Set 9:
CMB and Large Scale Structure
CMB Temperature Anisotropy

- Planck 2015 map of the temperature anisotropy (first discovered by COBE) from recombination:
CMB Temperature Anisotropy

- Power spectrum shows characteristic scales where the intensity of variations peak - reveals geometry and contents of the universe:
# CMB Parameter Inferences

- Spectrum constrains the matter-energy contents of the universe

<table>
<thead>
<tr>
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</thead>
<tbody>
<tr>
<td>$\Omega_b h^2$</td>
<td>0.02222 ± 0.00023</td>
<td>0.02228 ± 0.00025</td>
<td>0.0240 ± 0.0013</td>
<td>0.02225 ± 0.00016</td>
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<tr>
<td>$\Omega_c h^2$</td>
<td>0.1197 ± 0.0022</td>
<td>0.1187 ± 0.0021</td>
<td>0.1150$^{+0.0048}$,$^{-0.0055}$</td>
<td>0.1198 ± 0.0015</td>
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<tr>
<td>$100\theta_{MC}$</td>
<td>1.04085 ± 0.00047</td>
<td>1.04094 ± 0.00051</td>
<td>1.03988 ± 0.00094</td>
<td>1.04077 ± 0.00032</td>
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<tr>
<td>$\tau$</td>
<td>0.078 ± 0.019</td>
<td>0.053 ± 0.019</td>
<td>0.059$^{+0.022}$,$^{-0.019}$</td>
<td>0.079 ± 0.017</td>
</tr>
<tr>
<td>ln($10^{10} A_s$)</td>
<td>3.089 ± 0.036</td>
<td>3.031 ± 0.041</td>
<td>3.066$^{+0.046}$,$^{-0.041}$</td>
<td>3.094 ± 0.034</td>
</tr>
<tr>
<td>$n_s$</td>
<td>0.9655 ± 0.0062</td>
<td>0.965 ± 0.012</td>
<td>0.973 ± 0.016</td>
<td>0.9645 ± 0.0049</td>
</tr>
<tr>
<td>$H_0$</td>
<td>67.31 ± 0.96</td>
<td>67.73 ± 0.92</td>
<td>70.2 ± 3.0</td>
<td>67.27 ± 0.66</td>
</tr>
<tr>
<td>$\Omega_m$</td>
<td>0.315 ± 0.013</td>
<td>0.300 ± 0.012</td>
<td>0.286$^{+0.027}$,$^{-0.038}$</td>
<td>0.3156 ± 0.0091</td>
</tr>
<tr>
<td>$\sigma_8$</td>
<td>0.829 ± 0.014</td>
<td>0.802 ± 0.018</td>
<td>0.796 ± 0.024</td>
<td>0.831 ± 0.013</td>
</tr>
<tr>
<td>$10^9 A_s e^{-2\tau}$</td>
<td>1.880 ± 0.014</td>
<td>1.865 ± 0.019</td>
<td>1.907 ± 0.027</td>
<td>1.882 ± 0.012</td>
</tr>
</tbody>
</table>
Galaxy Redshift Surveys

- Galaxy redshift surveys (e.g. 2dF and SDSS) measure the three-dimensional distribution of galaxies today:

![Diagram showing the distribution of 6,476 LRGs (black) and 32,417 main galaxies (green/grey) within 1.25° of the Equatorial plane. The solid circles indicate the boundaries of NEAR, MID, and FAR subsamples. The "safe13" main galaxy sample is more local, extending out only to 600 h⁻¹ Mpc (dashed circle).]
Galaxy Power Spectrum

- SDSS LRG and Main power spectrum:

![Graph of power spectrum][1]

**FIG. 4:** Measured power spectra for the full LRG and main galaxy samples. Errors are uncorrelated and full window functions are shown in Figure 5. The solid curves correspond to the linear theory ΛCDM fits to WMAP3 alone from Table 5 of [7], normalized to galaxy bias $b = 1.9$ (top) and $b = 1.1$ (bottom) relative to the $\Omega = 0$ matter power. The dashed curves include the nonlinear correction of [29] for $A = 1.4$, with $Q_{nl} = 30$ for the LRGs and $Q_{nl} = 4.6$ for the main galaxies; see equation (4). The onset of nonlinear corrections is clearly visible for $k \sim 0.09 h/Mpc$ (vertical line).

Before using these measurements to constrain cosmological models, one faces important issues regarding their interpretation, related to evolution, nonlinearities and systematics.

**B. Clustering evolution**

The standard theoretical expectation is for matter clustering to grow over time and for bias (the relative clustering of galaxies and matter) to decrease over time [78–80] for a given class of galaxies. Bias is also...
Structure Formation

- Small perturbations from inflation over the course of the 14Gyr life of the universe are gravitationally enhanced into all of the structure seen today.

- Cosmic microwave background shows a snapshot at a few hundred thousand years old at recombination.

- Discovery in 1992 of cosmic microwave background anisotropy provided the observational breakthrough - convincing support for adiabatic initial density fluctuations of amplitude $10^{-5}$.

- Combine with galaxy clustering - large scale structure seen in galaxy surveys - right amplitude given cold dark matter.

- Following notes are at a slightly more advanced level than the book and are provided here for completeness.
Angular Power Spectrum

- Angular distribution of radiation is essentially the 3D temperature field projected onto a shell at the distance from the observer to recombination: called the last scattering surface.

- Take the radiation distribution at last scattering to also be described by an isotropic temperature fluctuation field $\Theta(x)$ and recombination to be instantaneous.

\[ \Theta(\hat{n}) = \int dD \Theta(x) \delta(D - D_*) \]

where $D$ is the comoving distance and $D_*$ denotes recombination.

