Set 8:

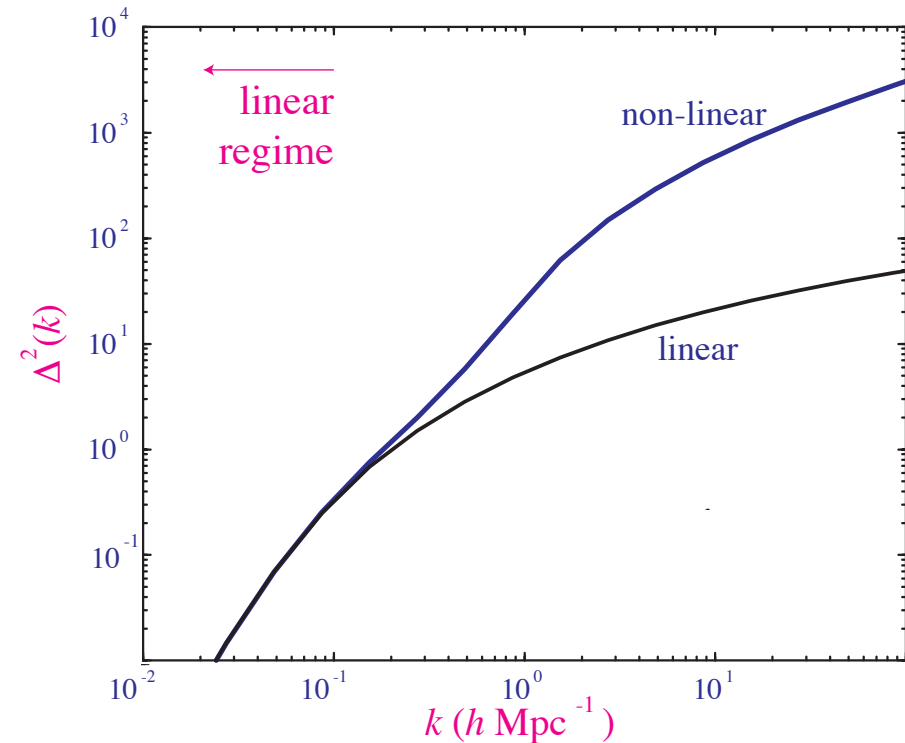## Nonlinear Structure

Nonlinear regime

- Inflationary initial perturbations provide density perturbations $\delta = \delta\rho/\rho$ that grow as $\delta \propto a$ in the linear regime

- $\Delta^2 = k^3 P(k)/2\pi^2$ contribution to variance $\langle \delta^2 \rangle$ per $d \ln k$

- Linear theory would predict that for $k > 0.1h\text{Mpc}^{-1}$, $\langle \delta^2 \rangle > 1$.

- Linear approximation breaks down at this point and we must follow the nonlinear equations

- Nonlinearities further enhance the formation of structure

Cosmological Simulations

- To evolve structure further requires cosmological simulations (see video)

- Many simulation results can be understood using simple analytic arguments

- High density fluctuations break away from the cosmological expansion and form bound objects called dark matter halos

- Halo formation can be understood in the spherical collapse model as an FRW background expansion in a slightly closed (positive curvature) universe

- If observable properties such as galaxies (stars) and gas can be associated with halos, then they can be modeled in a "halo model"
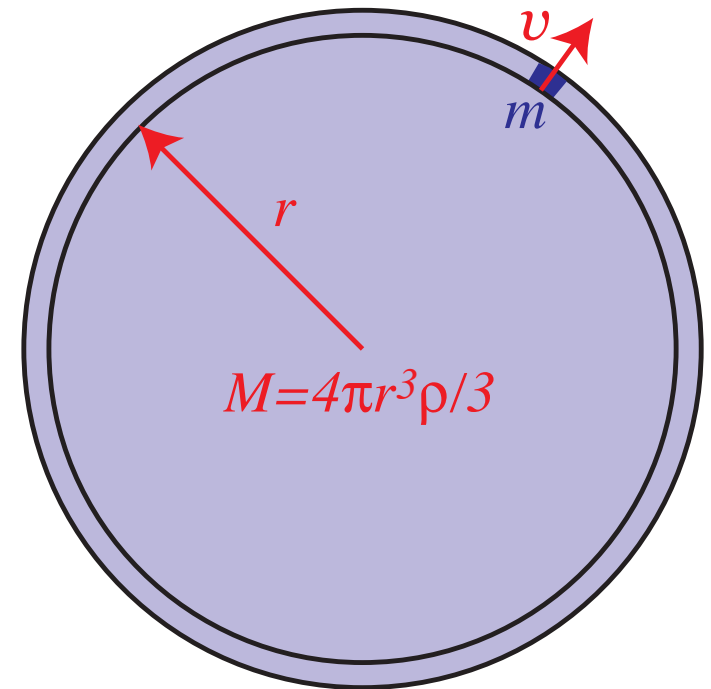
Newtonian Cosmology

- Recall that in the Newtonian interpretation of the expansion we can model the dynamics of a spherical volume of constant density using mass conservation

- Energy conservation

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r} = \text{const}$$

$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2 - \frac{GM}{r} = \text{const}$$

$$\frac{1}{2}\left(\frac{1}{r}\frac{dr}{dt}\right)^2 - \frac{GM}{r^3} = \frac{\text{const}}{r^2}$$

$$H^2 = \frac{8\pi G\rho}{3} - \frac{\text{const}}{a^2}$$

$M=4\pi r^3\rho/3$

v

m

r

Newtonian Energy Interpretation

- Constant determines whether the system recollapses or expands forever

- These equations define the evolution of not just the homogeneous cosmology but also a spherically symmetric "top hat" or spatially constant density perturbation in a matter dominated universe

- An overdense region will eventually collapse and form a "dark matter halo" just like a positive curvature universe
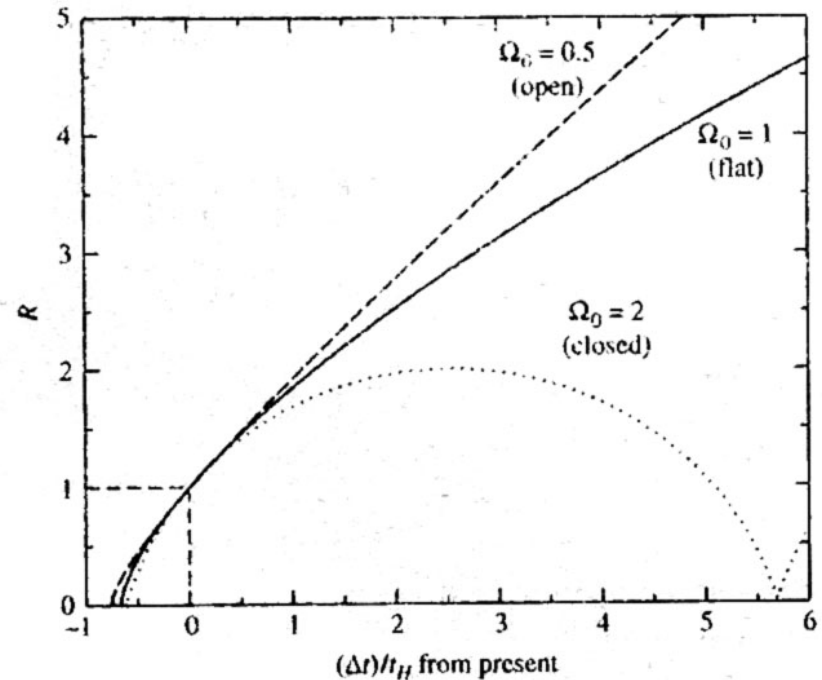
Closed Universe

