Set 2:

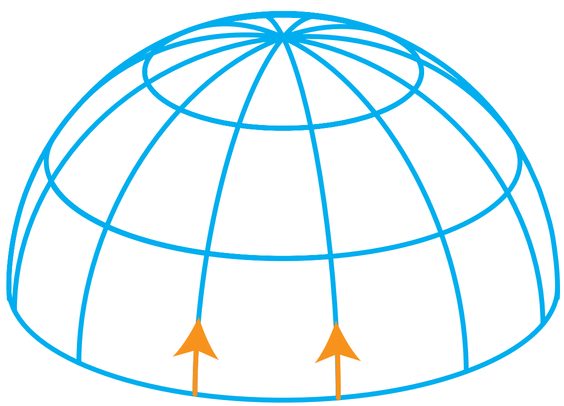
Cosmic Geometry

Newton vs Einstein

- Even though locally Newtonian gravity is an excellent approximation to General Relativity, in cosmology we deal with spatial and temporal scales across which the global picture benefits from a basic understanding of General Relativity.
- An example is: as we have seen in the previous set of notes, it is much more convenient to think of the space between galaxies expanding rather than galaxies receding through space
- While latter is a good description locally, its preferred coordinates place us at center and does not allow us to talk about distances beyond which galaxies are receding faster than light - though these distances as we shall see are also not directly observable
- To get a global picture of the expansion of the universe we need to think geometrically, like Einstein not Newton
- Best of both: think globally (Einstein), act locally (Newton)

Gravity as Geometry

- Einstein says Gravity as a force is really the geometry of spacetime
- Force between objects is a fiction of geometry - imagine the curved space of the 2-sphere - e.g. the surface of the earth
- Two people walk from equator to pole on lines of constant longitude
- Intersect at poles as if an attractive force exists between them
- Both walk on geodesics or straight lines of the shortest distance



Gravity as Geometry

- General relativity has two aspects
  - – A metric theory: geometry tells matter how to move
  - – Field equations: matter tells geometry how to curve
- Metric defines distances or separations in the spacetime and freely falling matter moves on a path that extremizes the distance
- Expansion of the universe carries two corresponding pieces
  - – Friedmann-Robertson-Walker geometry or metric tells matter, including light, how to move – allows us to chart out the expansion with light
  - – Matter content of the universe tells it how to expand – allows us to infer the components of the universe
- Useful to separate out these two pieces both conceptually and for understanding alternate cosmologies

FRW Geometry

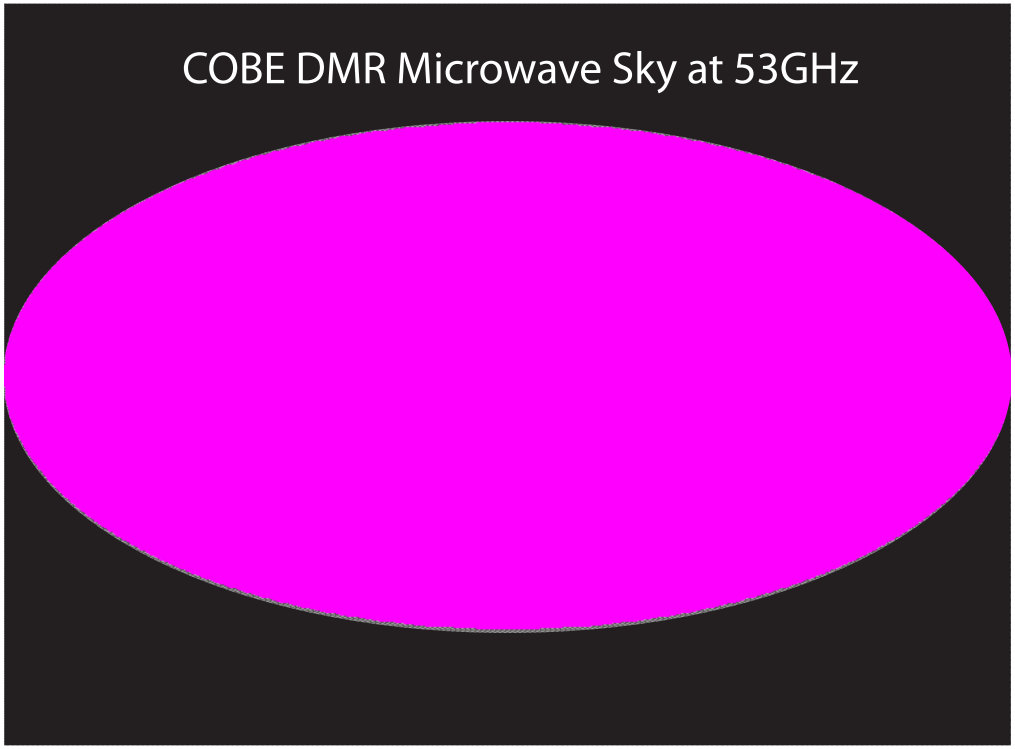
- FRW geometry = homogeneous and isotropic on large scales
- Universe observed to be nearly isotropic (e.g. CMB, radio point sources, galaxy surveys)
- Copernican principle: we're not special, must be isotropic to all observers (all locations)

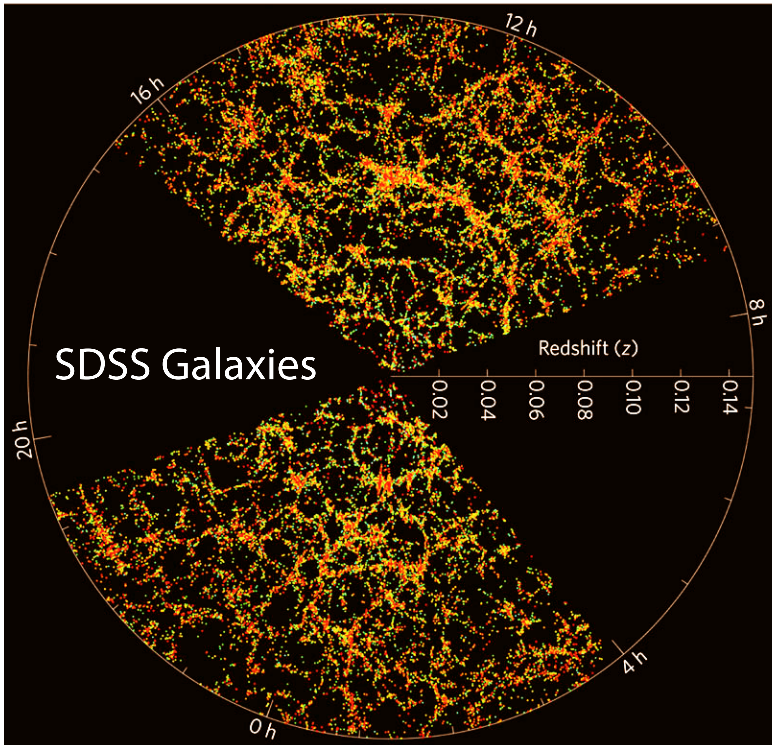
  Implies homogeneity

  Verified through galaxy redshift surveys
- FRW cosmology (homogeneity, isotropy & field equations) generically implies the expansion of the universe, except for special unstable cases