- Describe the temperature field by its Fourier moments.

\[ \Theta(x) = \int \frac{d^3 k}{(2\pi)^3} \Theta(k) e^{ik\cdot x} \]
Angular Power Spectrum

- Power spectrum

\[ \langle \Theta(k)^* \Theta(k') \rangle = (2\pi)^3 \delta(k - k') P_T(k) \]

\[ \Delta_T^2 = k^3 P_T / 2\pi^2 \]

- Temperature field

\[ \Theta(\hat{n}) = \int \frac{d^3 k}{(2\pi)^3} \Theta(k) e^{i k \cdot D^* \cdot \hat{n}} \]

- Multipole moments \( \Theta(\hat{n}) = \sum_{\ell m} \Theta_{\ell m} Y_{\ell m} \)

- Expand out plane wave in spherical coordinates

\[ e^{i k \cdot D^* \cdot \hat{n}} = 4\pi \sum_{\ell m} i^\ell j_\ell(k D^*) Y_{\ell m}^*(k) Y_{\ell m}(\hat{n}) \]
Angular Power Spectrum

- Power spectrum

\[ \Theta_{\ell m} = \int \frac{d^3 k}{(2\pi)^3} \Theta(k) 4\pi i^{\ell} j_\ell(k D_*) Y_{\ell m}(k) \]

\[ \langle \Theta^*_{\ell m} \Theta_{\ell' m'} \rangle = \int \frac{d^3 k}{(2\pi)^3} (4\pi)^2 (i)^{\ell-\ell'} j_\ell(k D_*) j_{\ell'}(k D_*) Y^*_{\ell m}(k) Y_{\ell' m'}(k) P_T(k) \]

\[ = \delta_{\ell \ell'} \delta_{mm'} 4\pi \int d \ln k \ j_\ell^2(k D_*) \Delta_T^2(k) \]

with \( \int_0^\infty j_\ell^2(x) d \ln x = 1/(2\ell(\ell + 1)) \), slowly varying \( \Delta_T^2 \)

- Angular power spectrum:

\[ C_\ell = \frac{4\pi \Delta_T^2(\ell/D_*)}{2\ell(\ell + 1)} = \frac{2\pi}{\ell(\ell + 1)} \Delta_T^2(\ell/D_*) \]
Thomson Scattering

- Thomson scattering of photons off of free electrons is the most important CMB process with a cross section (averaged over polarization states) of

\[ \sigma_T = \frac{8\pi\alpha^2}{3m_e^2} = 6.65 \times 10^{-25}\text{cm}^2 \]

- Density of free electrons in a fully ionized \( x_e = 1 \) universe

\[ n_e = (1 - Y_p/2)x_e n_b \approx 10^{-5}\Omega_b h^2(1+z)^3\text{cm}^{-3}, \]

where \( Y_p \approx 0.24 \) is the Helium mass fraction, creates a high (comoving) Thomson opacity

\[ \dot{\tau} \equiv n_e \sigma_T a \]

where dots are conformal time \( \eta \equiv \int dt/a \) derivatives and \( \tau \) is the optical depth.
Tight Coupling Approximation

- Near recombination $z \approx 10^3$ and $\Omega_b h^2 \approx 0.02$, the (comoving) mean free path of a photon

$$\lambda_C \equiv \frac{1}{\tau} \sim 2.5\text{Mpc}$$

small by cosmological standards!

- On scales $\lambda \gg \lambda_C$ photons are tightly coupled to the electrons by Thomson scattering which in turn are tightly coupled to the baryons by Coulomb interactions

- Specifically, their bulk velocities are defined by a single fluid velocity $v_\gamma = v_b$ and the photons carry no anisotropy in the rest frame of the baryons

- → No heat conduction or viscosity (anisotropic stress) in fluid
Zeroth Order Approximation

- Momentum density of a fluid is \((\rho + p)v\), where \(p\) is the pressure.
- Neglect the momentum density of the baryons.

\[ R \equiv \frac{(\rho_b + p_b)v_b}{(\rho_\gamma + p_\gamma)v_\gamma} = \frac{\rho_b + p_b}{\rho_\gamma + p_\gamma} = \frac{3\rho_b}{4\rho_\gamma} \]

\[ \approx 0.6 \left( \frac{\Omega_b h^2}{0.02} \right) \left( \frac{a}{10^{-3}} \right) \]

Since \(\rho_\gamma \propto T^4\) is fixed by the CMB temperature \(T = 2.73 (1 + z) K\) – OK substantially before recombination.

- Neglect radiation in the expansion.

\[ \frac{\rho_m}{\rho_r} = 3.6 \left( \frac{\Omega_m h^2}{0.15} \right) \left( \frac{a}{10^{-3}} \right) \]

- Neglect gravity.
Fluid Equations

- Density $\rho_\gamma \propto T^4$ so define temperature fluctuation $\Theta$

  \[ \delta_\gamma = 4 \frac{\delta T}{T} \equiv 4\Theta \]

- Real space continuity equation transformed to Fourier space

  \[ \nabla \rightarrow ik \]

  \[ \dot{\delta}_\gamma = -(1 + w_\gamma)kv_\gamma \]

  \[ \dot{\Theta} = -\frac{1}{3}kv_\gamma \]

- Euler equation (neglecting gravity)
\[
\dot{v}_\gamma = -(1 - 3w_\gamma) \frac{\dot{a}}{a} v + \frac{k c_s^2}{1 + w_\gamma} \delta_\gamma \\
\dot{v}_\gamma = k c_s^2 \frac{3}{4} \delta_\gamma = 3c_s^2 k \Theta
\]
Oscillator: Take One

- Combine these to form the **simple harmonic oscillator** equation

\[
\ddot{\Theta} + c_s^2 k^2 \Theta = 0
\]

where the sound speed is adiabatic

\[
c_s^2 = \frac{\delta p}{\delta \rho} = \frac{\dot{p}_\gamma}{\dot{\rho}_\gamma}
\]

here \(c_s^2 = 1/3\) since we are photon-dominated

- General solution:

\[
\Theta(\eta) = \Theta(0) \cos(ks) + \frac{\dot{\Theta}(0)}{kc_s} \sin(ks)
\]

where the **sound horizon** is defined as \(s \equiv \int c_s d\eta\)
Harmonic Extrema

• All modes are frozen in at recombination (denoted with a subscript *) yielding temperature perturbations of different amplitude for different modes. For the adiabatic (curvature mode) $\dot{\Theta}(0) = 0$

\[ \Theta(\eta_*) = \Theta(0) \cos(k_s*) \]