- Friedmann equation in a closed universe $K > 0$

$$H^2 + \frac{K}{a^2} = \frac{8\pi G}{3}\rho$$

when $K/a^2 = 8\pi G\rho/3$ then $H = 0$ and the expansion of the universe turns around into recollapse. In cosmo parameters

$$\frac{1}{a}\frac{da}{dt} = H_0 \left(\Omega_m a^{-3} - (\Omega_m - 1)a^{-2}\right)^{1/2}$$

this would occur when $a = \Omega_m/(\Omega_m - 1)$

- Now consider an initially small local overdensity

$$\rho(\mathbf{x}, \mathbf{t}) = \bar{\rho}(t)[1 + \delta(\mathbf{x}, t)]$$

in a globally $\Omega_m = 1$ universe- a local observer would see this as a slightly closed universe with an $\Omega_m > 1$

Spherical Collapse

- Spherical collapse calculation makes use of this remapping by matching the initial density perturbations to the local expansion parameters and determining the epoch of collapse

- Basic idea: mass M enclosed by the region $r = ar_0$ remains constant so solve for $r(t)$ and match to the initial density perturbation δ_i at a_i

- Completing this step (details below) we have the following parametric solution

$$
\begin{aligned}
r(\theta) &= \frac{3}{10} \frac{r_i}{\delta_i} (1 - \cos \theta) \\[2ex]
t(\theta) &= \frac{1}{2 H_0 \Omega_m^{1/2}} \left(\frac{3}{5} \frac{a_i}{\delta_i} \right)^{3/2} (\theta - \sin \theta)
\end{aligned}
$$

where θ is the development angle with $\theta = \pi$ the turnaround point, $\theta = 2\pi$ the collapse point

Spherical Collapse Relations

- Scale factor $a \propto t^{2/3}$

$$a = \left(\frac{3}{4}\right)^{2/3} \left(\frac{3}{5}\frac{a_i}{\delta_i}\right) (\theta - \sin\theta)^{2/3}$$

- At collapse $\theta = 2\pi$

$$a_{\mathrm{col}} = \left(\frac{3}{4}\right)^{2/3} \left(\frac{3}{5}\frac{a_i}{\delta_i}\right) (2\pi)^{2/3} \approx 1.686 \frac{a_i}{\delta_i}$$

- In linear theory and a matter dominated universe $\delta \propto a$ so $\delta_{\mathrm{linear}} = (a_{\mathrm{col}}/a_i)\delta_i = 1.686$ – Rule of thumb to remember – perturbation collapses when linear theory predicts $\delta_c \equiv 1.686$

- Interpretation: when linear theory predicts an $\mathcal{O}(1)$ density perturbation, in the real universe, that density perturbation has already collapsed to a nonlinear object

Spherical Collapse

- Derivation for advanced students: since $r = ar_0$ we can use the Friedmann solution for a in a closed universe to find $r(t)$

- In terms of development angle $\theta = H_0 \eta (\Omega_m - 1)^{1/2}$, scaled conformal time η

$$
\begin{aligned}
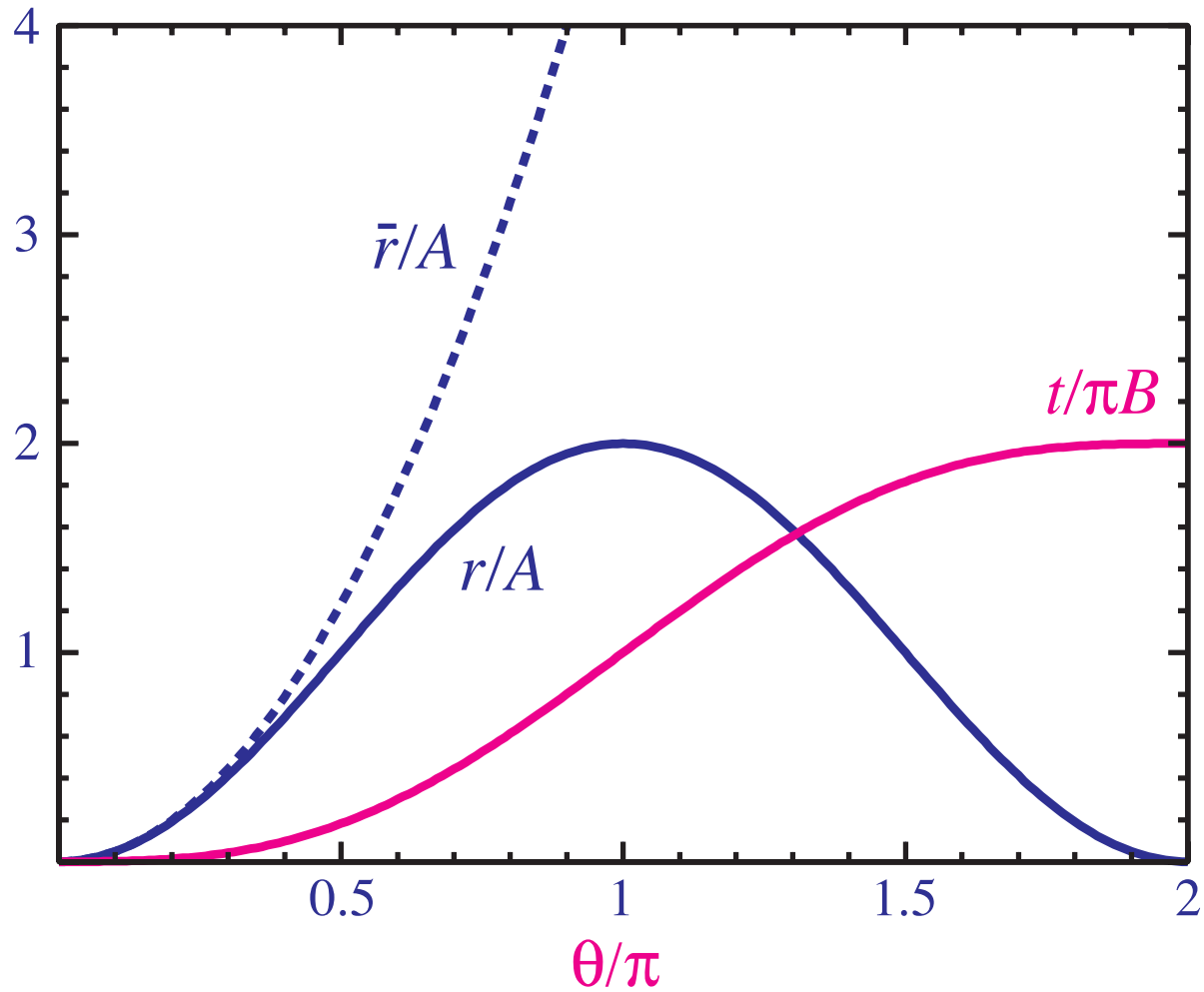r(\theta) &= A(1 - \cos\theta) \\
t(\theta) &= B(\theta - \sin\theta)
\end{aligned}
$$