Isotropy & Homogeneity

- Isotropy: CMB isotropic to 10^{-3}, 10^{-5} if dipole subtracted
- Redshift surveys show return to homogeneity on the $>$100Mpc scale





FRW Geometry

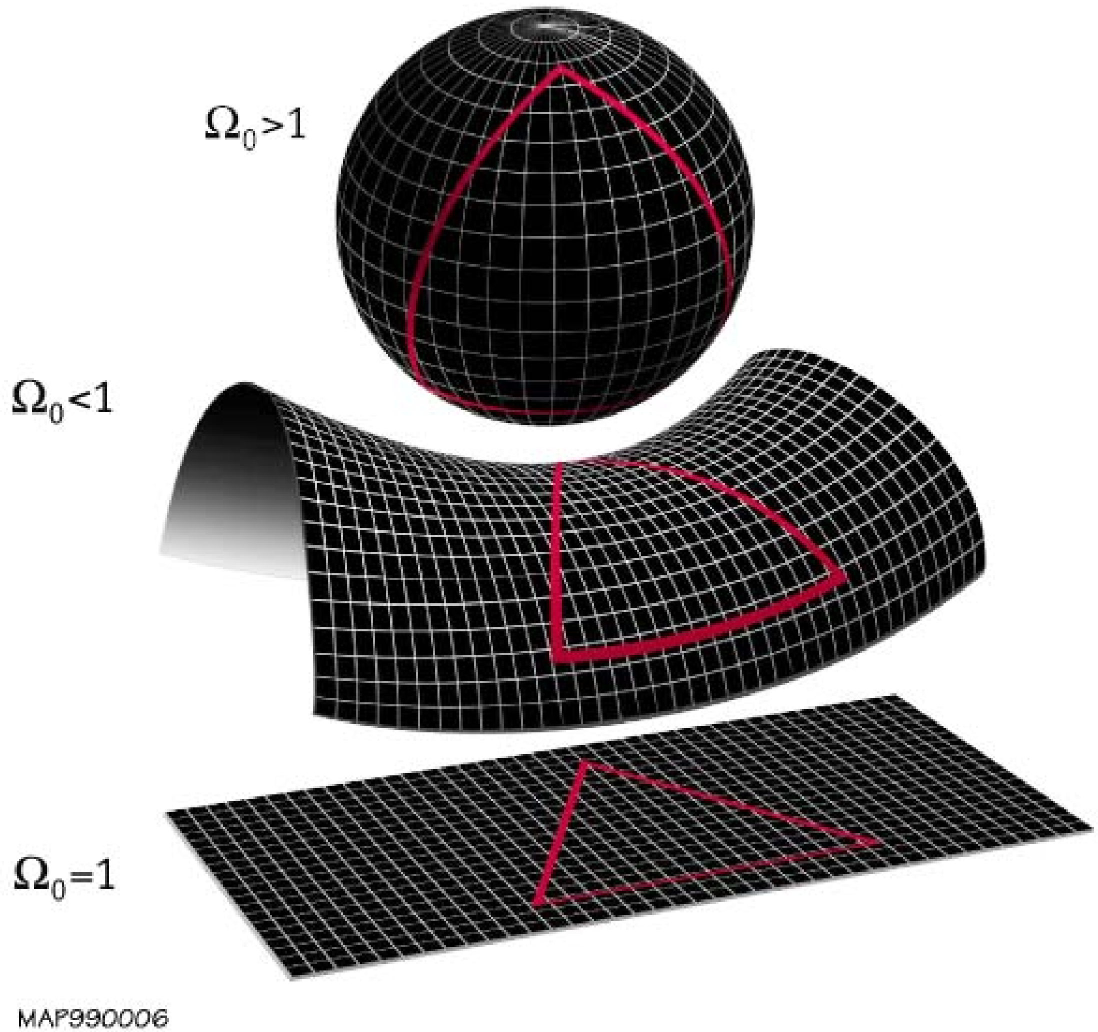
- Spatial geometry is that of constant curvature

