- When stress fluctuations dominate, perturbations are stabilized by the Jeans mechanism
- Hybrid Poisson equation: Newtonian curvature, comoving density perturbation $\Delta \equiv (\delta\rho/\rho)_{\rm com}$ implies $\Phi$ decays

$$(k^2 - 3K)\Phi = 4\pi G\rho\Delta \sim \eta^{-2}\Delta$$

# Transfer Function

- Freezing of $\Delta$ stops at $\eta_{\rm eq}$

$$\Phi \sim (k\eta_{\rm eq})^{-2}\Delta_H \sim (k\eta_{\rm eq})^{-2}\Phi_{\rm init}$$

- Transfer function has a $k^{-2}$ fall-off beyond $k_{\rm eq} \sim \eta_{\rm eq}^{-1}$

$$\eta_{\rm eq} = 15.7(\Omega_m h^2)^{-1}\left(\frac{T}{2.7K}\right)^2 \rm{Mpc}$$

- Small correction since growth with a smooth radiation component is logarithmic not frozen
- Transfer function is a direct output of an Einstein-Boltzmann code

# Fitting Function

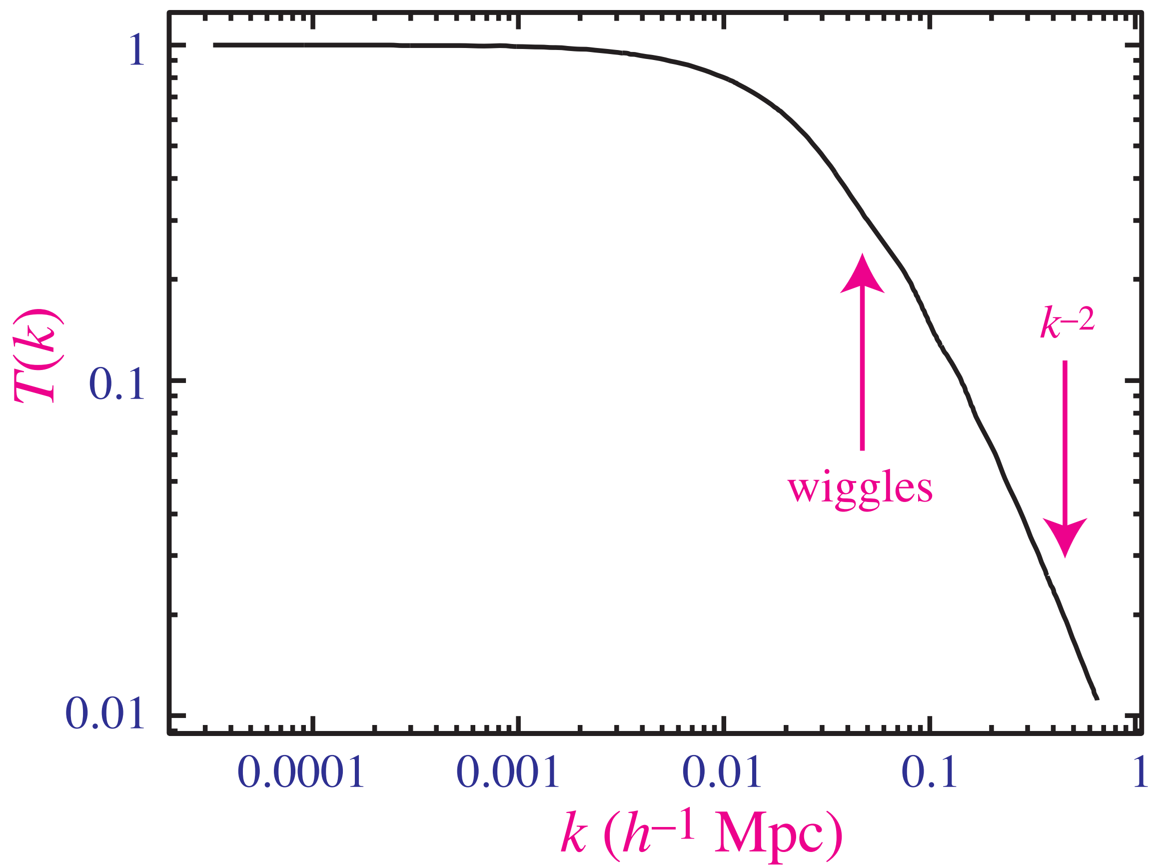
- Alternately accurate fitting formula exist, e.g. pure CDM form:

$$
\begin{aligned}
T(k(q)) &= \frac{L(q)}{L(q) + C(q)q^2} \\
L(q) &= \ln(e + 1.84q) \\
C(q) &= 14.4 + \frac{325}{1 + 60.5q^{1.11}} \\
q &= k/\Omega_m h^2 \mathrm{Mpc}^{-1} (T_{\rm CMB}/2.7K)^2
\end{aligned}
$$