• Modes caught in the extrema of their oscillation will have enhanced fluctuations

\[ k_n s_* = n\pi \]

yielding a fundamental scale or frequency, related to the inverse sound horizon

\[ k_A = \pi / s_* \]

and a harmonic relationship to the other extrema as $1 : 2 : 3 \ldots$
Peak Location

- The fundamental physical scale is translated into a fundamental angular scale by simple projection according to the angular diameter distance $D_A$

\[
\theta_A = \frac{\lambda_A}{D_A} \\
\ell_A = k_A D_A
\]

- In a flat universe, the distance is simply $D_A = D \equiv \eta_0 - \eta_* \approx \eta_0$, the horizon distance, and $k_A = \pi / s_* = \sqrt{3\pi / \eta_*}$ so

\[
\theta_A \approx \frac{\eta_*}{\eta_0}
\]

- In a matter-dominated universe $\eta \propto a^{1/2}$ so $\theta_A \approx 1/30 \approx 2^\circ$ or

\[
\ell_A \approx 200
\]
Curvature

- In a curved universe, the apparent or angular diameter distance is no longer the conformal distance \( D_A = R \sin(D/R) \neq D \).
- Objects in a closed universe are further than they appear! gravitational lensing of the background...
- Curvature scale of the universe must be substantially larger than current horizon.
- Flat universe indicates critical density and implies missing energy given local measures of the matter density “dark energy”.
- \( D \) also depends on dark energy density \( \Omega_{DE} \) and equation of state \( w = \frac{p_{DE}}{\rho_{DE}} \).
- Expansion rate at recombination or matter-radiation ratio enters into calculation of \( k_A \).
Doppler Effect

- **Bulk motion** of fluid changes the observed temperature via Doppler shifts

\[
\left( \frac{\Delta T}{T} \right)_{\text{dop}} = \hat{n} \cdot \mathbf{v}_\gamma
\]

- Averaged over directions

\[
\left( \frac{\Delta T}{T} \right)_{\text{rms}} = \frac{\mathbf{v}_\gamma}{\sqrt{3}}
\]

- Acoustic solution

\[
\frac{\mathbf{v}_\gamma}{\sqrt{3}} = -\frac{\sqrt{3}}{k} \dot{\Theta} = \frac{\sqrt{3}}{k} k c_s \Theta(0) \sin(ks)
\]

\[
= \Theta(0) \sin(ks)
\]
• Doppler effect for the photon dominated system is of equal amplitude and $\pi/2$ out of phase: extrema of temperature are turning points of velocity

• Effects add in quadrature:

$$\left(\frac{\Delta T}{T}\right)^2 = \Theta^2(0)[\cos^2(ks) + \sin^2(ks)] = \Theta^2(0)$$

• No peaks in $k$ spectrum! However the Doppler effect carries an angular dependence that changes its projection on the sky

$$\hat{n} \cdot \mathbf{v}_\gamma \propto \hat{n} \cdot \hat{k}$$

• Coordinates where $\hat{z} \parallel \hat{k}$

$$Y_{10}Y_{\ell 0} \rightarrow Y_{\ell \pm 1 0}$$

recoupling $j'_\ell Y_{\ell 0}$: no peaks in Doppler effect
Restoring Gravity

- Take a simple photon dominated system with gravity
- Continuity altered since a gravitational potential represents a stretching of the spatial fabric that dilutes number densities – formally a spatial curvature perturbation
- Think of this as a perturbation to the scale factor $a \rightarrow a(1 + \Phi)$ so that the cosmogical redshift is generalized to

$$\frac{\dot{a}}{a} \rightarrow \frac{\dot{a}}{a} + \dot{\Phi}$$

so that the continuity equation becomes

$$\dot{\Theta} = -\frac{1}{3}kv_\gamma - \dot{\Phi}$$
Restoring Gravity

- **Gravitational force** in momentum conservation $F = -m \nabla \Psi$ generalized to momentum density modifies the Euler equation to

$$\dot{v}_\gamma = k(\Theta + \Psi)$$

- General relativity says that $\Phi$ and $\Psi$ are the relativistic analogues of the Newtonian potential and that $\Phi \approx -\Psi$.

- In our matter-dominated approximation, $\Phi$ represents matter density fluctuations through the cosmological Poisson equation

$$k^2 \Phi = 4\pi G a^2 \rho_m \Delta_m$$

where the difference comes from the use of comoving coordinates for $k$ ($a^2$ factor), the removal of the background density into the background expansion ($\rho \Delta_m$) and finally a coordinate subtlety that enters into the definition of $\Delta_m$. 
Constant Potentials

- In the matter dominated epoch potentials are constant because infall generates velocities as \( v_m \sim k\eta\Psi \)

- Velocity divergence generates density perturbations as
  \( \Delta_m \sim -k\eta v_m \sim -(k\eta)^2\Psi \)

- And density perturbations generate potential fluctuations as
  \( \Phi \sim \Delta_m/(k\eta)^2 \sim -\Psi \), keeping them constant. Note that because of the expansion, density perturbations must grow to keep potentials constant.

- Here we have used the Friedman equation \( H^2 = 8\pi G\rho_m/3 \) and \( \eta = \int d\ln a/(aH) \sim 1/(aH) \)

- More generally, if stress perturbations are negligible compared with density perturbations (\( \delta p \ll \delta \rho \)) then potential will remain roughly constant – more specifically a variant called the Bardeen or comoving curvature \( R \) is constant
Oscillator: Take Two

- Combine these to form the simple harmonic oscillator equation

\[ \ddot{\Theta} + c_s^2 k^2 \Theta = -\frac{k^2}{3} \Psi - \ddot{\Phi} \]

- In a CDM dominated expansion \( \dot{\Phi} = \dot{\Psi} = 0 \). Also for photon domination \( c_s^2 = 1/3 \) so the oscillator equation becomes

\[ \ddot{\Theta} + \ddot{\Psi} + c_s^2 k^2 (\Theta + \Psi) = 0 \]

- Solution is just an offset version of the original

\[ [\Theta + \Psi](\eta) = [\Theta + \Psi](0) \cos(ks) \]

- \( \Theta + \Psi \) is also the observed temperature fluctuation since photons lose energy climbing out of gravitational potentials at recombination
Effective Temperature

