

Set 6:

Inflation

Horizon Problem

- The horizon in a decelerating universe scales as $\eta \propto a^{(1+3w)/2}$, $w > -1/3$. For example in a matter dominated universe

$$\eta \propto a^{1/2}$$

- CMB decoupled at $a_* = 10^{-3}$ so subtends an angle on the sky

$$\frac{\eta_*}{\eta_0} = a_*^{1/2} \approx 0.03 \approx 2^\circ$$

- So why is the CMB sky isotropic to 10^{-5} in temperature if it is composed of $\sim 10^4$ causally disconnected regions
- If smooth by fiat, why are there 10^{-5} fluctuations correlated on superhorizon scales

Flatness & Relic Problems

- Flatness problem: why is the radius of curvature larger than the observable universe. (Before the CMB determinations, why is it at least comparable to observable universe $|\Omega_K| \lesssim \Omega_m$)
- Also phrased as a coincidence problem: since $\rho_K \propto a^{-2}$ and $\rho_m \propto a^{-3}$, why would they be comparable today – modern version is dark energy coincidence $\rho_\Lambda = \text{const.}$
- Relic problem – why don't relics like monopoles dominate the energy density
- Inflation is a theory that solves all three problems at once and also supplies a source for density perturbations

Accelerating Expansion

- In a matter or radiation dominated universe, the horizon grows as a power law in a so that there is no way to establish causal contact on a scale longer than the inverse Hubble length ($1/aH$, comoving coordinates) at any given time: general for a decelerating universe

$$\eta = \int d \ln a \frac{1}{aH(a)}$$

- $H^2 \propto \rho \propto a^{-3(1+w)}$, $aH \propto a^{-(1+3w)/2}$, critical value of $w = -1/3$, the division between acceleration and deceleration
- In an accelerating universe, the Hubble length shrinks in comoving coordinates and so the horizon gets its contribution at the earliest times, e.g. in a cosmological constant universe, the horizon saturates to a constant value

Causal Contact

- True horizon always grows meaning that one always sees more and more of the universe. But the comoving Hubble length decreases: the difference in conformal time, the distance a photon can travel between two epochs denoted by the scale factor decreases. Regions that were in causal contact, leave causal contact.
- Horizon problem solved if the universe was in an acceleration phase up to η_i and the conformal time since then is shorter than the total conformal age

$$\eta_0 \gg \eta_0 - \eta_i$$

total distance \gg distance traveled since inflation
apparent horizon

Flatness & Relic

- Comoving radius of curvature is constant and can even be small compared to the full horizon $R \ll \eta_0$ yet still $\eta_0 \gg R \gg \eta_0 - \eta_i$
- In physical coordinates, the rapid expansion of the universe makes the current observable universe much smaller than the curvature scale
- Likewise, the number density of relics formed before the accelerating (or inflationary) epoch is diluted to make them rare in the current observable volume
- Common to reference time to the end of inflation $\tilde{\eta} \equiv \eta - \eta_i$. Here conformal time is negative during inflation and its value (as a difference in conformal time)

Scalar Fields

- Unlike a true cosmological constant, the period of exponential expansion must end to produce the hot big bang phase
- A cosmological constant is like potential energy - so imagine a ball rolling slowly in into a valley eventually converting potential into kinetic energy
- Technically, this is a scalar field: where the position on the hill is ϕ and the height of the potential is $V(\phi)$
- In spacetime $\phi(\mathbf{x}, t)$ is a function of position: different spacetime points can be at different field positions

Scalar Fields

- Inflation ends when the field rolls sufficiently down the potential that its kinetic energy becomes comparable to its potential energy
- The field then oscillates at the bottom of the potential and small couplings to standard model particles “reheats” the universe converting the inflaton energy into particles
- Due to the uncertainty principle in quantum mechanics, the field cannot remain perfectly unperturbed
- The small field fluctuations mean that inflation ends at a slightly different time at different points in space - leaving fluctuations in the scale factor, which are curvature or gravitational potential fluctuations
- Gravitational attraction into these potential wells forms all of the structure in the universe

