Set 2:
Cosmic Geometry
Newton vs Einstein

• Even though locally Newtonian gravity is an excellent approximation to General Relativity, in cosmology we deal with spatial and temporal scales across which the global picture benefits from a basic understanding of General Relativity.
• An example is: as we have seen in the previous set of notes, it is much more convenient to think of the space between galaxies expanding rather than galaxies receding through space.
• While latter is a good description locally, its preferred coordinates place us at center and does not allow us to talk about distances beyond which galaxies are receding faster than light - though these distances as we shall see are also not directly observable.
• To get a global picture of the expansion of the universe we need to think geometrically, like Einstein not Newton.
• Best of both: think globally (Einstein), act locally (Newton)
Gravity as Geometry

- Einstein says Gravity as a force is really the geometry of spacetime.
- Force between objects is a fiction of geometry - imagine the curved space of the 2-sphere - e.g. the surface of the earth.
- Two people walk from equator to pole on lines of constant longitude.
- Intersect at poles as if an attractive force exists between them.
- Both walk on geodesics or straight lines of the shortest distance.
Gravity as Geometry

- General relativity has two aspects
  - A metric theory: geometry tells matter how to move
  - Field equations: matter tells geometry how to curve

- Metric defines distances or separations in the spacetime and freely falling matter moves on a path that extremizes the distance

- Expansion of the universe carries two corresponding pieces
  - Friedmann-Robertson-Walker geometry or metric tells matter, including light, how to move – allows us to chart out the expansion with light
  - Matter content of the universe tells it how to expand – allows us to infer the components of the universe

- Useful to separate out these two pieces both conceptually and for understanding alternate cosmologies
FRW Geometry

- **FRW geometry** = homogeneous and isotropic on large scales

- Universe observed to be nearly isotropic (e.g. CMB, radio point sources, galaxy surveys)

- **Copernican principle**: we’re not special, must be isotropic to all observers (all locations)
  
  Implies homogeneity

  Verified through galaxy redshift surveys

- FRW cosmology (homogeneity, isotropy & field equations) generically implies the expansion of the universe, except for special unstable cases
Isotropy & Homogeneity

- Isotropy: CMB isotropic to $10^{-3}$, $10^{-5}$ if dipole subtracted
- Redshift surveys show return to homogeneity on the $>100\text{Mpc}$ scale
FRW Geometry

- **Spatial geometry** is that of constant curvature
  - Positive: sphere
  - Negative: saddle
  - Flat: plane

- **Metric** tells us how to measure distances on this surface
FRW Geometry

- Closed: sphere of radius $R$ and (real) curvature $K = 1/R^2$
- Suppress 1 dimension $\alpha$ represents total angular separation between two points on the sky $(\theta_1, \phi_1)$ and $(\theta_2, \phi_2)$
FRW Geometry

- Geometry tells matter how to move: take (null) geodesic motion for light along this generalized sense of longitude or radial distance $D$.
- This arc distance is the distance our photon traveler sees.
- We receive light from two different trajectories as observer at pole.
- Compared with our Euclidean expectation that the angle between the rays should be related to the separation at emission $\Sigma$ as $d\alpha \approx \Sigma/D$ the angular size appears larger because of the “lensing” magnification of the background.
- This leads to the so-called angular diameter distance - the most relevant sense of distance for the observer.
- In General Relativity, we are free to use any distance coordinate we like but the two have distinct uses.
FRW Geometry

- To define the angular diameter distance, look for a $D_A$ such that

$$d\Sigma = D_A d\alpha$$

Draw a circle at the distance $D$, its radius is $D_A = R \sin(D/R)$
FRW Geometry

- Angular diameter distance

- **Positively curved** geometry $D_A < D$ and objects are further than they appear

- **Negatively curved** universe $R$ is imaginary and

  $$R \sin(D/R) = i|R| \sin(D/i|R|) = |R| \sinh(D/|R|)$$

  and $D_A > D$ objects are closer than they appear

- **Flat** universe, $R \to \infty$ and $D_A = D$
FRW Geometry

- Now add that point 2 may have a different radial distance.
- What is the distance $d\Sigma$ between points 1 $(\theta_1, \phi_1, D_1)$ and point 2 $(\theta_2, \phi_2, D_2)$, separated by $d\alpha$ in angle and $dD$ in distance?

$$D_A = R \sin \left( \frac{D}{R} \right)$$
Angular Diameter Distance

• For small angular and radial separations, space is nearly flat so that the Pythagorean theorem holds for differentials

\[ d\Sigma^2 = dD^2 + D_A^2 d\alpha^2 \]