- In $h$ Mpc$^{-1}$, the critical scale depends on $\Gamma \equiv \Omega_m h$ also known as the shape parameter

# Transfer Function

- Numerical calculation



# Baryon Wiggles

- Baryons caught up in the acoustic oscillations of the CMB and impart acoustic wiggles to the transfer function. Density enhancements are produced kinematically through the continuity equation $\delta_b \sim (k\eta)v_b$ and hence are out of phase with CMB temperature peaks
- Dissipation of the acoustic oscillations eliminates both the CMB and baryon perturbations – known as Silk damping for the baryons. This suppression and the general fact that baryons are caught up with photons was one of the main arguments for CDM
- Detected first (so far only) in the SDSS LRG survey.
- An excellent standard ruler for angular diameter distance $D_A(z)$ since it does not evolve in redshift in linear theory
- Radial extent of wiggles gives $H(z)$ (not yet seen in data)

# Massive Neutrinos

- Neutrino dark matter suffers similar effects and hence cannot be the main component of dark matter in the universe
- Relativistic stresses of a light neutrino slow the growth of structure
- Neutrino species with cosmological abundance contribute to matter as $\Omega_\nu h^2 = \sum m_\nu / 94\mathrm{eV}$, suppressing power as $\Delta P/P \approx -8\Omega_\nu/\Omega_m$
- Current data from SDSS galaxy survey and CMB indicate $\sum m_\nu < 1.7\mathrm{eV}$ (95% CL) and with Ly$\alpha$ forest $< 0.42$ eV.

# Growth Function

- Same physics applies to the dark energy dominated universe
- Under the dark energy sound horizon or Jeans scale, dark energy density frozen. Potential decays at the same rate for all scales

$$G(a) = \frac{\Phi(k_{\rm norm}, a)}{\Phi(k_{\rm norm}, a_{\rm init})} \qquad {}' \equiv \frac{d}{d\ln a}$$

- Continuity + Euler + Poisson

$$G'' + \left(1 - \frac{\rho''}{\rho'} + \frac{1}{2}\frac{\rho_c'}{\rho_c}\right) G' + \left(\frac{1}{2}\frac{\rho_c' + \rho'}{\rho_c} - \frac{\rho''}{\rho'}\right) G = 0$$

where $\rho$ is the Jeans unstable matter and $\rho_c$ is the critical density

# Dark Energy Growth Suppression

- Pressure growth suppression: $\delta \equiv \delta\rho_m/\rho_m \propto aG$

$$\frac{d^2G}{d\ln a^2} + \left[\frac{5}{2} - \frac{3}{2}w(z)\Omega_{DE}(z)\right]\frac{dG}{d\ln a} + \frac{3}{2}[1 - w(z)]\Omega_{DE}(z)G = 0\,,$$

where $w \equiv p_{DE}/\rho_{DE}$ and $\Omega_{DE} \equiv \rho_{DE}/(\rho_m + \rho_{DE})$ with initial conditions $G = 1$, $dG/d\ln a = 0$

- As $\Omega_{DE} \to 0$ $g =$const. is a solution. The other solution is the decaying mode, elimated by initial conditions
- As $\Omega_{DE} \to 1$ $g \propto a^{-1}$ is a solution. Corresponds to a frozen density field.

# Power Spectrum Normalization

- Present (or matter dominated) vs inflationary initial conditions (normalized by CMB):

$$\Delta^2_\Phi(k,a) \approx \frac{9}{25}\Delta^2_{\zeta_i}(k_{\rm norm})G^2(a)T^2(k)\left(\frac{k}{k_{\rm norm}}\right)^{n-1}$$

- Density field

$$\begin{aligned}
k^2\Phi &= 4\pi G a^2 \Delta\rho_m \\
&= \frac{3}{2}H_0^2\Omega_m\frac{\Delta\rho_m}{\rho_m}\frac{1}{a} \\
\Delta^2_\Phi &= \frac{9}{4}\left(\frac{H_0}{k}\right)^4\Omega_m^2 a^{-2}\Delta^2_m \\
\Delta^2_m &= \frac{4}{25}\Delta^2_{\zeta_i}(k_{\rm norm})\Omega_m^{-2}a^2G^2(a)T^2(k)\left(\frac{k}{k_{\rm norm}}\right)^{n-1}\left(\frac{k}{H_0}\right)^4
\end{aligned}$$