Sufficient Inflation

- If the accelerating component has equation of state $w = -1$, $\rho = \text{const.}$, $H = H_i \text{ const.}$ so that $a \propto \exp(Ht)$

$$\begin{aligned}\tilde{\eta} &= - \int_a^{a_i} d \ln a \frac{1}{aH} = \frac{1}{aH_i} \Big|_a^{a_i} \\ &\approx -\frac{1}{aH_i} \quad (a_i \gg a)\end{aligned}$$

- In particular, the current horizon scale $H_0 \tilde{\eta}_0 \approx 1$ exited the horizon during inflation at

$$\begin{aligned}\tilde{\eta}_0 &\approx H_0^{-1} = \frac{1}{a_H H_i} \\ a_H &= \frac{H_0}{H_i}\end{aligned}$$

Sufficient Inflation

- Current horizon scale must have exited the horizon during inflation so that the start of inflation could not be after a_H - inflation must have lasted at least ~ 60 e-folds

$$\frac{a_H}{a_i} = \frac{H_0}{H_i a_i}$$
$$\frac{H_0}{H_i} = \sqrt{\frac{\rho_c}{\rho_i}}, \quad a_i = \frac{T_{\text{CMB}}}{T_i}$$

- $\rho_c^{1/4} = 3 \times 10^{-12} \text{ GeV}$, $T_{\text{CMB}} = 3 \times 10^{-13} \text{ GeV}$

$$\frac{a_H}{a_i} = 3 \times 10^{-29} \left(\frac{\rho_i^{1/4}}{10^{14} \text{ GeV}} \right)^{-2} \left(\frac{T_i}{10^{10} \text{ GeV}} \right)$$
$$\ln \frac{a_i}{a_H} = 65 + 2 \ln \left(\frac{\rho_i^{1/4}}{10^{14} \text{ GeV}} \right) - \ln \left(\frac{T_i}{10^{10} \text{ GeV}} \right)$$

Scalar Fields

- Mathematically, the scalar field obeys the Klein-Gordon equation in an expanding universe

$$\frac{d^2\phi}{dt^2} + 3H\frac{d\phi}{dt} + V' = 0$$

where $V' = dV/d\phi$ is the slope of the potential - the first and third term look like the equations of motion of a ball rolling down a hill - acceleration = gradient of potential

- The second $d\phi/dt$ term is a friction term provided by the expansion - “Hubble friction” - just like particle numbers and energy density dilute with the expansion, so too does the kinetic energy of the scalar field
- Kinetic energy is $(d\phi/dt)^2/2$ so, without the V' forcing term, it would decay as a^{-6} , or $w_{\text{kinetic}} = +1$, whereas $w_{\text{potential}} = -1$

Scalar Fields

- As the field rolls it slowly loses total energy to friction, which defines the slow roll parameter

$$\epsilon_H = -\frac{d \ln H}{d \ln a} = \frac{3}{2}(1 + w_\phi)$$

- Requirement that inflation last for the sufficient ~ 60 e-folds requires that $\epsilon_H \lesssim 1/60 \ll 1$
- This requirement also means that ϵ_H must also be slowly varying so as not to grow much during these 60 e-folds

$$\delta_1 = \frac{1}{2} \frac{d \ln \epsilon_H}{d \ln a} - \epsilon_H$$

with $|\delta_1| \ll 1$ (advanced students: its defined this way since it also determines how close the roll is to friction dominated

$$3H d\phi/dt \approx -V')$$

Perturbation Generation

- Horizon scale $\tilde{\eta}$ during inflation acts like an event horizon - things leaving causal contact
- Result of treating field fluctuations as a quantum simple harmonic oscillator (advanced students: see supplement) is that the uncertainty principle leads to inevitable fluctuations
- Fluctuations freeze in when the wavelength $\lambda = 2\pi/k$ becomes larger than the comoving horizon, so that parts of the fluctuation are no longer in causal contact with itself, i.e. when $k \approx 1/aH$

$$\delta\phi \approx \frac{H}{2\pi}$$

- We can also view this as an “origins” problem. Quantum fluctuations behave as a simple harmonic oscillator with frequency or rate $\omega \approx k/a$ and freezeout occurs when $\omega = H$, so $k/a = H$

