Set 3:

Cosmic Dynamics

FRW Dynamics

- This is as far as we can go on FRW geometry alone - we still need to know how the scale factor $a(t)$ evolves given matter-energy content
- General relativity: matter tells geometry how to curve, scale factor determined by content
- This next part is for advanced students and will not be required for problem sets or exams but included so you get a flavor of general relativity – cosmology is the simplest application of general relativity possible
- After this brief aside, we'll return to explain this by Newtonian mechanics - even for the cosmological expansion, gravity is locally Newtonian

General Relativity

- Build the Einstein tensor $G_{\mu\nu}$ out of the metric and use Einstein equation (overdots conformal time derivative)

$$G_{\mu\nu}(= R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R) = 8\pi G T_{\mu\nu}$$

- Easier to work with mixed upper and lower indices since the metric $g^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu}$
- For the FRW metric

$$\begin{aligned} G^0{}_0 &= -3\left(H^2 + \frac{K}{a^2}\right) \\ G^i{}_j - G^0{}_0 \frac{\delta^i{}_j}{3} &= -\frac{2}{a^2}\left(\frac{\ddot{a}}{a} - a^2 H^2\right)\delta^i{}_j = -\frac{2}{a}\frac{d^2 a}{dt^2}\delta^i{}_j , \end{aligned}$$

where recall the curvature $K = 1/R^2$ and overdots are $d/d\eta$

Matter as Curvature Source

- Likewise isotropy demands that the stress-energy tensor take the form

$$T^0{}_0 = -\rho, \quad T^i{}_j = p\delta^i{}_j \quad \rightarrow \quad T^i{}_j - T^0{}_0 \frac{\delta^i{}_j}{3} = p + \rho/3$$

where ρ is the energy density and p is the pressure

- It is not necessary to assume that the content is a perfect fluid - consequence of FRW symmetry
- This concludes our GR aside for advanced students – you are not responsible for that part

Friedmann Equations

- Einstein equations given the FRW symmetries become the Friedmann equations

$$
\begin{aligned}
H^2 + \frac{K}{a^2} &= \frac{8\pi G}{3}\rho \\
\frac{1}{a}\frac{d^2 a}{dt^2} &= -\frac{4\pi G}{3}(\rho + 3p)
\end{aligned}
$$

- Acceleration source is $\rho + 3p$ requiring $p < -\rho/3$ for positive acceleration
- Curvature as an effective energy density component

$$
\rho_K = -\frac{3}{8\pi G}\frac{K}{a^2} \propto a^{-2}
$$

Positive curvature gives negative effective energy density

Critical Density

- Friedmann equation for H then reads

$$H^2(a) = \frac{8\pi G}{3}(\rho + \rho_K) \equiv \frac{8\pi G}{3}\rho_c$$

defining a critical density today ρ_c in terms of the expansion rate

- In particular, its value today is given by the Hubble constant as

$$\rho_{\rm c}(z=0) = \frac{3H_0^2}{8\pi G} = 1.8788 \times 10^{-29} h^2 {\rm g\,cm}^{-3}$$

or about $10^{-5}h^2$ protons per cm^3 - really empty

- Energy density today is given as a fraction of critical

$$\Omega \equiv \frac{\rho}{\rho_c(z=0)}$$

- Note that physical energy density $\propto \Omega h^2$ (g cm^{-3})

Critical Density

- Likewise radius of curvature then given by

$$\Omega_K = (1-\Omega) = -\frac{1}{H_0^2 R^2} \to R = (H_0\sqrt{\Omega - 1})^{-1}$$

- If $\Omega \approx 1$, then true density is near critical $\rho \approx \rho_c$ and

$$\rho_K \ll \rho_c \leftrightarrow H_0 R \ll 1$$

Universe is flat across the Hubble distance

- $\Omega > 1$ positively curved

$$D_A = R\sin(D/R) = \frac{1}{H_0\sqrt{\Omega - 1}}\sin(H_0 D\sqrt{\Omega - 1})$$

- $\Omega < 1$ negatively curved

$$D_A = R\sin(D/R) = \frac{1}{H_0\sqrt{1-\Omega}}\sinh(H_0 D\sqrt{1-\Omega})$$

Newtonian Cosmology

- Now let’s try to understand the Friedmann equation from a Newtonian perspective
- First let’s use energy conservation reasoning – this is not quite right, but gives you an easy way of deriving the Friedmann equation if you forget it
- Next we’ll see the real Newtonian cosmology derivation which involves forces - these act locally and we don’t need to consider separations where general relativity is necessary
- This gives a perfectly correct derivation of the dynamics of the scale factor and since it determines the global expansion, we evade having to work with the field equations of general relativity directly

Newtonian Energy Interpretation

- Consider a test particle of mass m as part of expanding spherical shell of radius r and total mass M.