- Photons climb out of potential wells at last scattering
- Lose energy to gravitational redshifts
- Observed or effective temperature

$$\Theta + \Psi$$

- Effective temperature oscillates around zero with amplitude given by the initial conditions
- Note: initial conditions are set when the perturbation is outside of horizon, need inflation or other modification to matter-radiation FRW universe.
- GR says that initial temperature is given by initial potential
Sachs-Wolfe Effect and the Magic 1/3

- A gravitational potential is a perturbation to the temporal coordinate [formally a gauge transformation]

\[
\frac{\delta t}{t} = \Psi
\]

- Convert this to a perturbation in the scale factor,

\[
t = \int \frac{da}{aH} \propto \int \frac{da}{a \rho^{1/2}} \propto a^{3(1+w)/2}
\]

where \(w \equiv p/\rho\) so that during matter domination

\[
\frac{\delta a}{a} = \frac{2}{3} \frac{\delta t}{t}
\]

- CMB temperature is cooling as \(T \propto a^{-1}\) so

\[
\Theta + \Psi \equiv \frac{\delta T}{T} + \Psi = -\frac{\delta a}{a} + \Psi = \frac{1}{3} \Psi
\]
Baryon Loading

- Baryons add extra mass to the photon-baryon fluid
- Controlling parameter is the momentum density ratio:

\[
R \equiv \frac{p_b + \rho_b}{p_\gamma + \rho_\gamma} \approx 30\Omega_b^2 h^2 \left( \frac{a}{10^{-3}} \right)
\]

of order unity at recombination

- Momentum density of the joint system is conserved

\[
(p_\gamma + p_\gamma)v_\gamma + (\rho_b + p_b)v_b \approx (p_\gamma + p_\gamma + \rho_b + \rho_\gamma)v_\gamma
\]

\[
= (1 + R)(\rho_\gamma + p_\gamma)v_\gamma b
\]

where the controlling parameter is the momentum density ratio:

\[
R \equiv \frac{p_b + \rho_b}{p_\gamma + \rho_\gamma} \approx 30\Omega_b^2 h^2 \left( \frac{a}{10^{-3}} \right)
\]

of order unity at recombination
New Euler Equation

- Momentum density ratio enters as

\[ [(1 + R)v_{\gamma b}] \cdot = k\Theta + (1 + R)k\Psi \]

- Photon continuity remains the same

\[ \dot{\Theta} = -\frac{k}{3}v_{\gamma b} - \dot{\Phi} \]

- Modification of oscillator equation

\[ [(1 + R)\dot{\Theta}] \cdot + \frac{1}{3}k^2\Theta = -\frac{1}{3}k^2(1 + R)\Psi - [(1 + R)\dot{\Phi}] \cdot \]
Oscillator: Take Three

- Combine these to form the not-quite-so simple harmonic oscillator equation

\[
c_s^2 \frac{d}{d\eta}(c_s^{-2} \dot{\Theta}) + c_s^2 k^2 \Theta = -\frac{k^2}{3} \Psi - c_s^2 \frac{d}{d\eta}(c_s^{-2} \dot{\Phi})
\]

where \(c_s^2 \equiv \frac{\dot{p}_{\gamma b}}{\dot{\rho}_{\gamma b}}\)

\[
c_s^2 = \frac{1}{3} \frac{1}{1 + R}
\]

- In a CDM dominated expansion \(\dot{\Phi} = \dot{\Psi} = 0\) and the adiabatic approximation \(\dot{R}/R \ll \omega = kc_s\)

\[
[\Theta + (1 + R)\Psi](\eta) = [\Theta + (1 + R)\Psi](0) \cos(k s)
\]
Baryon Peak Phenomenology

- Photon-baryon ratio enters in **three ways**
- Overall larger **amplitude**:

\[
[\Theta + (1 + R)\Psi](0) = \frac{1}{3}(1 + 3R)\Psi(0)
\]

- Even-odd peak **modulation** of effective temperature

\[
[\Theta + \Psi]_{\text{peaks}} = \pm(1 + 3R) - 3R \frac{1}{3} \Psi(0)
\]

\[
[\Theta + \Psi]_1 - [\Theta + \Psi]_2 = [-6R] \frac{1}{3} \Psi(0)
\]

- Shifting of the **sound horizon** down or \( \ell_A \) up

\[
\ell_A \propto \sqrt{1 + R}
\]

- Actual effects **smaller** since \( R \) evolves
Phonon Baryon Ratio Evolution

- Oscillator equation has time *evolving mass*
  \[ c_s^2 \frac{d}{d\eta} \left( c_s^{-2} \dot{\Theta} \right) + c_s^2 k^2 \Theta = 0 \]

- Effective mass is is \( m_{\text{eff}} = 3c_s^{-2} = (1 + R) \)

- Adiabatic invariant
  \[ \frac{E}{\omega} = \frac{1}{2} m_{\text{eff}} \omega A^2 = \frac{1}{2} 3c_s^{-2} k c_s A^2 \propto A^2 (1 + R)^{1/2} = \text{const.} \]

- Amplitude of oscillation \( A \propto (1 + R)^{-1/4} \) decays adiabatically as the photon-baryon ratio changes
Oscillator: Take Three and a Half

- The not-quite-so simple harmonic oscillator equation is a forced harmonic oscillator

\[ c_s^2 \frac{d}{d\eta} (c_s^{-2} \dot{\Theta}) + c_s^2 k^2 \Theta = -\frac{k^2}{3} \Psi - c_s^2 \frac{d}{d\eta} (c_s^{-2} \Phi) \]

changes in the gravitational potentials alter the form of the acoustic oscillations

- If the forcing term has a temporal structure that is related to the frequency of the oscillation, this becomes a driven harmonic oscillator

- Term involving \( \Psi \) is the ordinary gravitational force

- Term involving \( \Phi \) involves the \( \dot{\Phi} \) term in the continuity equation as a (curvature) perturbation to the scale factor
Potential Decay

- Matter-to-radiation ratio

\[ \frac{\rho_m}{\rho_r} \approx 24 \Omega_m h^2 \left( \frac{a}{10^{-3}} \right) \]

of order unity at recombination in a low $\Omega_m$ universe

- Radiation is not stress free and so impedes the growth of structure

\[ k^2 \Phi = 4\pi G a^2 \rho_r \Delta_r \]

$\Delta_r \sim 4\Theta$ oscillates around a constant value, $\rho_r \propto a^{-4}$ so the Newtonian curvature decays.