  Positive: sphere

  Negative: saddle

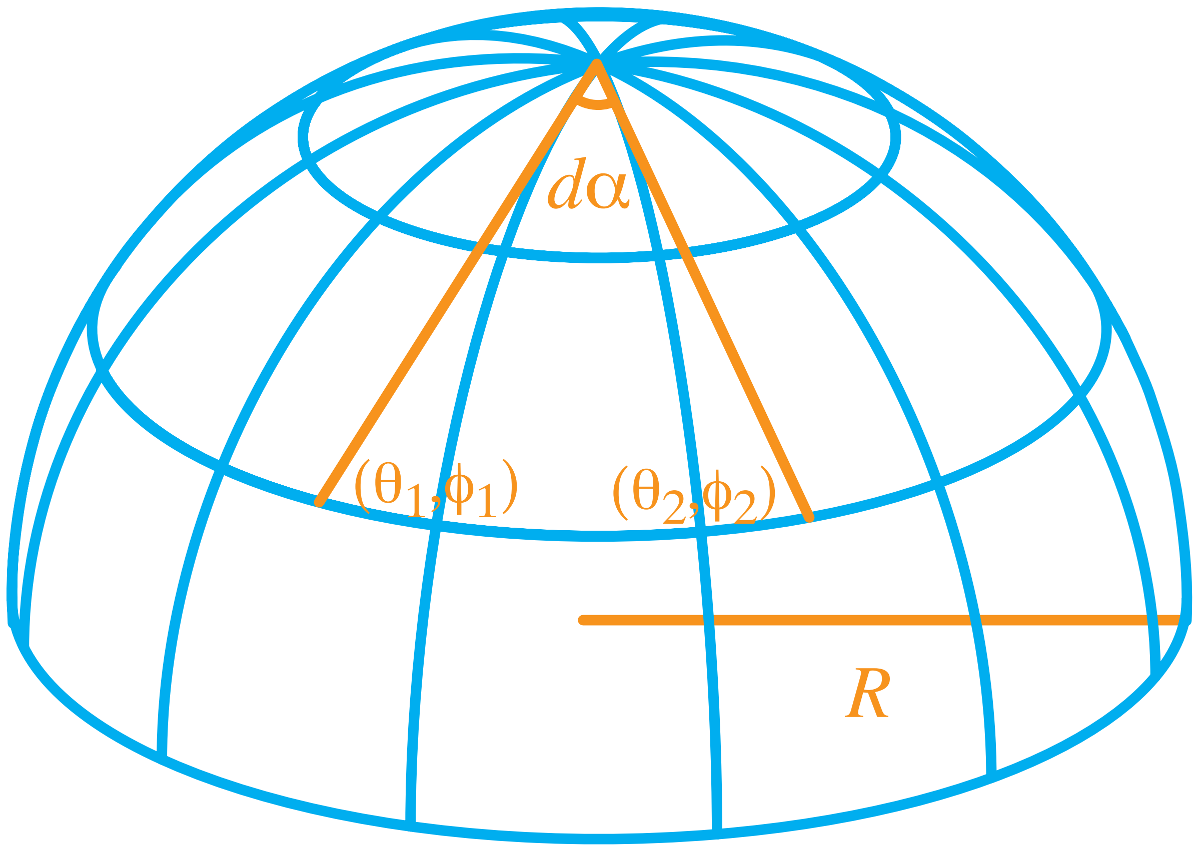
  Flat: plane

- Metric tells us how to measure distances on this surface



FRW Geometry

- Closed: sphere of radius R and (real) curvature $K = 1/R^2$
- Suppress 1 dimension α represents total angular separation between two points on the sky (θ_1, ϕ_1) and (θ_2, ϕ_2)



FRW Geometry

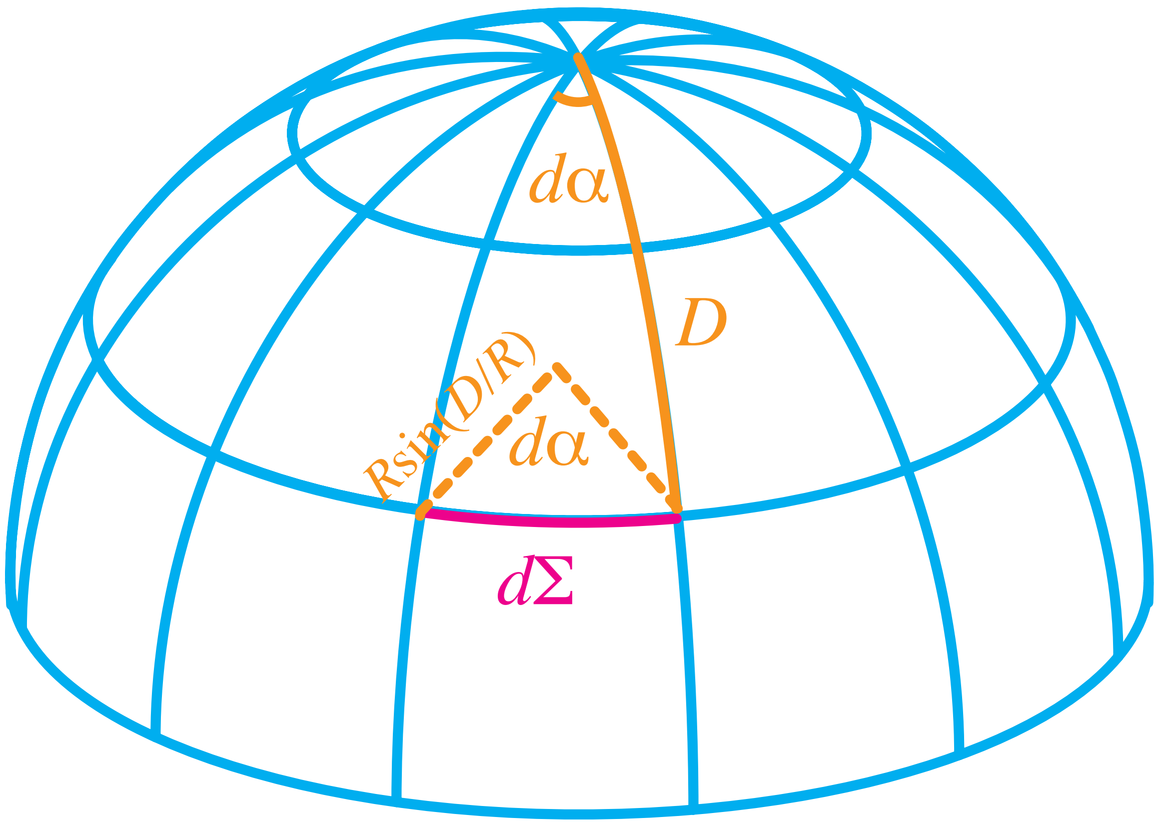
- Geometry tells matter how to move: take (null) geodesic motion for light along this generalized sense of longitude or radial distance D
- This arc distance is the distance our photon traveler sees
- We receive light from two different trajectories as observer at pole
- Compared with our Euclidean expectation that the angle between the rays should be related to the separation at emission Σ as $d\alpha \approx \Sigma/D$ the angular size appears larger because of the "lensing" magnification of the background
- This leads to the so called angular diameter distance - the most relevant sense of distance for the observer
- In General Relativity, we are free to use any distance coordinate we like but the two have distinct uses