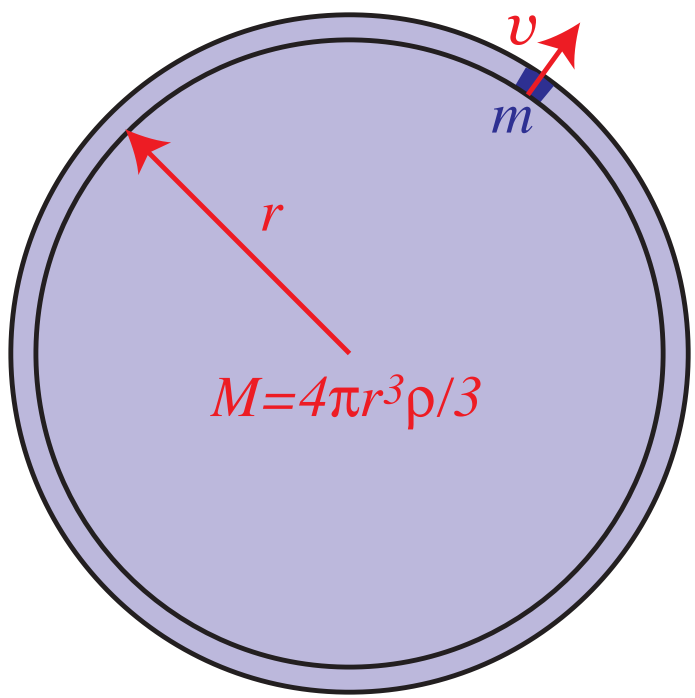
- Energy conservation

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r} = \text{const}$$

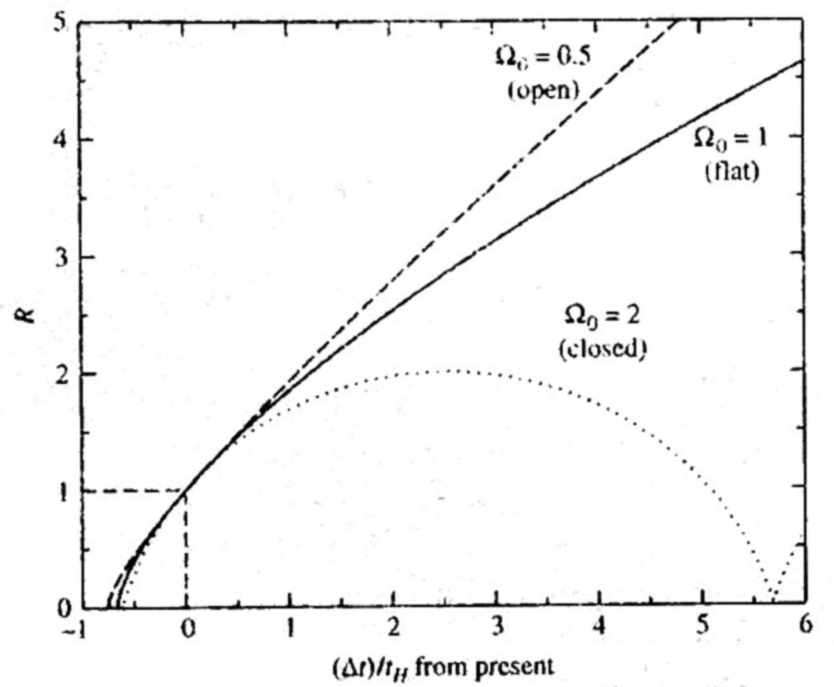
$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2 - \frac{GM}{r} = \text{const}$$

$$\frac{1}{2}\left(\frac{1}{r}\frac{dr}{dt}\right)^2 - \frac{GM}{r^3} = \frac{\text{const}}{r^2}$$

$$H^2 = \frac{8\pi G\rho}{3} - \frac{\text{const}}{a^2}$$



Newtonian Energy Interpretation



- Constant determines whether the system is bound and in the Friedmann equation is associated with curvature – not general since neglects pressure
- Nonetheless Friedmann equation is the same with pressure - but mass-energy within expanding shell is not constant
- Instead, rely on the fact that gravity in the weak field regime is Newtonian and forces unlike energies are unambiguously defined locally.

Newtonian Force Interpretation

- An alternate, more general Newtonian derivation, comes about by realizing that locally around an observer, gravity must look Newtonian.
- Given Newton's iron sphere theorem, the gravitational acceleration due to a spherically symmetric distribution of mass outside some radius r is zero (Birkhoff theorem in GR)
- We can determine the acceleration simply from the enclosed mass

$$\nabla \Psi_N = \frac{GM_N}{r^2} = \frac{4\pi G}{3}(\rho + 3p)r$$

where $\rho + 3p$ reflects the active gravitational mass provided by pressure.