where $A = r_0 \Omega_m / 2(\Omega_m - 1)$, $B = H_0^{-1} \Omega_m / 2(\Omega_m - 1)^{3/2}$.

- Turn around at $\theta = \pi$, $r = 2A$, $t = B\pi$.

- Collapse at $\theta = 2\pi$, $r \to 0$, $t = 2\pi B$

- Now we need to find the A and B constants given an initial density perturbation

Spherical Collapse

- Parametric Solution:

Correspondence

- Eliminate cosmological correspondence in A and B in terms of enclosed mass M

$$M = \frac{4\pi}{3} r_0^3 \Omega_m \rho_c = \frac{4\pi}{3} r_0^3 \Omega_m \frac{3H_0^2}{8\pi G}$$

- Related as $A^3 = GMB^2$, and to initial perturbation

$$\lim_{\theta \to 0} r(\theta) = A\left(\frac{1}{2}\theta^2 - \frac{1}{24}\theta^4\right)$$

$$\lim_{\theta \to 0} t(\theta) = B\left(\frac{1}{6}\theta^3 - \frac{1}{120}\theta^5\right)$$

- Leading Order: $r = A\theta^2/2$, $t = B\theta^3/6$

$$r = \frac{A}{2}\left(\frac{6t}{B}\right)^{2/3}$$

Next Order

- Leading order is unperturbed matter dominated expansion
  $r \propto a \propto t^{2/3}$

- Iterate r and t solutions

$$\lim_{\theta \to 0} t(\theta) = \frac{\theta^3}{6} B \left[1 - \frac{1}{20} \left(\frac{6t}{B} \right)^{2/3} \right]$$

$$\theta \approx \left(\frac{6t}{B} \right)^{1/3} \left[1 + \frac{1}{60} \left(\frac{6t}{B} \right)^{2/3} \right]$$

Next Order

- Substitute back into $r(\theta)$

$$
\begin{aligned}
r(\theta) &= A\frac{\theta^2}{2}\left(1 - \frac{\theta^2}{12}\right) \\
&= \frac{A}{2}\left(\frac{6t}{B}\right)^{2/3}\left[1 - \frac{1}{20}\left(\frac{6t}{B}\right)^{2/3}\right] \\
&= \frac{1}{2}(6t)^{2/3}(GM)^{1/3}\left[1 - \frac{1}{20}\left(\frac{6t}{B}\right)^{2/3}\right]
\end{aligned}
$$

Density Correspondence

- Density

$$\rho_m = \frac{M}{\frac{4}{3}\pi r^3}$$

$$= \frac{1}{6\pi t^2 G}\left[1 + \frac{3}{20}\left(\frac{6t}{B}\right)^{2/3}\right]$$

- Density perturbation

$$\delta \equiv \frac{\rho_m - \bar{\rho}_m}{\bar{\rho}_m} \approx \frac{3}{20}\left(\frac{6t}{B}\right)^{2/3}$$

Density Correspondence

- Time → scale factor

$$t = \frac{2}{3H_0\Omega_m^{1/2}}a^{3/2}$$

$$\delta = \frac{3}{20}a\left(4/BH_0\Omega_m^{1/2}\right)^{2/3}$$

- A and B constants → initial cond.

$$B = \frac{1}{2H_0\Omega_m^{1/2}}\left(\frac{3}{5}\frac{a_i}{\delta_i}\right)^{3/2}$$

$$A = \frac{3}{10}\frac{r_i}{\delta_i}$$

- End derivation for advanced student

Virialization

- A real density perturbation is neither spherical nor homogeneous

- Shell crossing if δ_i doesn't monotonically decrease

- Collapse does not proceed to a point but reaches virial equilibrium

$$U \;\; = \;\; -2K, \qquad E = U + K = U(r_{\text{max}}) = \frac{1}{2}U(r_{\text{vir}})$$