FRW Geometry

- To define the angular diameter distance, look for a D_A such that

$$d\Sigma = D_A d\alpha$$

Draw a circle at the distance D, its radius is $D_A = R\sin(D/R)$



FRW Geometry

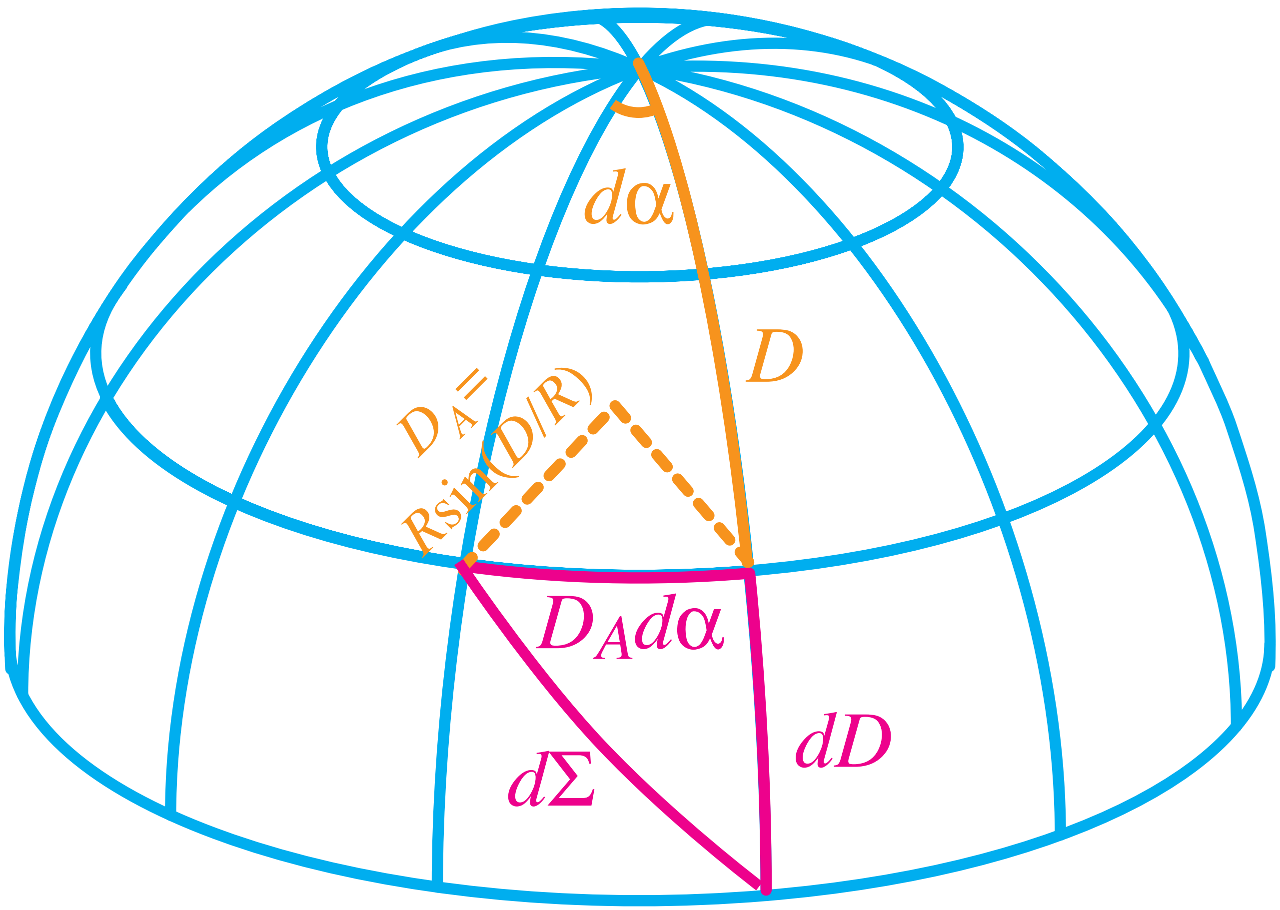
- Angular diameter distance
- Positively curved geometry $D_A < D$ and objects are further than they appear
- Negatively curved universe R is imaginary and

  $$R \sin(D/R) = i|R| \sin(D/i|R|) = |R| \sinh(D/|R|)$$

  and $D_A > D$ objects are closer than they appear
- Flat universe, $R \to \infty$ and $D_A = D$

FRW Geometry

- Now add that point 2 may have a different radial distance
- What is the distance $d\Sigma$ between points 1 (θ_1, ϕ_1, D_1) and point 2 (θ_2, ϕ_2, D_2), separated by $d\alpha$ in angle and dD in distance?



Angular Diameter Distance

- For small angular and radial separations, space is nearly flat so that the Pythagorean theorem holds for differentials

$$d\Sigma^2 = dD^2 + D_A^2 d\alpha^2$$

- Now restore the fact that the angular separation can involve two angles on the sky - the curved sky is just a copy of the spherical geometry with unit radius that we were suppressing before

$$\begin{aligned} d\Sigma^2 &= dD^2 + D_A^2 d\alpha^2 \\ &= dD^2 + D_A^2 (d\theta^2 + \sin^2\theta d\phi^2) \end{aligned}$$

- D_A useful for describing observables (flux, angular positions)
- D useful for theoretical constructs (causality, relationship to temporal evolution)

Alternate Notation

- Aside: line element is often also written using D_A as the coordinate distance

$$dD_A^2 = \left(\frac{dD_A}{dD}\right)^2 dD^2$$

$$\left(\frac{dD_A}{dD}\right)^2 = \cos^2(D/R) = 1 - \sin^2(D/R) = 1 - (D_A/R)^2$$

$$dD^2 = \frac{1}{1-(D_A^2/R)^2} dD_A^2$$

and defining the curvature $K = 1/R^2$ the line element becomes

$$d\Sigma^2 = \frac{1}{1 - D_A^2 K} dD_A^2 + D_A^2(d\theta^2 + \sin^2\theta d\phi^2)$$

where $K < 0$ for a negatively curved space

Line Element or Metric Uses

- Metric also defines the volume element

$$
\begin{aligned}
dV &= (dD)(D_A d\theta)(D_A \sin\theta d\phi) \\
&= D_A^2 dD d\Omega
\end{aligned}
$$

  where $d\Omega = \sin\theta d\theta d\phi$ is solid angle

- Most of classical cosmology boils down to these three quantities, (comoving) radial distance, (comoving) angular diameter distance, and volume element

- For example, distance to a high redshift supernova, angular size of the horizon at last scattering and BAO feature, number density of clusters...

Comoving Coordinates

- Remaining degree of freedom (preserving homogeneity and isotropy) is the temporal evolution of overall scale factor
- Relates the geometry (fixed by the radius of curvature R) to physical coordinates – a function of time only

$$d\sigma^2 = a^2(t)d\Sigma^2$$

  our conventions are that the scale factor today $a(t_0) \equiv 1$

- Similarly physical distances are given by $d(t) = a(t)D$, $d_A(t) = a(t)D_A$.
- Distances in upper case are comoving; lower, physical

  Comoving coordinates do not change with time and

  Simplest coordinates to work out geometrical effects

Time and Conformal Time

- Spacetime separation (with $c = 1$)

$$\begin{aligned} ds^2 &= -dt^2 + d\sigma^2 \\ &= -dt^2 + a^2(t) d\Sigma^2 \end{aligned}$$

- Taking out the scale factor in the time coordinate

$$ds^2 \equiv a^2(t)\left(-d\eta^2 + d\Sigma^2\right)$$

$d\eta = dt/a$ defines conformal time – useful in that photons travelling radially from observer on null geodesics $ds^2 = 0$

$$\Delta D = \Delta\eta = \int \frac{dt}{a}$$

so that time and distance may be interchanged

FRW Metric

- Aside for advanced students: Relationship between coordinate differentials and space-time separation defines the metric $g_{\mu\nu}$

$$ds^2 \equiv g_{\mu\nu} dx^\mu dx^\nu = a^2(\eta)(-d\eta^2 + d\Sigma^2)$$

  Einstein summation - repeated lower-upper pairs summed

- Usually we will use comoving coordinates and conformal time as the x^μ unless otherwise specified – metric for other choices are related by $a(t)$

- Aside: scale factor plays the role of a conformal rescaling (which preserves spacetime “angles”, i.e. light cone and causal structure - hence conformal time)

Horizon

- Distance travelled by a photon in the whole lifetime of the universe defines the horizon
- Since $ds = 0$, the horizon is simply the elapsed conformal time

$$D_{\text{horizon}}(t) = \int_0^t \frac{dt'}{a} = \eta(t)$$

- Horizon always grows with time
- Always a point in time before which two observers separated by a distance D could not have been in causal contact
- Horizon problem: why is the universe homogeneous and isotropic on large scales especially for objects seen at early times, e.g. CMB, when horizon small