Newtonian Force Interpretation

- Hence the gravitational acceleration

$$\frac{\ddot{r}}{r} = -\frac{1}{r}\nabla\Psi_N = -\frac{4\pi G}{3}(\rho + 3p)$$

- We'll come back to this way of viewing the effect of the expansion on the formation of structure - in particular the evolution of a spherically symmetric density perturbation

Conservation Law

- The two Friedmann equation are redundant in that one can be derived from the other via energy conservation
  - Advanced students: consequence of Bianchi identities in GR: $\nabla^\mu G_{\mu\nu} = 0$
  - Think of this as an adiabatically expanding gas

$$
\begin{aligned}
d\rho V + pdV &= 0 \\
d\rho a^3 + pda^3 &= 0 \\
\dot\rho a^3 + 3\frac{\dot a}{a}\rho a^3 + 3\frac{\dot a}{a} p a^3 &= 0 \\
\frac{\dot\rho}{\rho} = -3(1 + \frac{p}{\rho})\frac{\dot a}{a}
\end{aligned}
$$

Equation of State Parameter

- Time evolution depends on "equation of state" $w(a) = p/\rho$
- If $w =$ const. then the energy density depends on the scale factor as $\rho \propto a^{-3(1+w)}$
- Different particle species have different equations of state
- Even non-particle species like curvature and dark energy have effective equations of state defined by the average pressure / average energy density - in these cases w does not define a real (local) equation of state of a real expanding gas

Multicomponent Universe

Special cases:

- nonrelativistic matter $\rho_m = m n_m \propto a^{-3}$, $w_m = 0$
- ultrarelativistic radiation $\rho_r = E n_r \propto n_r/\lambda \propto a^{-4}$, $w_r = 1/3$
- (cosmological) constant energy density $\rho_\Lambda \propto a^0$, $w_\Lambda = -1$
- total energy density summed over above

$$\rho(a) = \sum_i \rho_i(a) = \rho_c(a=1) \sum_i \Omega_i a^{-3(1+w_i)}$$

  – again think of curvature as fictitious energy density
- curvature $\rho_K \propto a^{-2}$, $w_K = -1/3$
  – again all components sum up to critical density

$$\rho_c = \rho + \rho_K \to 1 = \sum_i \Omega_i + \Omega_K$$

  – likewise for p_c and $w_c = p_c/\rho_c$

Multicomponent Universe

- For the Friedmann equation we can always think of a multicomponent universe as a single component universe with a complicated equation of state $w_c(a) = p_c(a)/\rho_c(a)$
- Now let's relate the two Friedmann equations with energy conservation

Acceleration Equation

- Time derivative of (first) Friedmann equation

$$
\begin{aligned}
\frac{dH^2}{dt} &= \frac{8\pi G}{3}\frac{d\rho_c}{dt} \\
2H\left[\frac{1}{a}\frac{d^2a}{dt^2} - H^2\right] &= \frac{8\pi G}{3} H[-3(1+w_c)\rho_c] \\
\left[\frac{1}{a}\frac{d^2a}{dt^2} - 2\frac{4\pi G}{3}\rho_c\right] &= -\frac{4\pi G}{3}[3(1+w_c)\rho_c] \\
\frac{1}{a}\frac{d^2a}{dt^2} &= -\frac{4\pi G}{3}[(1+3w_c)\rho_c] \\
&= -\frac{4\pi G}{3}(\rho+\rho_K+3p+3p_K) \\
&= -\frac{4\pi G}{3}(1+3w)\rho
\end{aligned}
$$

- Acceleration if $w < -1/3$
- Reverse: Newtonian acceleration implies Friedmann equation

Expansion Required

- Friedmann equations "predict" the expansion of the universe. Non-expanding conditions $da/dt = 0$ and $d^2a/dt^2 = 0$ require

$$\rho = -\rho_K \qquad \rho = -3p$$

  i.e. a positive curvature and a total equation of state $w \equiv p/\rho = -1/3$

- Since matter is known to exist, one can in principle achieve this by adding a balancing cosmological constant

$$\rho = \rho_m + \rho_\Lambda = -\rho_K = -3p = 3\rho_\Lambda$$
$$\rho_\Lambda = -\frac{1}{3}\rho_K\,, \quad \rho_m = -\frac{2}{3}\rho_K$$

  Einstein introduced ρ_Λ for exactly this reason – "biggest blunder"; but note that this balance is unstable: ρ_m can be perturbed but ρ_Λ, a true constant cannot