$r_{\text{vir}} = \frac{1}{2}r_{\text{max}}$ since $U \propto r^{-1}$. Thus $\theta_{\text{vir}} = \frac{3}{2}\pi$
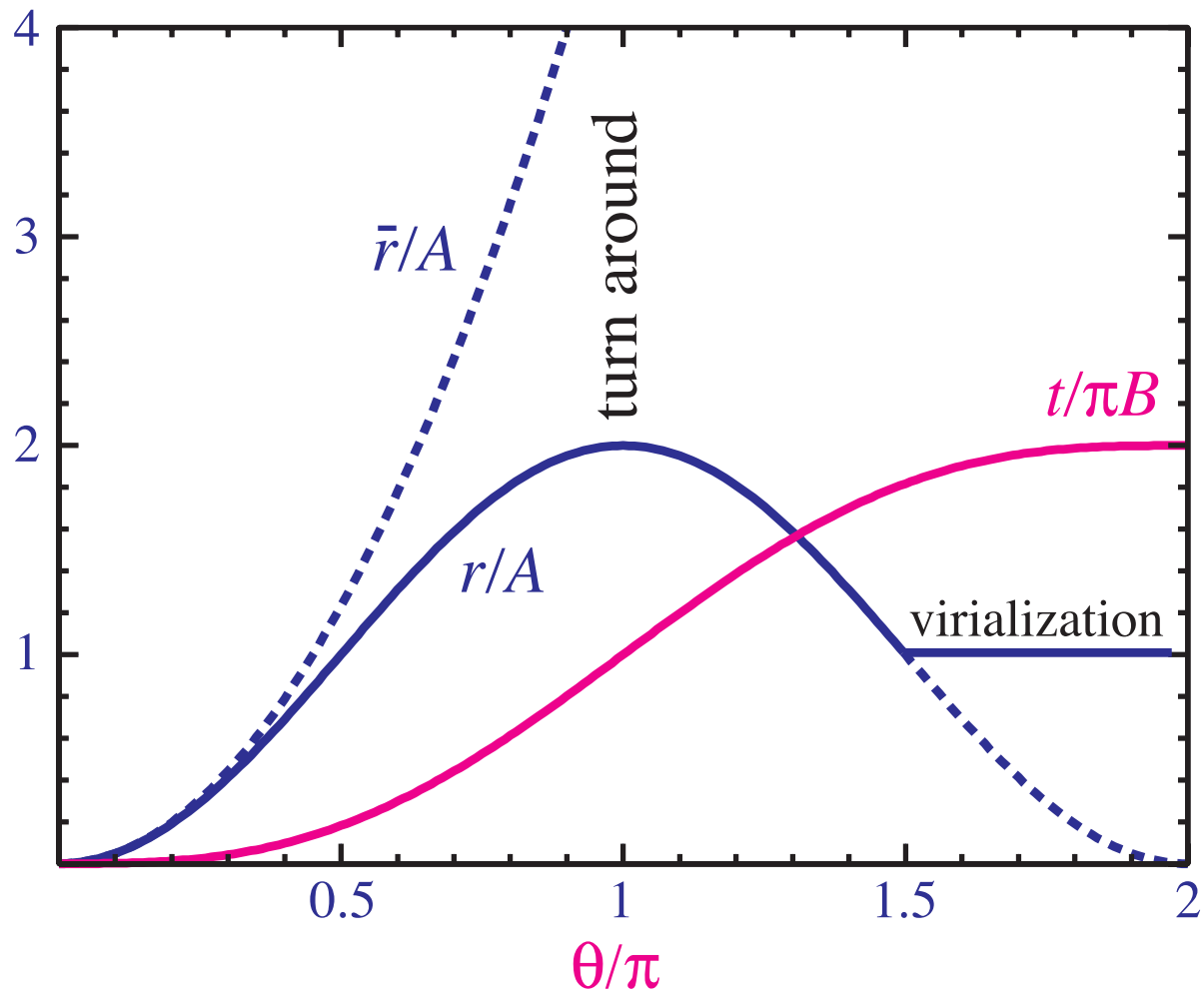
- Overdensity at virialization

$$\frac{\rho_m(\theta = 3\pi/2)}{\bar{\rho}_m(\theta = 2\pi)} = 18\pi^2 \approx 178$$

- Threshold $\Delta_v = 178$ often used to define a collapsed object

- Equivalently relation between virial mass, radius, overdensity:
$M_v = \frac{4\pi}{3}r_v^3 \rho_m \Delta_v$

Virialization

- Schematic Picture:

The Mass Function

- Spherical collapse predicts the end state as virialized halos given an initial density perturbation

- Initial density perturbation is a Gaussian random field

- Compare the variance in the linear density field to threshold $\delta_c = 1.686$ to determine collapse fraction

- Combine to form the mass function, the number density of halos in a range dM around M.

- Halo density defined entirely by linear theory

Press-Schechter Formalism

- **Smooth** linear density density field on mass scale M with tophat

$$R = \left(\frac{3M}{4\pi} \right)^{1/3}$$

- Result is a Gaussian random field with variance $\sigma^2(M)$

- Fluctuations above the threshold δ_c correspond to collapsed regions. The fraction in halos $> M$ becomes

$$\frac{1}{\sqrt{2\pi}\sigma(M)} \int_{\delta_c}^{\infty} d\delta \exp \left(-\frac{\delta^2}{2\sigma^2(M)} \right) = \frac{1}{2}\text{erfc} \left(\frac{\nu}{\sqrt{2}} \right)$$

  where $\nu \equiv \delta_c/\sigma(M)$

- **Problem:** even as $\sigma(M) \to \infty$, $\nu \to 0$, collapse fraction $\to 1/2$ – only overdense regions participate in spherical collapse.

- **Multiply by 2!** Justified by underdensity within overdensity

Press-Schechter Mass Function

- Differentiate in M to find fraction in range dM and multiply by ρ_m/M the number density of halos if all of the mass were composed of such halos $\rightarrow$ differential number density of halos

$$\frac{dn}{d\ln M} = \frac{\rho_m}{M}\frac{d}{d\ln M}\mathrm{erfc}\left(\frac{\nu}{\sqrt{2}}\right)$$

$$= \sqrt{\frac{2}{\pi}}\frac{\rho_m}{M}\frac{d\ln\sigma^{-1}}{d\ln M}\nu\exp(-\nu^2/2)$$

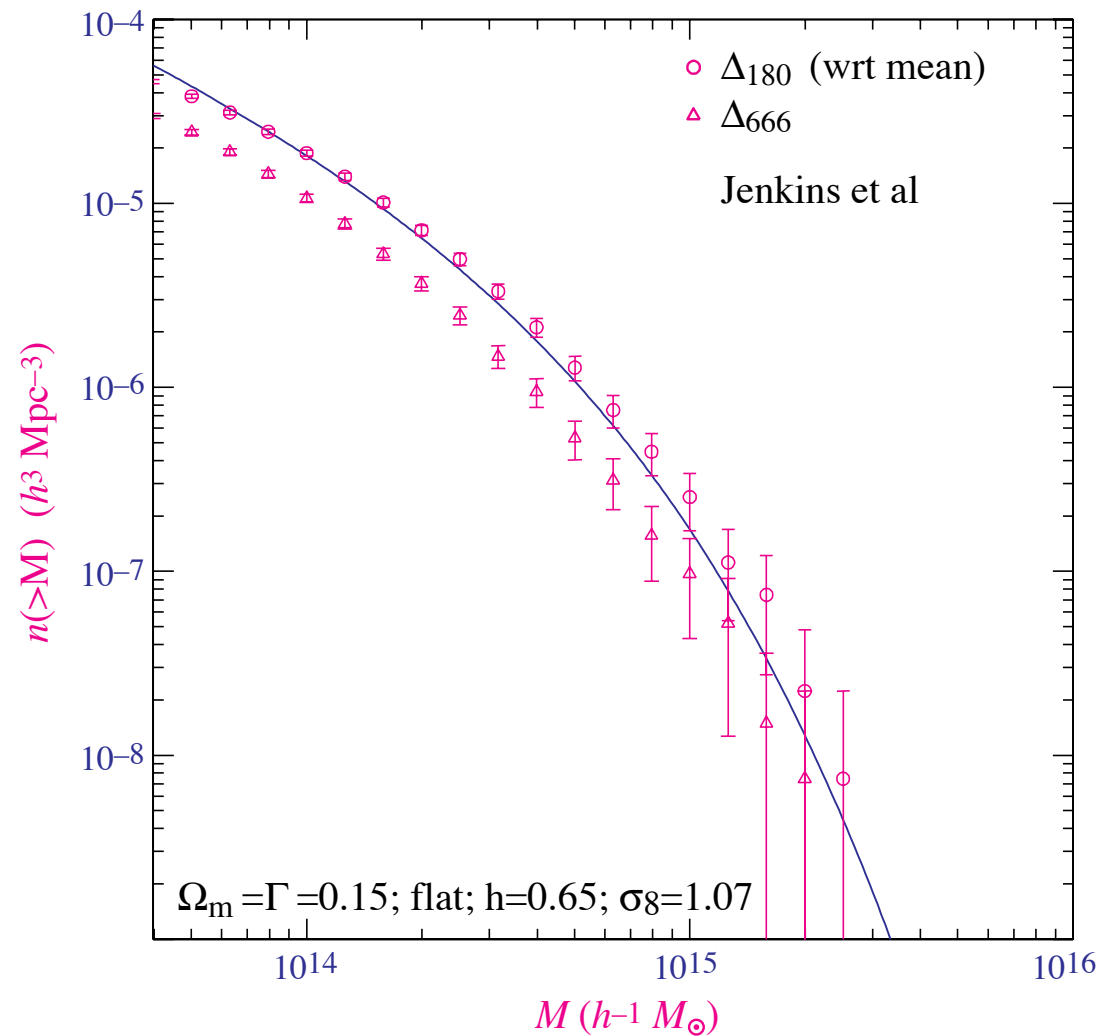- High mass: exponential cut off above M_* where $\sigma(M_*) = \delta_c$