Special vs. General Relativity

- From our class perspective, the big advantage of comoving coordinates and conformal time is that we have largely reduced general relativity to special relativity
- In these coordinates, aside from the difference between D and D_A, we can think of photons propagating in flat spacetime
- Now let's relate this discussion to observables
- Rule of thumb to avoid dealing with the expansion directly:
  - Convert from physical quantities to conformal-comoving quantities at emission
  - In conformal-comoving coordinates, light propagates as usual
  - At reception $a = 1$, conformal-comoving coordinates are physical, so interpret as usual

Hubble Parameter

- Useful to define the expansion rate or Hubble parameter

$$H(t) \equiv \frac{1}{a}\frac{da}{dt} = \frac{d\ln a}{dt}$$

fractional change in the scale factor per unit time - $\ln a = N$ is also known as the e-folds of the expansion

- Cosmic time becomes

$$t = \int dt = \int \frac{d\ln a}{H(a)}$$

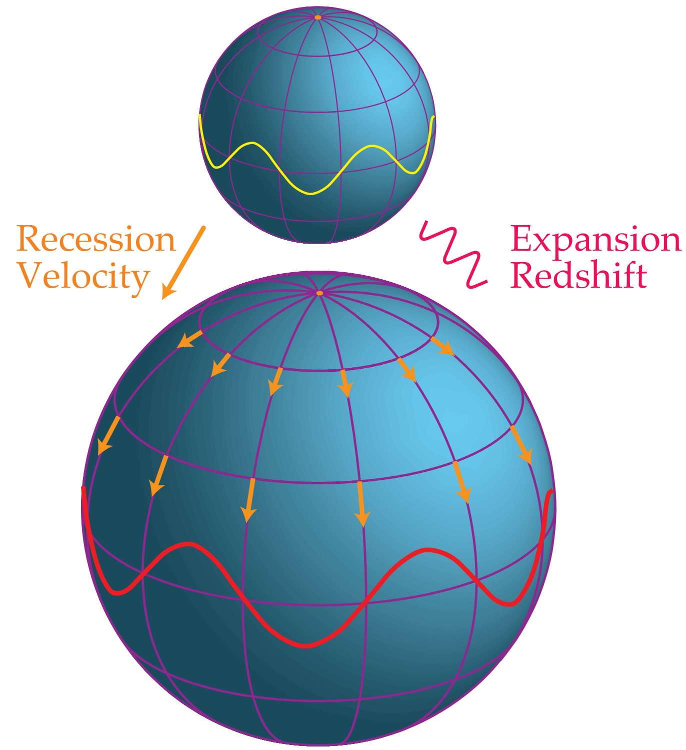
- Conformal time becomes

$$\eta = \int \frac{dt}{a} = \int \frac{d\ln a}{aH(a)}$$

- Advantageous since conservation laws give matter evolution with a; $a = (1+z)^{-1}$ is a direct observable...

Redshift

- Wavelength of light "stretches" with the scale factor
- The physical wavelength $\lambda_{\rm emit}$ associated with an observed wavelength today $\lambda_{\rm obs}$ (or comoving=physical units today) is



$$\frac{\lambda_{\rm emit}}{\lambda_{\rm obs}} = \frac{a(t_{\rm emit})}{a(t_{\rm obs})} = a(t_{\rm emit})$$

so that the redshift of spectral lines measures the scale factor of the universe at t, $1+z=1/a$.

- Interpreting the redshift as a Doppler shift, objects recede in an expanding universe

Distance-Redshift Relation

- Given atomically known rest wavelength $\lambda_{\rm emit}$, redshift can be precisely measured from spectra
- Combined with a measure of distance, distance-redshift $D(z) \equiv D(z(a))$ can be inferred - given that photons travel $D = \Delta\eta$ this tells us how the scale factor of the universe evolves with time.
- Related to the expansion history as

$$D(a) = \int dD = \int_a^1 \frac{d\ln a'}{a' H(a')}$$

$$[d\ln a' = -d\ln(1+z) = -a' dz]$$

$$D(z) = -\int_z^0 \frac{dz'}{H(z')} = \int_0^z \frac{dz'}{H(z')}$$

Hubble Law

- Note limiting case is the Hubble law

$$\lim_{z \to 0} D(z) = z/H(z=0) \equiv z/H_0$$

independently of the geometry and expansion dynamics

- Hubble constant usually quoted as as dimensionless h

$$H_0 = 100h\,\mathrm{km\,s^{-1}Mpc^{-1}}$$

- Observationally $h \sim 0.7$ (see below)
- With $c = 1$, $H_0^{-1} = 9.7778\,(h^{-1}$ Gyr) defines the time scale (Hubble time, $\sim$ age of the universe)
- As well as $H_0^{-1} = 2997.9\,(h^{-1}$ Mpc) a length scale (Hubble scale $\sim$ Horizon scale)

Standard Ruler

- Standard Ruler: object of known physical size λ
- Let's apply our rule of thumb: at emission the comoving size is Λ :

$$\lambda = a(t)\Lambda$$

Now everything about light is normal: the object of comoving size Λ subtends an observed angle α on the sky α

$$\alpha = \frac{\Lambda}{D_A(z)}$$

- This is the easiest way of thinking about it. But we could also define an effective physical distance $d_A(z)$ which corresponds to what we would infer in a non expanding spacetime

$$\alpha \equiv \frac{\lambda}{d_A(z)} = \frac{a\Lambda}{d_A(z)} = \frac{\Lambda}{D_A(z)} \to d_A(z) = aD_A(z) = \frac{D_A(z)}{1+z}$$

Standard Ruler

- Since $D_A \to D_A(D_{\rm horizon})$ whereas $(1+z)$ unbounded, angular size of a fixed physical scale at high redshift actually increases with radial distance
- Paradox: the further away something is, the bigger it appears
  - Easily resolved by thinking about comoving coordinates - a fixed physical scale λ as the universe shrinks and $a \to 0$ will eventually encompass the whole observable universe out to the horizon in comoving coordinates so of course it subtends a big angle on the sky!
  - But there are no such bound objects in the early universe - there is no causal way such bigger-than-the-horizon objects could form
- Knowing λ or Λ and measuring α and z allows us to infer the comoving angular diameter distance $D_A(z)$