- General rule: potential decays if the dominant energy component has substantial stress fluctuations, i.e. below the generalized sound horizon or Jeans scale
Radiation Driving

- Decay is timed precisely to drive the oscillator - close to fully coherent

\[
[\Theta + \Psi](\eta) = [\Theta + \Psi](0) + \Delta \Psi - \Delta \Phi \\
= \frac{1}{3} \Psi(0) - 2\Psi(0) = \frac{5}{3} \Psi(0)
\]

- 5× the amplitude of the Sachs-Wolfe effect!

- Coherent approximation is exact for a photon-baryon fluid but reality is reduced to \( \sim 4\times \) because of neutrino contribution to radiation

- Actual initial conditions are \( \Theta + \Psi = \Psi/2 \) for radiation domination but comparison to matter dominated SW correct
Damping

• Tight coupling equations assume a perfect fluid: no viscosity, no heat conduction

• Fluid imperfections are related to the mean free path of the photons in the baryons

\[ \lambda_C = \dot{\tau}^{-1} \quad \text{where} \quad \dot{\tau} = n_e \sigma_T a \]

is the conformal opacity to Thomson scattering

• Dissipation is related to the diffusion length: random walk approximation

\[ \lambda_D = \sqrt{N} \lambda_C = \sqrt{\eta / \lambda_C} \lambda_C = \sqrt{\eta \lambda_C} \]

the geometric mean between the horizon and mean free path

• \( \lambda_D / \eta_* \sim \text{few \%} \), so expect the peaks \( \gg 3 \) to be affected by dissipation
Equations of Motion

- Continuity

\[ \dot{\Theta} = -\frac{k}{3} v_\gamma - \dot{\Phi}, \quad \dot{\delta}_b = -k v_b - 3\dot{\Phi} \]

where the photon equation remains unchanged and the baryons follow number conservation with \( \rho_b = m_b n_b \)

- Euler

\[ \dot{v}_\gamma = k (\Theta + \Psi) - \frac{k}{6} \pi_\gamma - \dot{\tau} (v_\gamma - v_b) \]

\[ \dot{v}_b = -\frac{\dot{a}}{a} v_b + k \Psi + \dot{\tau} (v_\gamma - v_b)/R \]

where the photons gain an anisotropic stress term \( \pi_\gamma \) from radiation viscosity and a momentum exchange term with the baryons and are compensated by the opposite term in the baryon Euler equation
**Viscosity**

- **Viscosity** is generated from radiation **streaming** from hot to cold regions.

- Expect

\[ \pi_\gamma \sim v_\gamma \frac{k}{\dot{T}} \]

generated by streaming, suppressed by **scattering** in a wavelength of the fluctuation. **Radiative transfer** says

\[ \pi_\gamma \approx 2A_v v_\gamma \frac{k}{\dot{T}} \]

where \( A_v = 16/15 \)

\[ \dot{v}_\gamma = k(\Theta + \Psi) - \frac{k}{3} A_v \frac{k}{\dot{T}} v_\gamma \]
Oscillator: Penultimate Take

- Adiabatic approximation \((\omega \gg \dot{a}/a)\)
  \[
  \dot{\Theta} \approx -\frac{k}{3}v_\gamma
  \]

- Oscillator equation contains a \(\dot{\Theta}\) damping term
  \[
  c_s^2 \frac{d}{d\eta} (c_s^{-2} \dot{\Theta}) + \frac{k^2 c_s^2}{\dot{\tau}} A_v \dot{\Theta} + k^2 c_s^2 \Theta = -\frac{k^2}{3}\Psi - c_s^2 \frac{d}{d\eta} (c_s^{-2} \dot{\Phi})
  \]

- Heat conduction term similar in that it is proportional to \(v_\gamma\) and is suppressed by scattering \(k/\dot{\tau}\). Expansion of Euler equations to leading order in \(k/\dot{\tau}\) gives
  \[
  A_h = \frac{R^2}{1 + R}
  \]
  since the effects are only significant if the baryons are dynamically important
Oscillator: Final Take

- **Final oscillator equation**

\[ c_s^2 \frac{d}{d\eta} (c_s^{-2} \dot{\Theta}) + \frac{k^2 c_s^2}{\dot{\tau}} [A_v + A_h] \dot{\Theta} + k^2 c_s^2 \Theta = -\frac{k^2}{3} \Psi - c_s^2 \frac{d}{d\eta} (c_s^{-2} \dot{\Phi}) \]

- Solve in the **adiabatic approximation**

\[ \Theta \propto \exp(i \int \omega d\eta) \]

\[ -\omega^2 + \frac{k^2 c_s^2}{\dot{\tau}} (A_v + A_h) i\omega + k^2 c_s^2 = 0 \] (1)
Dispersion Relation

- Solve

\[ \omega^2 = k^2 c_s^2 \left[ 1 + i \frac{\omega}{\dot{\tau}} (A_v + A_h) \right] \]

\[ \omega = \pm kc_s \left[ 1 + \frac{i}{2} \frac{\omega}{\dot{\tau}} (A_v + A_h) \right] \]

\[ = \pm kc_s \left[ 1 \pm \frac{i}{2} \frac{k c_s}{\dot{\tau}} (A_v + A_h) \right] \]

- Exponentiate

\[ \exp(i \int \omega d\eta) = e^{\pm iks} \exp[-k^2 \int d\eta \frac{1}{2} \frac{c_s^2}{\dot{\tau}} (A_v + A_h)] \]

\[ = e^{\pm iks} \exp[-(k/k_D)^2] \quad (2) \]

- Damping is exponential under the scale \( k_D \)
Diffusion Scale

• Diffusion wavenumber

\[ k_D^{-2} = \int d\eta \frac{1}{\tau} \frac{1}{6(1 + R)} \left( \frac{16}{15} + \frac{R^2}{1 + R} \right) \]

• Limiting forms

\[ \lim_{R \to 0} k_D^{-2} = \frac{1}{6} \frac{1}{15} \int d\eta \frac{1}{\tau} \]

\[ \lim_{R \to \infty} k_D^{-2} = \frac{1}{6} \int d\eta \frac{1}{\tau} \]

• Geometric mean between horizon and mean free path as expected from a random walk

\[ \lambda_D = \frac{2\pi}{k_D} \sim \frac{2\pi}{\sqrt{6}} \left( \eta \dot{\tau}^{-1} \right)^{1/2} \]
Thomson Scattering

- Polarization state of radiation in direction \( \hat{n} \) described by the intensity matrix \( \langle E_i(\hat{n}) E_j^*(\hat{n}) \rangle \), where \( \mathbf{E} \) is the electric field vector and the brackets denote time averaging.