$$M_* \sim 10^{13}h^{-1}M_\odot \quad \text{today}$$

- Low mass divergence: (too many for the observations?)

$$\frac{dn}{d\ln M} \propto\sim M^{-1}$$

Numerical Mass Function

- Fit cosmological simulations to Press Schechter motivated form

Δ_{180} (wrt mean)
Δ_{666}

Jenkins et al

$n(>M)$ $(h^3\ \mathrm{Mpc}^{-3})$

$M\ (h^{-1}\ M_\odot)$

$\Omega_m = \Gamma = 0.15$; flat; h=0.65; σ_8=1.07

Halo Bias

- If halos are formed without regard to the underlying density fluctuation and move under the gravitational field then their number density is an unbiased tracer of the dark matter density fluctuation

$$\left(\frac{\delta n}{n}\right)_{\text{halo}} = \left(\frac{\delta \rho}{\rho}\right)$$

- However spherical collapse says the probability of forming a halo depends on the initial density field

- Large scale density field acts as "background" enhancement of probability of forming a halo or "peak"

- Peak-Background Split (Efstathiou 1998; Cole & Kaiser 1989; Mo & White 1997)

Peak-Background Split

- Schematic Picture:
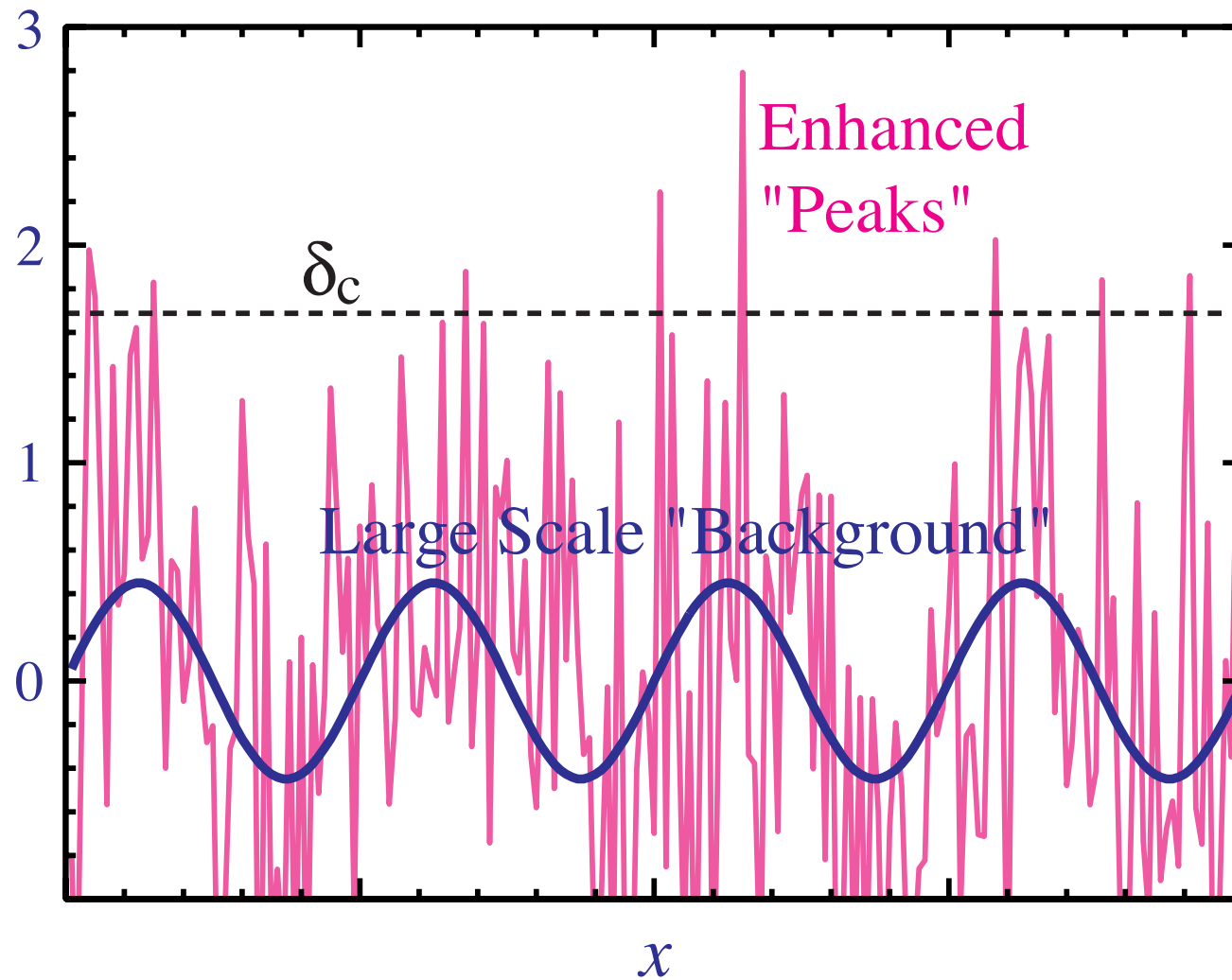
Perturbed Mass Function