# Antiquated Normalization Conventions

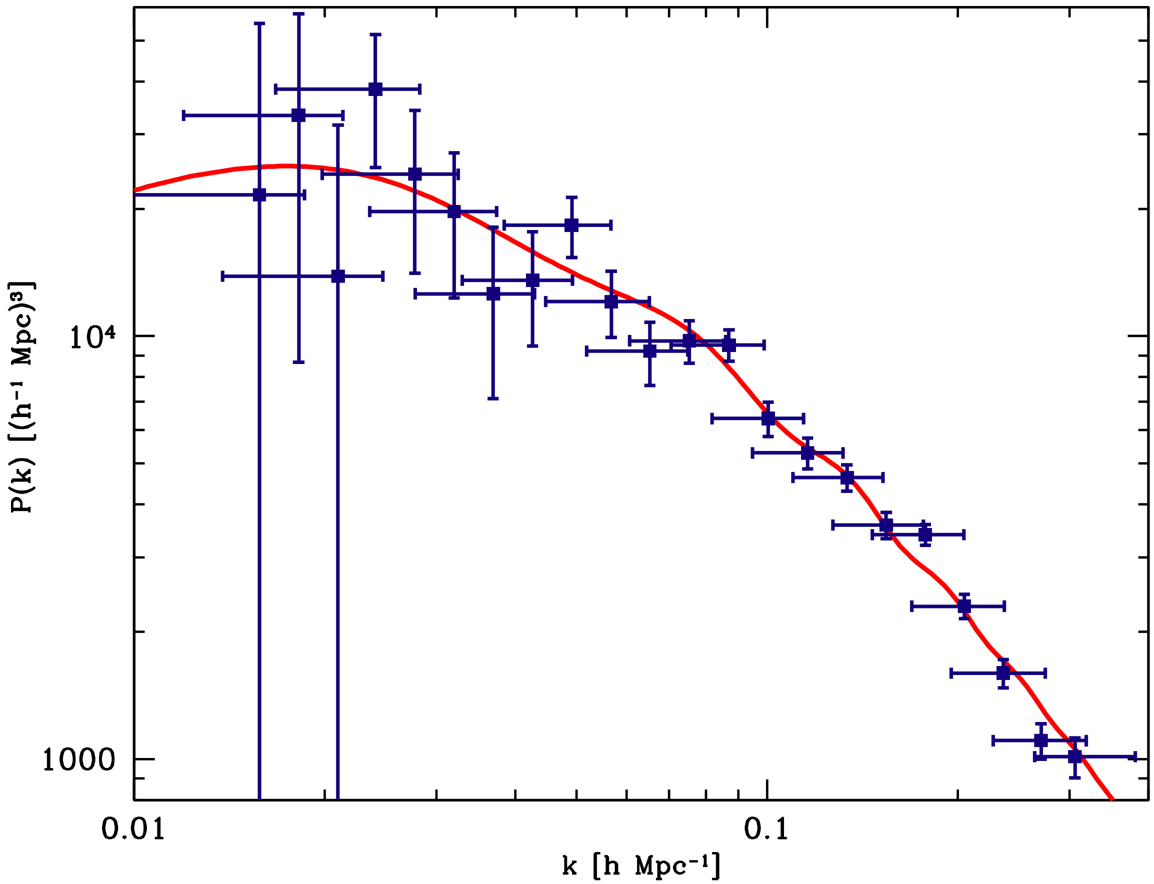
- Current density field on the horizon scale $k = H_0$

$$\delta_H^2 = \frac{4}{25}\Delta^2_{\zeta_i}(k_{\rm norm})\Omega_m^{-2}a^2G^2(a) = (2G(1)/\Omega_m \times 10^{-5})$$

- $\sigma_8$, RMS of density field filtered by tophat of $8h^{-1}$Mpc

# Power Spectrum

- SDSS data




- Power spectrum defines large scale structure observables: galaxy clustering, velocity field, Ly$\alpha$ forest clustering, cosmic shear

# Velocity field

- Continuity gives the velocity from the density field as

$$v = -\dot{\Delta}/k = -\frac{aH}{k}\frac{d\Delta}{d\ln a}$$
$$= -\frac{aH}{k}\Delta\frac{d\ln(aG)}{d\ln a}$$

- In a $\Lambda$CDM model or open model $d\ln(aG)/d\ln a \approx \Omega_m^{0.6}$
- Measuring both the density field and the velocity field (through distance determination and redshift) allows a measurement of $\Omega_m$
- Practically one measures $\beta = \Omega_m^{0.6}/b$ where $b$ is a bias factor for the tracer of the density field, i.e. with galaxy numbers $\delta n/n = b\Delta$
- Can also measure this factor from the redshift space power spectrum - the Kaiser effect where clustering in the radial direction is apparently enhanced by gravitational infall

# Redshift Space Power Spectrum

- Kaiser effect is separable from the real space clustering if one measures modes parallel and transverse to the line of sight. Redshift space distortions only modify the former
- 2D power spectrum in "s" or redshift space

$$P_s(k_\perp, k_\parallel) = \left[1 + \beta\left(\frac{k_\parallel}{k}\right)^2\right]^2 b^2 P(k)$$

where $k^2 = k_\parallel^2 + k_\perp^2$ and $k_\perp$ is a 2D vector transverse to the line of sight