Standard Candle

- Standard Candle: objects of same luminosity L, measured flux F
- Apply rules again: at emission in conformal-comoving coordinates
  - L is the energy per unit time at emission
  - Since $E \propto \lambda^{-1}$ and comoving wavelength $\Lambda \propto \lambda/a$ so comoving energy $\mathcal{E} \propto \Lambda^{-1} \propto aE$
  - Per unit time at emission $\Delta t = a\Delta\eta$ in conformal time
  - So observed luminosity today is $\mathcal{L} = \mathcal{E}/\Delta\eta = a^2 L$
  - All photons must pass through the sphere at a given distance, so the comoving surface area is $4\pi D_A^2$
- Put this together to the observed flux at $a = 1$

$$F = \frac{\mathcal{L}}{4\pi D_A^2} = \frac{L}{4\pi D_A^2}\frac{1}{(1+z)^2}$$

Notice the flux is diminished by two powers of $(1+z)$

Luminosity Distance

- We can again define a physical "luminosity" distance that corresponds to our non-expanding spacetime intuition

$$F \equiv \frac{L}{4\pi d_L^2}$$

- So luminosity distance

$$d_L = (1+z)D_A = (1+z)^2 d_A$$

- As $z \to 0$, $d_L = d_A = D_A$
- But as $z \to \infty$, $d_L \gg d_A$ - key to understanding Olber's paradox

Olber's Paradox Redux

- Surface brightness - object of physical size λ

$$S = \frac{F}{\Delta\Omega} = \frac{L}{4\pi d_L^2}\frac{d_A^2}{\lambda^2}$$

- In a non-expanding geometry (regardless of curvature), surface brightness is conserved $d_A = d_L$

$$S = \text{const.}$$

  – each site line in universe full of stars will eventually end on surface of star, night sky should be as bright as sun (not infinite)

- In an expanding universe

$$S \propto (1+z)^{-4}$$

Olber's Paradox Redux

- Second piece: age finite so even if stars exist in the early universe, not all site lines end on stars
- But even as age goes to infinity and the number of site lines goes to 100%, surface brightness of distant objects (of fixed physical size) goes to zero
  - Angular size increases
  - Redshift of "luminosity" i.e. energy and arrival time dilation

Measuring $D(z)$

- Astro units side: since flux ratios are very large in cosmology, its more useful to take the log

$$m_1 - m_2 = -2.5 \log_{10}(F_1/F_2)$$

  related to d_L by definition by inverse square law

$$m_1 - m_2 = 5 \log_{10}[d_L(z_1)/d_L(z_2)]$$

- To quote in terms of a single object, introduce absolute magnitude as the magnitude that would be measured for the object at 10 pc

$$m - M = 5 \log_{10}[d_L(z)/10\mathrm{pc}]$$

  Knowing absolute magnitude is the same as knowing the absolute distance, otherwise distances are relative

Measuring $D(z)$

- If absolute magnitude unknown, then both standard candles and standard rulers measure relative sizes and fluxes – ironically this means that measuring the change in H is easier than measuring H_0 – acceleration easier than rate!

  For standard candle, e.g. compare magnitudes low z_0 to a high z object - using the Hubble law $d_L(z_0) = z_0/H_0$ we have

$$\Delta m = m_z - m_{z_0} = 5\log_{10}\frac{d_L(z)}{d_L(z_0)} = 5\log_{10}\frac{H_0 d_L(z)}{z_0}$$

  Likewise for a standard ruler comparison at the two redshifts

$$\frac{d_A(z)}{d_A(z_0)} = \frac{H_0 d_A(z)}{z_0}$$

- Distances are measured in units of h^{-1} Mpc.

Measuring $D(z)$

- Since z is a direct observable, in both cases $H_0 D_A(z)$ is the measured quantity
- We can relate that back to $H_0 D(z)$ recalling that

$$H_0 D_A = H_0 R \sin(H_0 D / H_0 R)$$

and use h^{-1} Mpc as the unit for all lengths – furthermore, local observations are at distances much smaller than R so $H_0 D_A = H_0 D$ is a good approximation

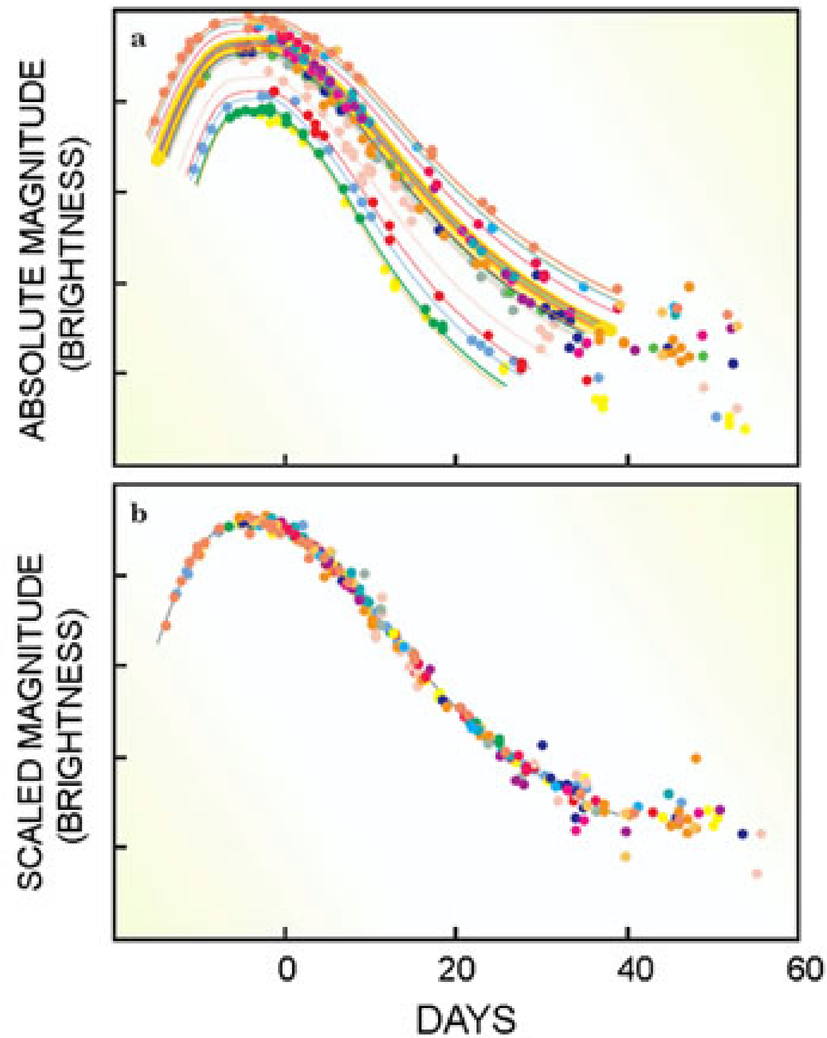
- Then since $D(z) = \int dz/H(z)$ we have

$$H_0 D(z) = \int dz \frac{H_0}{H(z)}$$

- Fundamentally our low to high z comparison tells us the change in expansion rate $H(z)/H_0$