- Density fluctuation split

$$\delta = \delta_b + \delta_p$$

- Lowers the threshold for collapse

$$\delta_{cp} = \delta_c - \delta_b$$

so that $\nu = \delta_{cp}/\sigma$

- Taylor expand number density $n_M \equiv dn/d\ln M$

$$n_M + \frac{dn_M}{d\nu}\frac{d\nu}{d\delta_b}\delta_b \ldots = n_M\left[1 + \frac{(\nu^2 - 1)}{\sigma\nu}\delta_b\right]$$

if mass function is given by Press-Schechter

$$n_M \propto \nu \exp(-\nu^2/2)$$

Halo Bias

- Halos are biased tracers of the "background" dark matter field with a bias $b(M)$ that is given by spherical collapse and the form of the mass function

- Combine the enhancement with the original unbiased expectation

$$\frac{\delta n_M}{n_M} = b(M)\delta_b$$

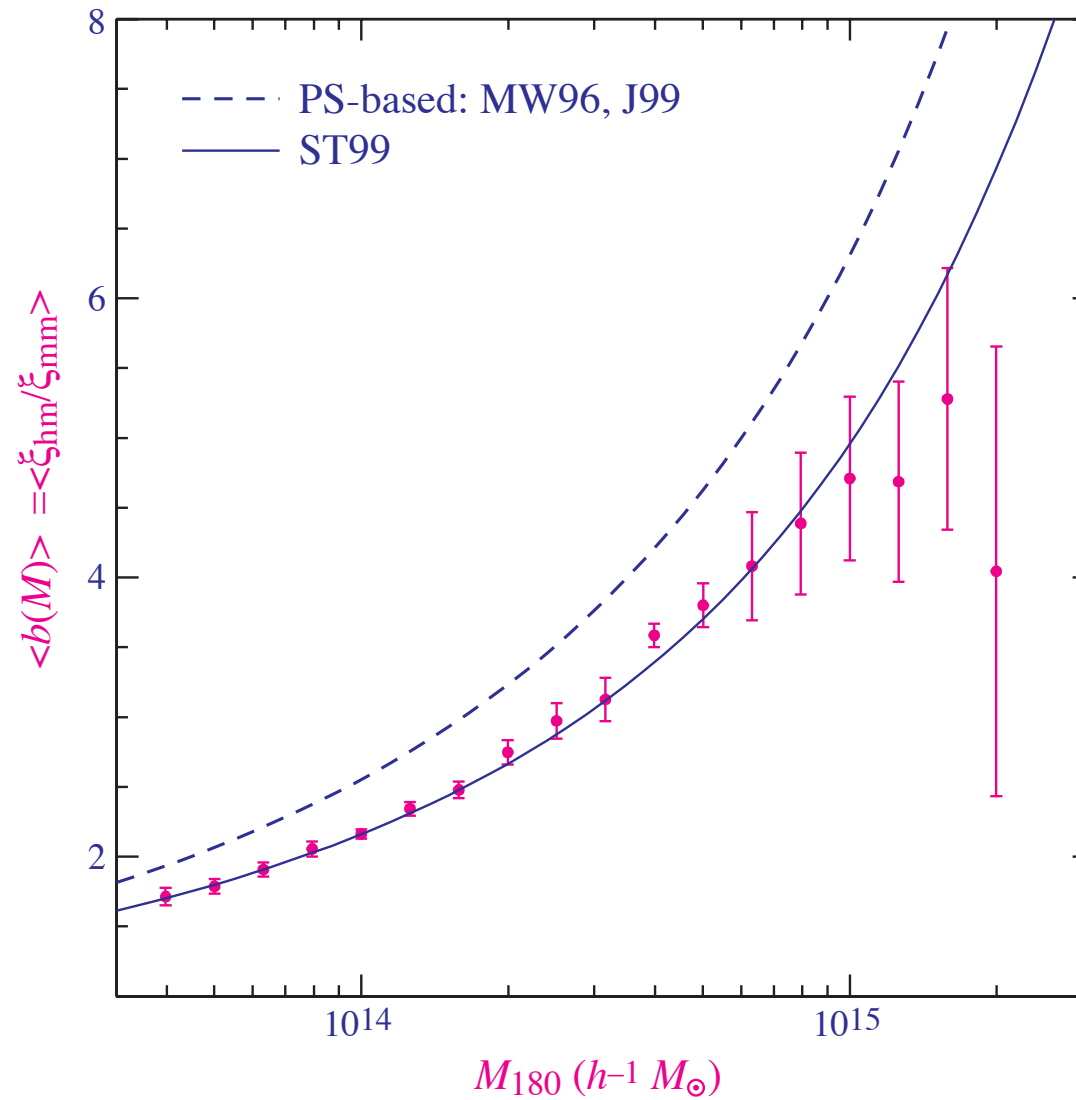- For Press-Schechter

$$b(M) = 1 + \frac{\nu^2 - 1}{\delta_c}$$

- Improved by the Sheth-Torman mass function

$$b(M) = 1 + \frac{a\nu^2 - 1}{\delta_c} + \frac{2p}{\delta_c[1 + (a\nu^2)^p]}$$

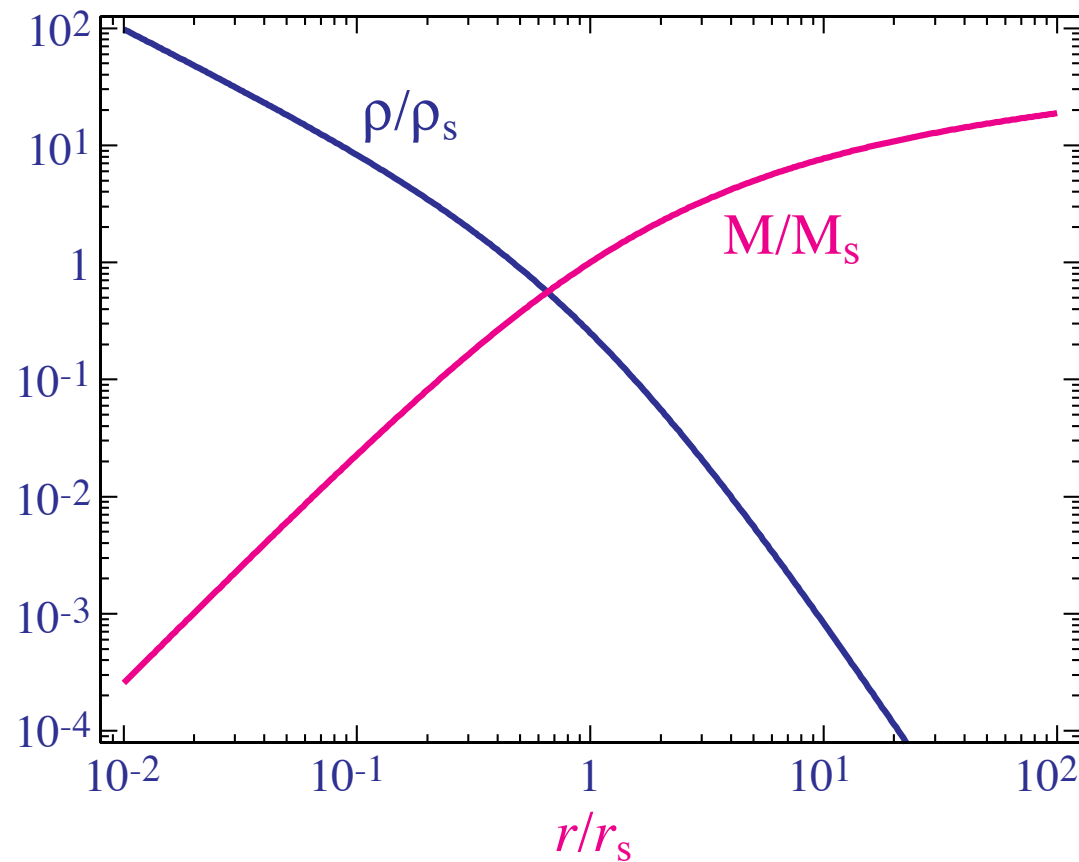with $a = 0.75$ and $p = 0.3$ to match simulations.

