Supplement
Inflationary Perturbations

Field perturbations

- Let's define the perturbed scalar field as $\phi = \phi_0 + \phi_1$ where ϕ_0 is the unperturbed field (i.e. $\phi_1 = \delta\phi$)
- Field fluctuations obey a damped harmonic oscillator equation (with dots referring to conformal time derivatives)

$$\ddot{\phi}_1 + 2\frac{\dot{a}}{a}\dot{\phi}_1 + k^2\phi_1 \approx 0$$

- We want a simple harmonic oscillator that we can then quantize so define $u \equiv a\phi_1$

$$\dot{u} = \dot{a}\phi_1 + a\dot{\phi}_1$$

$$\ddot{u} = \ddot{a}\phi_1 + 2\dot{a}\dot{\phi}_1 + a\ddot{\phi}_1$$

$$\ddot{u} + [k^2 - \frac{\ddot{a}}{a}]u = 0$$

- Note Friedmann equations say if $p = -\rho$, $\ddot{a}/a = 2(\dot{a}/a)^2$

Harmonic Oscillator

- Now let's look at the oscillator equation

$$\ddot{u} + [k^2 - 2\left(\frac{\dot{a}}{a}\right)^2]u = 0$$

or for conformal time measured from the end of inflation

$$\tilde{\eta} = \eta - \eta_{\rm end}$$

$$\tilde{\eta} = \int_{a_{\rm end}}^{a} \frac{da}{Ha^2} \approx -\frac{1}{aH}$$

- So we can rewrite this as

$$\ddot{u} + [k^2 - \frac{2}{\tilde{\eta}^2}]u = 0$$

Quantum Fluctuations

- Simple harmonic oscillator $\ll$ Hubble length

$$\ddot{u} + k^2 u = 0$$

- Quantize the simple harmonic oscillator

$$\hat{u} = u(k,\tilde{\eta})\hat{a} + u^*(k,\tilde{\eta})\hat{a}^\dagger$$

where $u(k,\tilde{\eta})$ satisfies classical equation of motion and the creation and annihilation operators satisfy

$$[a, a^\dagger] = 1, \qquad a|0\rangle = 0$$

- Normalize wavefunction $[\hat{u}, d\hat{u}/d\tilde{\eta}] = i$

$$u(k,\eta) = \frac{1}{\sqrt{2k}} e^{-ik\tilde{\eta}}$$

Quantum Fluctuations

- Zero point fluctuations of ground state

$$
\begin{aligned}
\langle u^2 \rangle &= \langle 0|u^\dagger u|0\rangle \\
&= \langle 0|(u^* \hat{a}^\dagger + u\hat{a})(u\hat{a} + u^* \hat{a}^\dagger)|0\rangle \\
&= \langle 0|\hat{a}\hat{a}^\dagger|0\rangle |u(k,\tilde{\eta})|^2 \\
&= \langle 0|[\hat{a},\hat{a}^\dagger] + \hat{a}^\dagger \hat{a}|0\rangle |u(k,\tilde{\eta})|^2 \\
&= |u(k,\tilde{\eta})|^2 = \frac{1}{2k}
\end{aligned}
$$

- Classical equation of motion take this quantum fluctuation outside horizon where it freezes in.

Slow Roll Limit

- Classical equation of motion then has the exact solution

$$u = \frac{1}{\sqrt{2k}}\left(1 - \frac{i}{k\tilde\eta}\right) e^{-ik\tilde\eta}$$

- For $|k\tilde\eta| \ll 1$ (late times, $\gg$ Hubble length) fluctuation freezes in

$$\lim_{|k\tilde\eta|\to 0} u = -\frac{1}{\sqrt{2k}}\frac{i}{k\tilde\eta} \approx \frac{iHa}{\sqrt{2k^3}}$$

$$\phi_1 = \frac{iH}{\sqrt{2k^3}}$$

- Power spectrum of field fluctuations

$$\Delta^2_{\phi_1} = \frac{k^3|\phi_1|^2}{2\pi^2} = \frac{H^2}{(2\pi)^2}$$