Perturbation Generation

- Interpretation: universe is expanding quickly enough that various parts of the wave cannot “find” each other to maintain “equilibrium” (continue oscillating)
- Can heuristically understand the freezout value in the same way as Hawking radiation from a black hole - virtual particles become real when separated by the horizon
- Here H defines the horizon area (or in black hole language the Hawking temperature) and dimensional analysis says the field fluctuation must scale with H , the only dimensionfull quantity
- Because H remains roughly constant during inflation the result is a scale invariant spectrum of fluctuations

Curvature Fluctuation

- Field fluctuations change the scale factor at which inflation ends

$$\mathcal{R} = -\delta \ln a = -\frac{d \ln a}{dt} \frac{dt}{d\phi} \delta\phi = -\frac{H^2}{2\pi} \frac{dt}{d\phi}$$

- Using the equation of state of ϕ

$$w_\phi = \frac{p_\phi}{\rho_\phi} = \frac{(d\phi/dt)^2/2 - V}{(d\phi/dt)^2/2 + V} \approx \frac{(d\phi/dt)^2/2}{V} - 1$$

and $H^2 \approx 8\pi G V/3$ from Friedmann, so

$$\epsilon_H \approx \frac{3}{2} \frac{(d\phi/dt)^2}{V} \approx 4\pi G \frac{(d\phi/dt)^2}{H^2}$$

- So the variance of fluctuations per log wavenumber $d \ln k$

$$\Delta_{\mathcal{R}}^2 \equiv \langle \mathcal{R}^2 \rangle \approx \frac{H^4}{4\pi^2} \frac{4\pi G}{H^2 \epsilon_H} \approx \frac{G}{\pi} \frac{H^2}{\epsilon_H}$$

Tilt

- Curvature power spectrum is scale invariant to the extent that H and ϵ_H are constant
- Scalar spectral index

$$\frac{d \ln \Delta_{\mathcal{R}}^2}{d \ln k} \equiv n_S - 1 = 2 \frac{d \ln H}{d \ln k} - \frac{d \ln \epsilon_H}{d \ln k}$$

- Evaluate at horizon crossing where fluctuation freezes $k = aH$

$$\begin{aligned} \frac{d \ln H}{d \ln k} &\approx \frac{d \ln H}{d \ln a} = -\epsilon_H \\ \frac{d \ln \epsilon}{d \ln k} &\approx \frac{d \ln \epsilon}{d \ln a} = 2(\delta_1 + \epsilon_H) \end{aligned}$$

- Tilt in the slow-roll approximation

$$n_S = 1 - 4\epsilon_H - 2\delta_1$$

Gravitational Waves

- Gravitational wave amplitude satisfies Klein-Gordon equation ($K = 0$), same as scalar field

$$\frac{d^2 h_{+, \times}}{dt^2} + 3H \frac{dh_{+, \times}}{dt} + \frac{k^2}{a^2} h_{+, \times} = 0.$$

- Acquires quantum fluctuations in same manner as ϕ . Lagrangian sets the normalization
- Scale-invariant gravitational wave amplitude

$$\Delta_{+, \times}^2 = 16\pi G \frac{H^2}{(2\pi)^2}$$

- Gravitational wave power $\propto H^2 \propto V \propto E_i^4$ where E_i is the energy scale of inflation

Gravitational Waves

- Tensor-scalar ratio

$$r \equiv 4 \frac{\Delta_+^2}{\Delta_{\mathcal{R}}^2} = 16\epsilon_H$$

- Tensor tilt:

$$\frac{d \ln \Delta_+^2}{d \ln k} \equiv n_T = 2 \frac{d \ln H}{d \ln k} = -2\epsilon_H$$

- Consistency relation between tensor-scalar ratio and tensor tilt

$$r = 16\epsilon = -8n_T$$

- Measurement of scalar tilt and gravitational wave amplitude constrains inflationary model in the slow roll context
- Comparison of tensor-scalar ratio and tensor tilt tests the idea of slow roll itself

Observability

- Gravitational waves from inflation can be measured via its imprint on the polarization of the CMB...