Numerical Bias

- Example of halo bias from a simulation

Plot with legend:
- - - PS-based: MW96, J99
—— ST99

Y-axis: $\langle b(M) \rangle = \langle \xi_{hm}/\xi_{mm} \rangle$
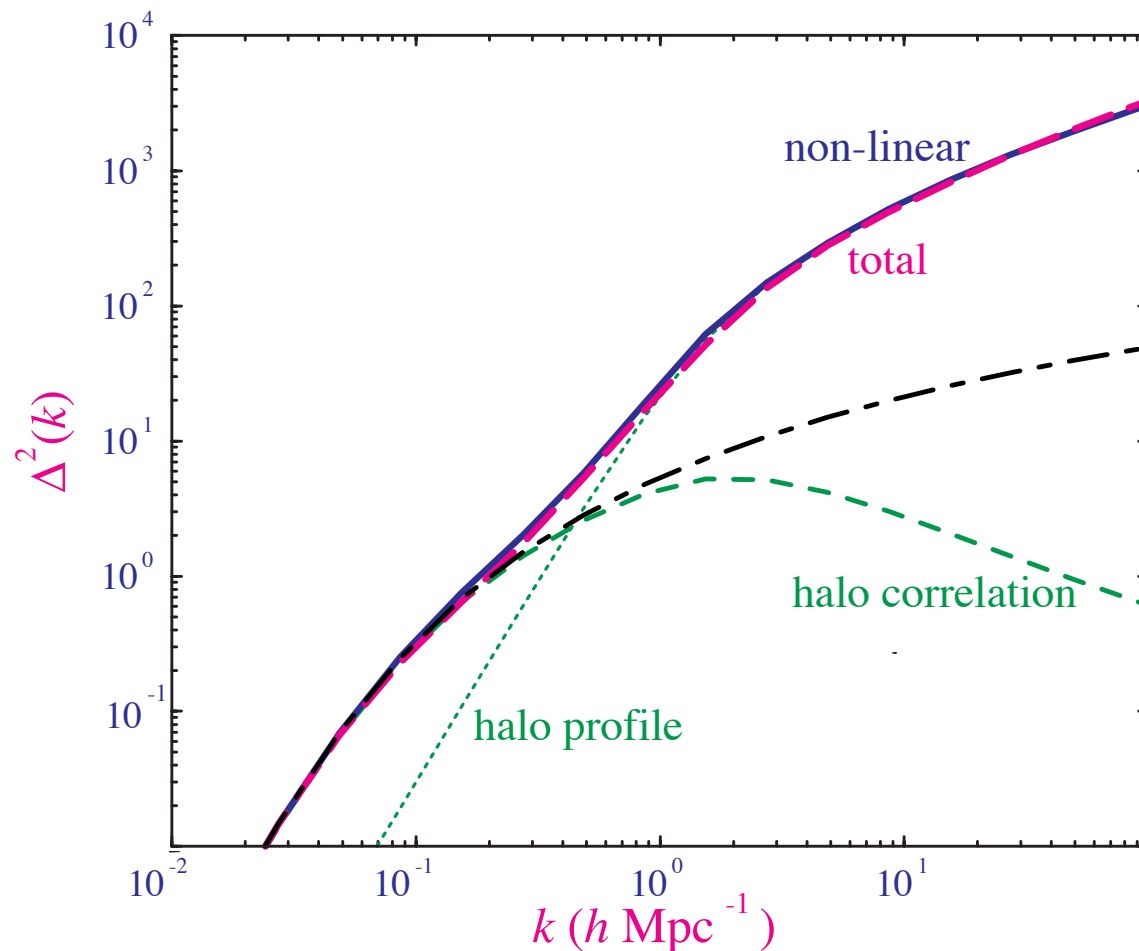
X-axis: M_{180} ($h^{-1} M_\odot$)

NFW Profile

- Density profile well-described by (Navarro, Frenk & White 1997)

$$\rho(r) = \frac{\rho_s}{(r/r_s)(1 + r/r_s)^2}$$

The Halo Model

- NFW halos, of abundance n_M given by mass function, clustered according to the halo bias $b(M)$ and the linear theory $P(k)$

- Power spectrum example:

Galaxy Power Spectrum

- For galaxies, one
  defines a halo occupation
  distribution which
  determines the number
  of galaxies (satisfying
  a certain observational
  criteria) that can occupy a
  halo of mass M

- Galaxy clustering or power spectrum composed from the
  distribution of galaxies in halos and the clustering of halos in large
  scale structure