Supernovae as Standard Candles

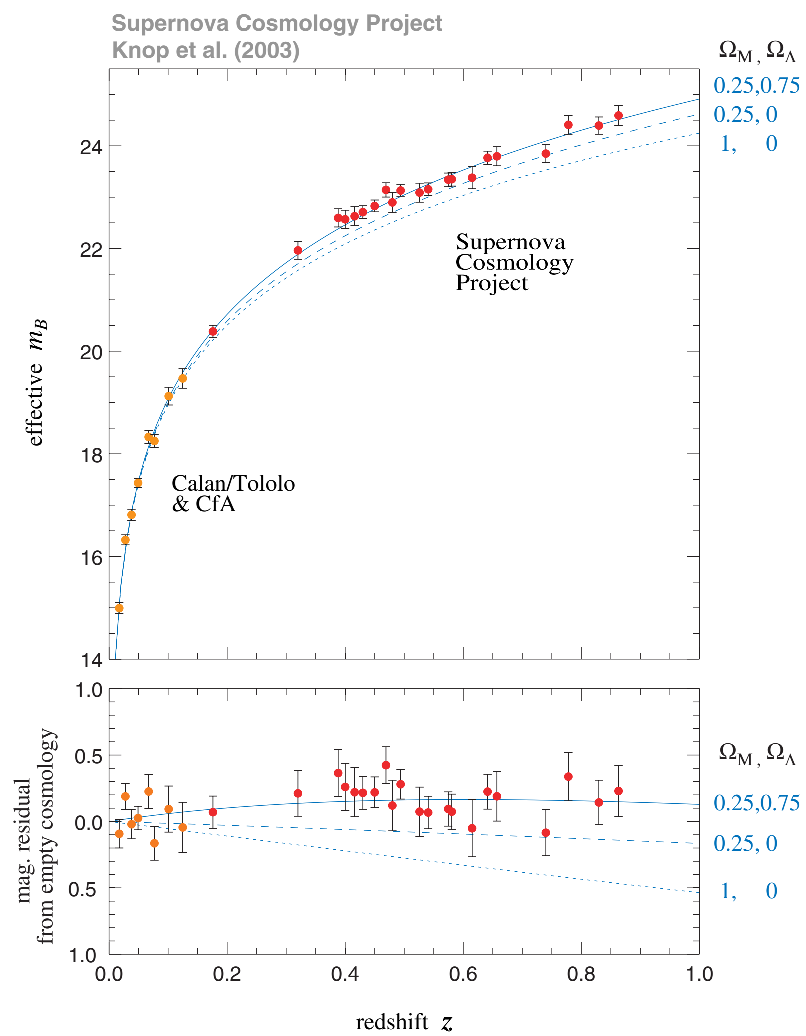
- Type 1A supernovae are white dwarfs that reach Chandrashekar mass where electron degeneracy pressure can no longer support the star, hence a very regular explosion
- Moreover, the scatter in absolute magnitude is correlated with the shape of the light curve - the rate of decline from peak light, empirical "Phillips relation"



- Higher ^{56}N, brighter SN, higher opacity, longer light curve duration

Beyond Hubble's Law

- Type 1A are therefore "standardizable" candles leading to a very low scatter $\delta m \sim 0.15$ and visible out to high redshift $z \sim 1$
- Two groups in 1999 found that SN more distant at a given redshift than expected
- Cosmic acceleration



Acceleration of the Expansion

- Using SN as a relative indicator (independent of absolute magnitude), comparison of low and high z gives

$$H_0 D(z) = \int dz \frac{H_0}{H}$$

  more distant implies that $H(z)$ not increasing at expect rate, i.e. is more constant

- Take the limiting case where $H(z)$ is a constant (a.k.a. de Sitter expansion)

$$\begin{aligned}
H &= \frac{1}{a}\frac{da}{dt} = \text{const} \\
\frac{dH}{dt} &= \frac{1}{a}\frac{d^2a}{dt^2} - H^2 = 0 \\
\frac{1}{a}\frac{d^2a}{dt^2} &= H^2 > 0
\end{aligned}$$

Acceleration of the Expansion

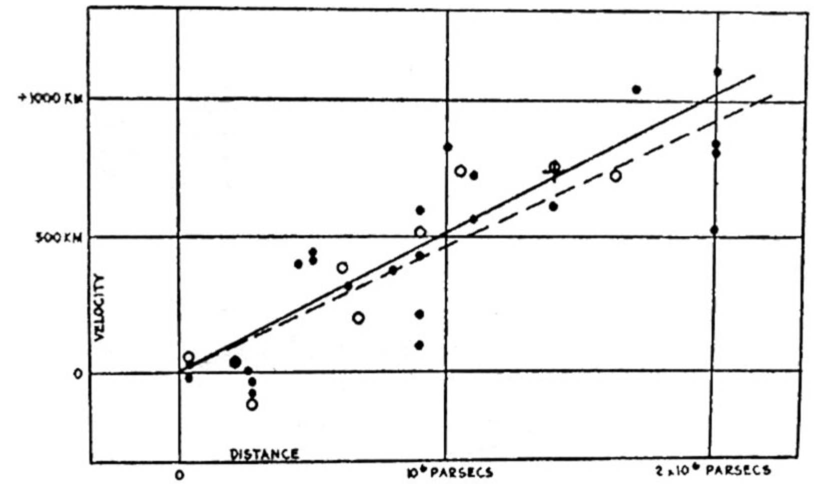
- Indicates that the expansion of the universe is accelerating
- Intuition tells us (FRW dynamics shows) ordinary matter decelerates expansion since gravity is attractive
- Ordinary expectation is that

$$H(z > 0) > H_0$$

  so that the Hubble parameter is higher at high redshift

  Or equivalently that expansion rate decreases as it expands
- Notice that this a purely geometric inference and does not yet say anything about what causes acceleration – topic of next set of lectures on cosmic dynamics

Hubble Constant



- Getting H_0 itself is harder since we need to know the absolute distance d_L to the objects: $H_0 = z_0/d_L$
- Hubble actually inferred too large a Hubble constant of $H_0 \sim 500$km/s/Mpc
- Miscalibration of the Cepheid distance scale - absolute measurement hard, checkered history
- Took 70 years to settle on this value with a factor of 2 discrepancy persisting until late 1990's - which is after the projects which discovered acceleration were conceived!
- H_0 now measured as 73.48 ± 1.66km/s/Mpc by SHOES calibrating off AGN water maser