- Differential cross section

\[
\frac{d\sigma}{d\Omega} = \frac{3}{8\pi} |\hat{\mathbf{E}}' \cdot \hat{\mathbf{E}}|^2 \sigma_T,
\]

where \( \sigma_T = \frac{8\pi \alpha^2}{3m_e} \) is the Thomson cross section, \( \hat{\mathbf{E}}' \) and \( \hat{\mathbf{E}} \) denote the incoming and outgoing directions of the electric field or polarization vector.

- Summed over angle and incoming polarization

\[
\sum_{i=1,2} \int d\hat{n}' \frac{d\sigma}{d\Omega} = \sigma_T
\]
Polarization Generation

- Heuristic: incoming radiation shakes an electron in direction of electric field vector $\hat{E}'$
- Radiates photon with polarization also in direction $\hat{E}'$
- But photon cannot be longitudinally polarized so that scattering into $90^\circ$ can only pass one polarization
- Linearly polarized radiation like polarization by reflection
- Unlike reflection of sunlight, incoming radiation is nearly isotropic
- Missing linear polarization supplied by scattering from direction orthogonal to original incoming direction
- Only quadrupole anisotropy generates polarization by Thomson scattering
Acoustic Polarization

- Break down of tight-coupling leads to quadrupole anisotropy of

\[ \pi_{\gamma} \approx \frac{k}{\dot{\tau}} v_{\gamma} \]

- Scaling \( k_D = (\dot{\tau}/\eta_*)^{1/2} \rightarrow \dot{\tau} = k_D^2 \eta_* \)

- Know: \( k_D s_* \approx k_D \eta_* \approx 10 \)

- So:

\[ \pi_{\gamma} \approx \frac{k}{k_D 10} v_{\gamma} \]

\[ \Delta_P \approx \frac{\ell}{\ell_D 10} \Delta_T \]
Acoustic Polarization

- Gradient of velocity is along direction of wavevector, so polarization is pure $E$-mode
- Velocity is $90^\circ$ out of phase with temperature – turning points of oscillator are zero points of velocity:
  $$\Theta + \Psi \propto \cos(ks); \quad v_\gamma \propto \sin(ks)$$
- Polarization peaks are at troughs of temperature power
Cross Correlation

- Cross correlation of temperature and polarization

\[(\Theta + \Psi)(v_\gamma) \propto \cos(ks) \sin(ks) \propto \sin(2ks)\]

- Oscillation at twice the frequency

- Correlation: radial or tangential around hot spots

- Partial correlation: easier to measure if polarization data is noisy, harder to measure if polarization data is high $S/N$ or if bands do not resolve oscillations

- Good check for systematics and foregrounds

- Comparison of temperature and polarization is proof against features in initial conditions mimicking acoustic features
CMB Normalization

- Normalization of potential, hence inflationary power spectrum, set by CMB observations, aka COBE or WMAP/Planck normalization

- Angular power spectrum:

\[
C_\ell = \frac{4\pi \Delta^2_T(\ell/D_*)}{2\ell(\ell + 1)} = \frac{2\pi}{\ell(\ell + 1)} \Delta^2_T(\ell/D_*)
\]

- \(\ell(\ell + 1)C_\ell/2\pi = \Delta^2_T\) is commonly used log power

- Sachs-Wolfe effect says \(\Delta^2_T = \Delta^2_\Phi/9\), \(\Phi = \frac{3}{5}\mathcal{R}\) initial

- Observed number at recombination

\[
\Delta^2_T = \left(\frac{28 \mu K}{2.725 \times 10^6 \mu K}\right)^2
\]
\[
\Delta^2_\Phi \approx (3 \times 10^{-5})^2
\]
\[
\Delta^2_\mathcal{R} \approx (5 \times 10^{-5})^2
\]
COBE to WMAP to Planck

- Given that the temperature response to an inflationary initial perturbation is known for all $k$ through the Boltzmann solution of the acoustic physics, one can translate $\Delta_T^2$ to $\Delta_R^2$ at the best measured $k \approx \ell/D_*$. 

- The CMB normalization was first extracted from COBE at $\ell \sim 10$ or $k \sim H_0$. A low $\ell$ normalization point suffers from cosmic variance: only $2\ell + 1$ samples of a given $\ell$ mode.

- WMAP measures very precisely the first acoustic peak at $\ell \approx 200$ which corresponds to $k \sim 0.02 \text{ Mpc}^{-1}$.

- Planck measures out to $\ell = 1000 - 2000$ precisely which corresponds to $k \sim 0.08 \text{ Mpc}^{-1}$.

- Taking out the CMB transfer function $\Delta_R^2$ nearly scale invariant with a small red tilt of $n_s - 1 \approx 0.03$. 
Transfer Function

- **Transfer function** transfers the initial Newtonian curvature to its value today (linear response theory)

\[ T(k) = \frac{\Phi(k, a = 1)}{\Phi(k, a_{\text{init}})} \frac{\Phi(k_{\text{norm}}, a_{\text{init}})}{\Phi(k_{\text{norm}}, a = 1)} \]

- Conservation of Bardeen curvature: Newtonian curvature is a constant when stress perturbations are negligible: above the horizon during radiation and dark energy domination, on all scales during matter domination

- When stress fluctuations dominate, perturbations are stabilized by the Jeans mechanism

- Hybrid Poisson equation: Newtonian curvature, comoving density perturbation \( \Delta \equiv (\delta \rho/\rho)_{\text{com}} \) implies \( \Phi \) decays

\[ (k^2 - 3K)\Phi = 4\pi G a^2 \rho \Delta \sim \eta^{-2} \Delta \]
Transfer Function