- Generalize to clustering statistic of any observable that is
  associated with dark matter haloes, e.g. gas, gravitational lensing
  etc.
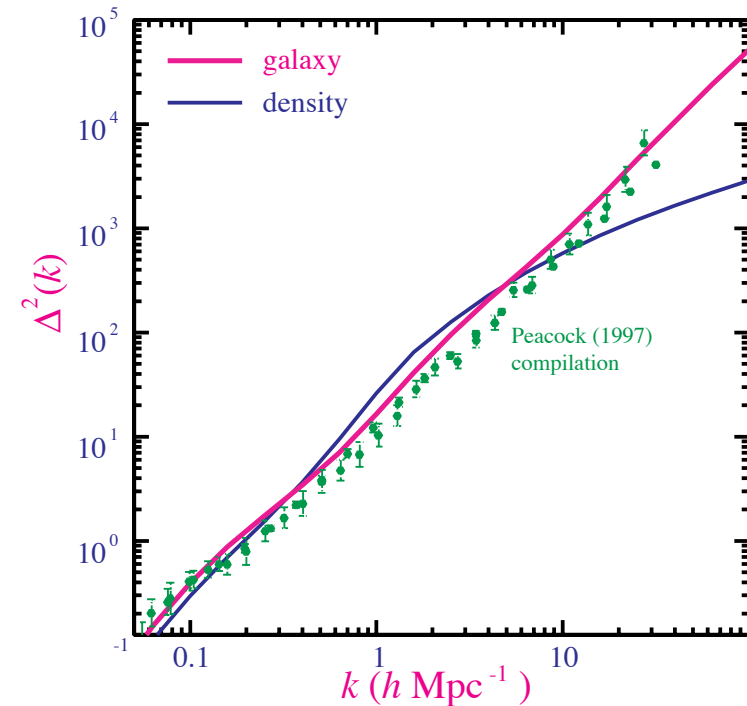
Hierarchical Structure Formation

- With the density power spectrum from linear theory implying that the variance increases as R decreases and time increases, small halos form first

- Exponential suppression of high mass halos starts at low masses and progresses to higher masses as halos merge (see video)

- Small halos are not hot enough to stimulate atomic line transitions and then cool and fragment by radiating

- Stars form in halos only at $z < 10$ and radiation from stars can then reionize the plasma

Galaxy Formation

- Merging proto-Galactic objects of $10^6 - 10^8 M_\odot$ can eventually assemble the galaxies of $10^{12} M_\odot$ we see today. Both lower and upper range determined by cooling.

- Proto-galactic objects can form if cooling is sufficiently rapid that the heating of the gas during collapse (which would prevent collapse due to pressure, internal motions) can be overcome

- Recall virial theorem supplies estimate of thermal kinetic energy

$$-2\langle K \rangle = \langle U \rangle$$

$$-2N\frac{1}{2}\mu m_H \sigma^2 = -\frac{3}{5}\frac{GMN\mu m_H}{R}$$

where μm_H is the average mass of particles in the gas, M is the total mass and σ is the rms velocity

Galaxy Formation

- Solve for velocity dispersion for a self gravitating system

$$\sigma = \left(\frac{3}{5} \frac{GM}{R} \right)^{1/2}$$

- Associate the average kinetic energy with a temperature, called the virial temperature

$$\frac{1}{2} \mu m_H \sigma^2 = \frac{3}{2} k T_{\text{virial}}$$

where μ is the mean molecular weight. Solve for virial temperature

$$T_{\text{virial}} = \frac{\mu m_H \sigma^2}{3k} = \frac{\mu m_H}{5k} \frac{GM}{R} \approx \frac{\mu m_H}{5k} GM^{2/3} \left(\frac{4\pi\rho}{3} \right)^{1/3}$$

- Cooling is a function of the gas temperature through the cooling function.

Galaxy Formation

- Cooling rate (luminosity) per volume

$$r_{\text{cool}} = n^2 \Lambda(T)$$

n^2 (number density squared) comes
from the fact that cooling is usually a
2 body process - for $T > 10^6$K thermal bremsstrahlung and
Compton scattering, for $T \sim 10^4 - 10^5$K from the collisional
excitation of atomic lines of hydrogen and helium
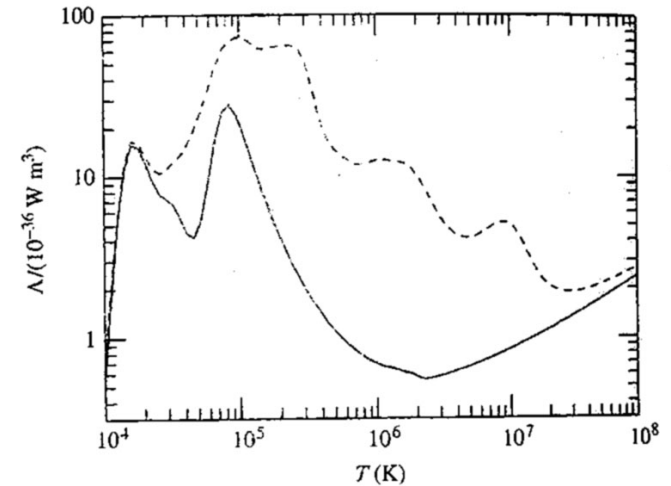
- Galaxy formation only starts when dark matter mass makes the
virial temperture exceed $T \sim 10^4$K when cooling becomes
efficient $M \sim 10^8 M_\odot$ -first objects and current dwarf ellipticals

Galaxy Formation

- Cooling time is the time required to radiate away all of the thermal energy of the gas

$$r_{\text{cool}} V t_{\text{cool}} = \frac{3}{2} N k T_{\text{virial}}$$

$$t_{\text{cool}} = \frac{3}{2} \frac{k T_{\text{virial}}}{n \Lambda}$$

- Compared with the free fall time - from our dimensional relation

$$GM \sim R v^2 \sim R(R^2/t_{\text{ff}}^2), \quad M \propto \rho R^3$$

we get $t_{\text{ff}} \propto (G\rho)^{-1/2}$ with the proportionality given for the time of collapse for a homogeneneous sphere of initial density ρ

$$t_{\text{ff}} = \left(\frac{3\pi}{32} \frac{1}{G\rho} \right)^{1/2}$$

Galaxy Formation

- If $t_{\text{cool}} < t_{\text{ff}}$ then the object will collapse essentially in free fall - fragment and form stars. If opposite, then gravitational potential energy heats the gas making it stabilized by pressure establishing virial equilibrium

$$\left(\frac{t_{\text{ff}}}{t_{\text{cool}}} \right) > \left(\frac{3\pi}{32} \frac{1}{G\rho} \right)^{1/2} \frac{2}{3} \frac{n\Lambda}{kT_{\text{virial}}}$$

- Taking typical numbers $T \sim 10^6 \text{K}$ and $n \sim 5 \times 10^4 \text{m}^{-3}$ and with the density of the collapsing medium being associated with the gas $\rho = \mu m_H n$ gives an upper limit on the gas mass that can cool of $10^{12} M_\odot$ comparable to a large galaxy.

Disk Formation

- Proto-galactic gas fragment and collide retaining initial angular momentum provided from torques from other proto-galactic systems

- Rotationally supported gas disk, cooling in dense regions until HI clouds form from which star formation occurs - thick disk

- Cool molecular gas settles to midplane of thick disk efficiently forming stars - thinness is self regulating - if disk continued to get thinner then density and star formation goes up heating the material and re-puffing out the disk