# Power Spectrum Errors

- The precision with which the power spectrum can be measured is ultimately limited by sample variance from having a finite survey volume $V = L^3$. This is basically a mode counting argument. The errors on the power spectrum are given by

$$\left(\frac{\Delta P_s}{P_s}\right)^2 = \frac{2}{N_k}$$

where $N_k$ is the number of modes in a range of $\Delta k_\perp$, $\Delta k_\parallel$. This is determined by the $k$-space volume and the fundamental mode of the box $k_0 = 2\pi/L$ which sets the cell size in the volume

$$\left(\frac{\Delta P_s}{P_s}\right)^2 = \frac{2}{\frac{V}{(2\pi)^3} 2\pi k_\perp \Delta k_\perp \Delta k_\parallel}$$

# Lyman-$\alpha$ Forest

- QSO spectra absorbed by neutral hydrogen through the Ly$\alpha$ transition.
- The optical depth to absorption is (with $ds$ in physical scale)

$$\tau(\nu) = \int ds x_{\rm HI} n_b \sigma_\alpha \sim \int ds x_{\rm HI} n_b \Gamma \phi(\nu) \lambda^2$$

  where $x_{\rm HI}$ is the neutral fraction, $\Gamma = 6.25 \times 10^8 s^{-1}$ is the transition rate and $\lambda = 1216$A is the Ly$\alpha$ wavelength and $\phi(\nu)$ is the Lorentz profile. For radiation at a given emitted frequency $\nu_0$ above the transition, it will redshift through the transition
- Resonant transition: lack of complete absorption, known as the lack of a Gunn-Peterson trough indicates that the universe is nearly fully ionized $x_{\rm HI} \ll 1$ out to the highest redshift quasar $z \sim 6$; indications that this is near the end of the reionization epoch

# Lyman-$\alpha$ Forest

- In ionization equilibrium, the Gunn-Peterson optical depth is a tracer of the underlying baryon density which itself is a tracer of the dark matter $\tau_{GP} \propto \rho_b^2 T^{-0.7}$ with $T(\rho_b)$.

$$\frac{d(1 - x_{\rm HI})}{dt} = -x_{\rm HI} \int d\nu \frac{4\pi J_\nu}{h\nu}\sigma_\nu + (1 - x_{\rm HI})^2 n_b R$$

  where $\sigma_\nu$ is the photoionization cross section (sharp edge at threshold and falling in frequency means $J_\nu \approx J_{21}$) and $R \propto T^{-0.7}$ is the recombination coefficient.

- Given an equation of state from simulations of $p \propto \rho^\gamma$

$$x_{\rm HI} \propto \frac{\rho_b R}{J_{21}} \propto \frac{\rho_b T^{-0.7}}{J_{21}}, \quad \tau_{GP} \propto \frac{\rho_b^{2-0.7(\gamma-1)}}{J_{\rm HI}}$$

- Clustering in the Ly$\alpha$ forest reflects the underlying power spectrum modulo an overall ionization intensity $J_{21}$

# Gravitational Lensing

- Gravitational potentials along the line of sight $\hat{\mathbf{n}}$ to some source at comoving distance $D_s$ lens the images according to (flat universe)

$$\phi(\hat{\mathbf{n}}) = 2\int dD \frac{D_s - D}{D D_s} \Phi(D\hat{\mathbf{n}}, \eta(D))$$

  remapping image positions as

$$\hat{\mathbf{n}}^I = \hat{\mathbf{n}}^S + \nabla_{\hat{\mathbf{n}}}\phi(\hat{\mathbf{n}})$$

- Since absolute source position is unknown, use image distortion defined by the Jacobian matrix

$$\frac{\partial n_i^I}{\partial n_j^S} = \delta_{ij} + \psi_{ij}$$

# Weak Lensing

- Small image distortions described by the convergence $\kappa$ and shear components $(\gamma_1, \gamma_2)$

$$\psi_{ij} = \begin{pmatrix} \kappa - \gamma_1 & -\gamma_2 \\ -\gamma_2 & \kappa + \gamma_1 \end{pmatrix}$$

where $\nabla_{\hat{\mathbf{n}}} = D\nabla$ and

$$\psi_{ij} = 2 \int dD \frac{D(D_s - D)}{D_s} \nabla_i \nabla_j \Phi(D\hat{\mathbf{n}}, \eta(D))$$

- In particular, through the Poisson equation the convergence (measured from shear) is simply the projected mass

$$\kappa = \frac{3}{2}\Omega_m H_0^2 \int dD \frac{D(D_s - D)}{D_s} \frac{\Delta(D\hat{\mathbf{n}}, \eta(D))}{a}$$