- Freezing of $\Delta$ stops at $\eta_{eq}$

$$\Phi \sim (k\eta_{eq})^{-2}\Delta_H \sim (k\eta_{eq})^{-2}\Phi_{\text{init}}$$

- Transfer function has a $k^{-2}$ fall-off beyond $k_{eq} \sim \eta_{eq}^{-1}$

$$\eta_{eq} = 15.7(\Omega_m h^2)^{-1} \left(\frac{T}{2.7K}\right)^2 \text{Mpc}$$

- Small correction since growth with a smooth radiation component is logarithmic not frozen

- Transfer function is a direct output of an Einstein-Boltzmann code
Fitting Function

- Alternately accurate fitting formula exist, e.g. pure CDM form:

\[
T(k(q)) = \frac{L(q)}{L(q) + C(q)q^2}
\]

\[
L(q) = \ln(e + 1.84q)
\]

\[
C(q) = 14.4 + \frac{325}{1 + 60.5q^{1.11}}
\]

\[
q = k/\Omega_m h^2 \text{Mpc}^{-1}(T_{\text{CMB}}/2.7K)^2
\]

- In $h$ Mpc$^{-1}$, the critical scale depends on $\Gamma \equiv \Omega_m h$ also known as the shape parameter.
Transfer Function

- Numerical calculation

![Graph of Transfer Function](image)

The graph represents the transfer function $T(k)$, which is a function of $k$ (in $h^{-1}$ Mpc). The function decreases with increasing $k$, following a $k^{-2}$ behavior at high $k$. The wiggles indicate fluctuations in the function as $k$ increases.
Baryon Acoustic Oscillations

- Baryons caught up in the acoustic oscillations of the CMB and impart acoustic oscillations to the transfer function. Density enhancements are produced kinematically through the continuity equation $\delta_b \sim (k\eta)v_b$ and hence are out of phase with CMB temperature peaks.

- Dissipation of the acoustic oscillations eliminates both the CMB and baryon perturbations – known as Silk damping for the baryons. This suppression and the general fact that baryons are caught up with photons was one of the main arguments for CDM.

- Detected first (so far only) in the SDSS LRG survey.

- An excellent standard ruler for angular diameter distance $D_A(z)$ since it does not evolve in redshift in linear theory.

- Radial extent of BAO gives $H(z)$ (not yet seen in data).
Massive Neutrinos

- Neutrino dark matter suffers similar effects and hence cannot be the main component of dark matter in the universe.
- Relativistic stresses of a light neutrino slow the growth of structure.
- Neutrino species with cosmological abundance contribute to matter as $\Omega_\nu h^2 = \sum m_\nu / 94\text{eV}$, suppressing power as $\Delta P/P \approx -8\Omega_\nu / \Omega_m$.
- Current data Planck and large scale structure indicate $\sum m_\nu < 0.23\text{eV}$ (95% CL).
Growth Function

- Same physics applies to the dark energy dominated universe
- Under the dark energy sound horizon or Jeans scale, dark energy density frozen. Potential decays at the same rate for all scales

\[ G(a) = \frac{\Phi(k_{\text{norm}}, a)}{\Phi(k_{\text{norm}}, a_{\text{init}})} , \equiv \frac{d}{d \ln a} \]

- Continuity + Euler + Poisson

\[ G'' + \left( 1 - \frac{\rho''}{\rho'} + \frac{1}{2} \frac{\rho'_c}{\rho_c} \right) G' + \left( \frac{1}{2} \frac{\rho'_c + \rho'}{\rho_c} - \frac{\rho''}{\rho'} \right) G = 0 \]

where \( \rho \) is the Jeans unstable matter and \( \rho_c \) is the critical density
Dark Energy Growth Suppression

- **Pressure growth suppression:** \( \delta \equiv \frac{\delta \rho_m}{\rho_m} \propto aG \)

\[
\frac{d^2 G}{d \ln a^2} + \left[ \frac{5}{2} - \frac{3}{2} w(z) \Omega_{DE}(z) \right] \frac{dG}{d \ln a} + \frac{3}{2} [1 - w(z)] \Omega_{DE}(z) G = 0,
\]

where \( w \equiv p_{DE}/\rho_{DE} \) and \( \Omega_{DE} \equiv \rho_{DE}/(\rho_m + \rho_{DE}) \) with initial conditions \( G = 1, \frac{dG}{d \ln a} = 0 \)

- As \( \Omega_{DE} \to 0 \) \( g = \text{const.} \) is a solution. The other solution is the decaying mode, eliminated by initial conditions

- As \( \Omega_{DE} \to 1 \) \( g \propto a^{-1} \) is a solution. Corresponds to a frozen density field.
Power Spectrum Normalization

- Present (or matter dominated) vs inflationary initial conditions (normalized by CMB):

\[ \Delta_\Phi^2(k, a) \approx \frac{9}{25} \Delta_{R_i}^2(k_{\text{norm}}) G^2(a) T^2(k) \left( \frac{k}{k_{\text{norm}}} \right)^{n-1} \]

- Density field

\[ k^2 \Phi = 4\pi G a^2 \Delta \rho_m \]

\[ = \frac{3}{2} H_0^2 \Omega_m \frac{\Delta \rho_m}{\rho_m} \frac{1}{a} \]

\[ \Delta_\Phi^2 = \frac{9}{4} \left( \frac{H_0}{k} \right)^4 \Omega_m^2 a^{-2} \Delta_m^2 \]

\[ \Delta_m^2 = \frac{4}{25} \Delta_{R_i}^2(k_{\text{norm}}) \Omega_m^{-2} a^2 G^2(a) T^2(k) \left( \frac{k}{k_{\text{norm}}} \right)^{n-1} \left( \frac{k}{H_0} \right)^4 \]
Antiquated Normalization Conventions

- Current density field on the horizon scale $k = H_0$

$$\delta^2_H = \frac{4}{25} \Delta^2_{\mathcal{R}_i}(k_{\text{norm}}) \Omega_m^{-2} a^2 G^2(a) = \left(\frac{2G(1)}{\Omega_m} \times 10^{-5}\right)$$