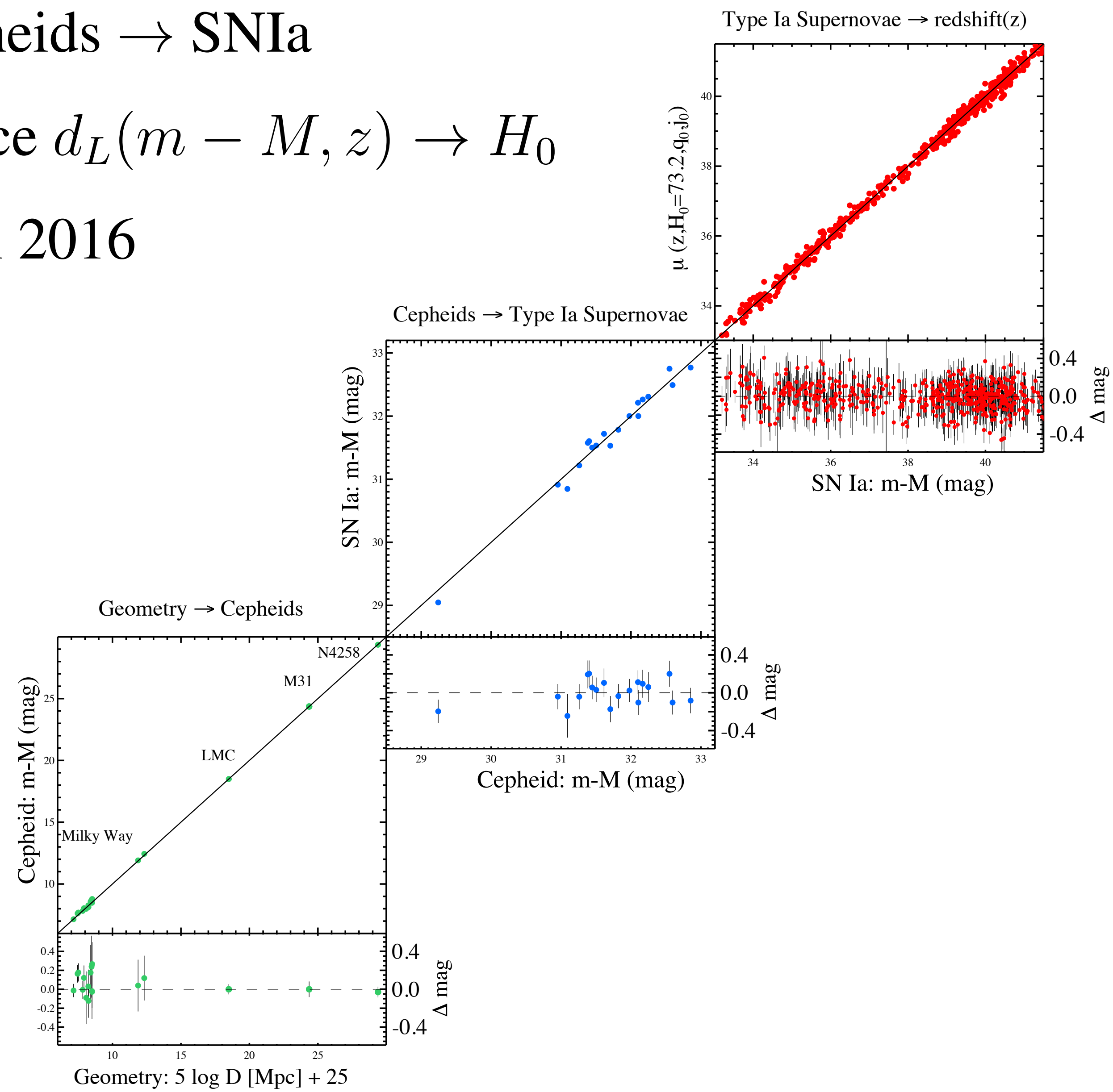
Hubble Constant History

- Difficult measurement since local galaxies have peculiar motions and so their velocity is not entirely due to the "Hubble flow"
- A "distance ladder" of cross calibrated measurements
- Primary distance indicators cepheids, novae planetary nebula, tip of red giant branch, or globular cluster luminosity function, AGN water maser
- Use more luminous secondary distance indications to go out in distance to Hubble flow

    Tully-Fisher, fundamental plane, surface brightness fluctuations, Type 1A supernova

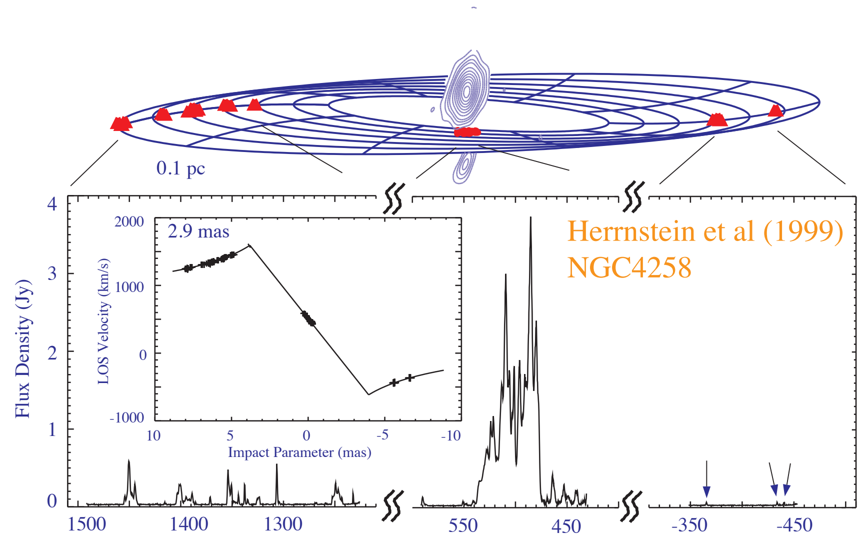
Modern Distance Ladder

- Geometry → Cepheids → SNIa
- Luminosity distance $d_L(m - M, z) \to H_0$
- SH0ES, Riess et al 2016



Maser-Cepheid-SN Distance Ladder

- Water maser around AGN, gas in Keplerian orbit
- Measure proper motion, radial velocity, acceleration of orbit



- Method 1: radial velocity plus orbit infer tangential velocity = distance × angular proper motion

$$v_t = d_A (d\alpha/dt)$$

- Method 2: centripetal acceleration and radial velocity from line infer physical size

$$a = v^2/R, \qquad R = d_A \theta$$

Maser-Cepheid-SN Distance Ladder

- Calibrate Cepheid period-luminosity relation in same galaxy
- SHOES project then calibrates SN distance in galaxies with Cepheids

  Also: consistent with recent HST parallax determinations of 10 galactic Cepheids (8% distance each) with $\sim 20\%$ larger H_0 error bars - normal metalicity as opposed to LMC Cepheids.

- Measure SN at even larger distances out into the Hubble flow
- Riess et al (2019) $H_0 = 74.03 \pm 1.42$ km/s/Mpc more precise (1.9%) than the HST Key Project calibration (11%).
- As of Spring 2019, this differs from the CMB/BAO distance ladder $H_0 = 67.66 \pm 0.42$ working from high redshifts at 4σ.....