- $\sigma_8$, RMS of density field filtered by tophat of $8h^{-1}\text{Mpc}$
Power Spectrum

- SDSS data

- Power spectrum defines large scale structure observables: galaxy clustering, velocity field, Ly$\alpha$ forest clustering, cosmic shear
Velocity field

- Continuity gives the velocity from the density field as

\[ v = -\frac{\Delta}{k} = -\frac{aH}{k} \frac{d\Delta}{d \ln a} \]

\[ = -\frac{aH}{k} \Delta \frac{d \ln (aG)}{d \ln a} \]

- In a \( \Lambda \)CDM model or open model \( d \ln (aG)/d \ln a \approx \Omega_m^{0.6} \)

- Measuring both the density field and the velocity field (through distance determination and redshift) allows a measurement of \( \Omega_m \)

- Practically one measures \( \beta = \Omega_m^{0.6}/b \) where \( b \) is a bias factor for the tracer of the density field, i.e. with galaxy numbers \( \delta n/n = b \Delta \)

- Can also measure this factor from the redshift space power spectrum - the Kaiser effect where clustering in the radial direction is apparently enhanced by gravitational infall
Redshift Space Power Spectrum

- Kaiser effect is separable from the real space clustering if one measures modes parallel and transverse to the line of sight. Redshift space distortions only modify the former.

- 2D power spectrum in “s” or redshift space

\[ P_s(k_\perp, k_\parallel) = \left[ 1 + \beta \left( \frac{k_\parallel}{k} \right)^2 \right]^2 b^2 P(k) \]

where \( k^2 = k_\parallel^2 + k_\perp^2 \) and \( k_\perp \) is a 2D vector transverse to the line of sight.
Power Spectrum Errors

- The precision with which the power spectrum can be measured is ultimately limited by sample variance from having a finite survey volume $V = L^3$. This is basically a mode counting argument. The errors on the power spectrum are given by

$$\left( \frac{\Delta P_s}{P_s} \right)^2 = \frac{2}{N_k}$$

where $N_k$ is the number of modes in a range of $\Delta k_\perp$, $\Delta k_\parallel$. This is determined by the $k$-space volume and the fundamental mode of the box $k_0 = 2\pi / L$ which sets the cell size in the volume

$$\left( \frac{\Delta P_s}{P_s} \right)^2 = \frac{2}{(2\pi)^3 2\pi k_\perp \Delta k_\perp \Delta k_\parallel}$$
Lyman-α Forest

- QSO spectra absorbed by neutral hydrogen through the Lyα transition.
- The optical depth to absorption is (with $ds$ in physical scale)

$$\tau(\nu) = \int ds x_{\text{HI}} n_b \sigma_\alpha \sim \int ds x_{\text{HI}} n_b \Gamma \phi(\nu) \lambda^2$$

where $x_{\text{HI}}$ is the neutral fraction, $\Gamma = 6.25 \times 10^8 \text{s}^{-1}$ is the transition rate and $\lambda = 1216\text{Å}$ is the Lyα wavelength and $\phi(\nu)$ is the Lorentz profile. For radiation at a given emitted frequency $\nu_0$ above the transition, it will redshift through the transition.

- Resonant transition: lack of complete absorption, known as the lack of a Gunn-Peterson trough indicates that the universe is nearly fully ionized $x_{\text{HI}} \ll 1$ out to the highest redshift quasar $z \sim 6$; indications that this is near the end of the reionization epoch.
Lyman-\(\alpha\) Forest

- In ionization equilibrium, the Gunn-Peterson optical depth is a tracer of the underlying baryon density which itself is a tracer of the dark matter \(\tau_{GP} \propto \rho_b^2 T^{-0.7}\) with \(T(\rho_b)\).

\[
\frac{d(1 - x_{\text{HI}})}{dt} = -x_{\text{HI}} \int d\nu \frac{4\pi J_\nu}{h\nu} \sigma_\nu + (1 - x_{\text{HI}})^2 n_b R
\]

where \(\sigma_\nu\) is the photoionization cross section (sharp edge at threshold and falling in frequency means \(J_\nu \approx J_{21}\)) and \(R \propto T^{-0.7}\) is the recombination coefficient.

- Given an equation of state from simulations of \(p \propto \rho^\gamma\)

\[x_{\text{HI}} \propto \frac{\rho_b R}{J_{21}} \propto \frac{\rho_b T^{-0.7}}{J_{21}}, \quad \tau_{GP} \propto \frac{\rho_b^{2-0.7(\gamma-1)}}{J_{\text{HI}}}
\]

- Clustering in the Ly\(\alpha\) forest reflects the underlying power spectrum modulo an overall ionization intensity \(J_{21}\)
Gravitational Lensing

- Gravitational potentials along the line of sight $\hat{n}$ to some source at comoving distance $D_s$ lens the images according to (flat universe)

$$\phi(\hat{n}) = 2 \int dD \frac{D_s - D}{DD_s} \Phi(D\hat{n}, \eta(D))$$

remapping image positions as

$$\hat{n}^I = \hat{n}^S + \nabla_{\hat{n}} \phi(\hat{n})$$

- Since absolute source position is unknown, use image distortion defined by the Jacobian matrix

$$\frac{\partial n_i^I}{\partial n_j^S} = \delta_{ij} + \psi_{ij}$$
Weak Lensing

- Small image distortions described by the convergence $\kappa$ and shear components $(\gamma_1, \gamma_2)$

$$\psi_{ij} = \begin{pmatrix} \kappa - \gamma_1 & -\gamma_2 \\ -\gamma_2 & \kappa + \gamma_1 \end{pmatrix}$$

where $\nabla_{\hat{n}} = D \nabla$ and

$$\psi_{ij} = 2 \int dD \frac{D(D_s - D)}{D_s} \nabla_i \nabla_j \Phi(D\hat{n}, \eta(D))$$

- In particular, through the Poisson equation the convergence (measured from shear) is simply the projected mass

$$\kappa = \frac{3}{2} \Omega_m H_0^2 \int dD \frac{D(D_s - D)}{D_s} \frac{\Delta(D\hat{n}, \eta(D))}{a}$$