• Now restore the fact that the angular separation can involve two angles on the sky - the curved sky is just a copy of the spherical geometry with unit radius that we were suppressing before

\[ d\Sigma^2 = dD^2 + D_A^2 d\alpha^2 \]
\[ = dD^2 + D_A^2 (d\theta^2 + \sin^2 \theta d\phi^2) \]

• \( D_A \) useful for describing observables (flux, angular positions)

• \( D \) useful for theoretical constructs (causality, relationship to temporal evolution)
**Alternate Notation**

- Aside: line element is often also written using $D_A$ as the coordinate distance

\[
dD_A^2 = \left( \frac{dD_A}{dD} \right)^2 dD^2
\]

\[
\left( \frac{dD_A}{dD} \right)^2 = \cos^2(D/R) = 1 - \sin^2(D/R) = 1 - (D_A/R)^2
\]

\[
dD^2 = \frac{1}{1 - (D_A/R)^2} dD_A^2
\]

and defining the curvature $K = 1/R^2$ the line element becomes

\[
d\Sigma^2 = \frac{1}{1 - D_A^2 K} dD_A^2 + D_A^2 (d\theta^2 + \sin^2 \theta d\phi^2)
\]

where $K < 0$ for a negatively curved space
Line Element or Metric Uses

- Metric also defines the volume element

\[ dV = (dD)(D_A d\theta)(D_A \sin \theta d\phi) \]

\[ = D_A^2 dD d\Omega \]

where \( d\Omega = \sin \theta d\theta d\phi \) is solid angle

- Most of classical cosmology boils down to these three quantities, (comoving) radial distance, (comoving) angular diameter distance, and volume element

- For example, distance to a high redshift supernova, angular size of the horizon at last scattering and BAO feature, number density of clusters...
Comoving Coordinates

- Remaining degree of freedom (preserving homogeneity and isotropy) is the temporal evolution of overall scale factor.
- Relates the geometry (fixed by the radius of curvature $R$) to physical coordinates – a function of time only.
  \[ d\sigma^2 = a^2(t) d\Sigma^2 \]
  - Our conventions are that the scale factor today $a(t_0) \equiv 1$.
- Similarly physical distances are given by $d(t) = a(t)D$, $d_A(t) = a(t)D_A$.
- Distances in upper case are comoving; lower, physical.
  - Comoving coordinates do not change with time and are simplest coordinates to work out geometrical effects.
**Time and Conformal Time**

- Spacetime separation (with $c = 1$)

  \[ ds^2 = -dt^2 + d\sigma^2 \]
  
  \[ = -dt^2 + a^2(t)d\Sigma^2 \]

- Taking out the scale factor in the time coordinate

  \[ ds^2 \equiv a^2(t)\left( -d\eta^2 + d\Sigma^2 \right) \]

  $d\eta = dt/a$ defines **conformal time** – useful in that photons travelling radially from observer on null geodesics $ds^2 = 0$

  \[ \Delta D = \Delta \eta = \int \frac{dt}{a} \]

  so that **time and distance** may be interchanged
FRW Metric

- Aside for advanced students: Relationship between coordinate differentials and space-time separation defines the metric $g_{\mu\nu}$

$$ds^2 \equiv g_{\mu\nu}dx^\mu dx^\nu = a^2(\eta)(-d\eta^2 + d\Sigma^2)$$

Einstein summation - repeated lower-upper pairs summed

- Usually we will use comoving coordinates and conformal time as the $x^\mu$ unless otherwise specified – metric for other choices are related by $a(t)$

- Aside: scale factor plays the role of a conformal rescaling (which preserves spacetime “angles”, i.e. light cone and causal structure - hence conformal time)
Horizon

- Distance travelled by a photon in the whole lifetime of the universe defines the horizon

- Since $ds = 0$, the horizon is simply the elapsed conformal time

\[
D_{\text{horizon}}(t) = \int_0^t \frac{dt'}{a} = \eta(t)
\]

- Horizon always grows with time

- Always a point in time before which two observers separated by a distance $D$ could not have been in causal contact

- Horizon problem: why is the universe homogeneous and isotropic on large scales especially for objects seen at early times, e.g. CMB, when horizon small
Special vs. General Relativity

- From our class perspective, the big advantage of comoving coordinates and conformal time is that we have largely reduced general relativity to special relativity.
- In these coordinates, aside from the difference between $D$ and $D_A$, we can think of photons propagating in flat spacetime.
- Now let’s relate this discussion to observables.
- Rule of thumb to avoid dealing with the expansion directly:
  - Convert from physical quantities to conformal-comoving quantities at emission.
  - In conformal-comoving coordinates, light propagates as usual.
  - At reception $a = 1$, conformal-comoving coordinates are physical, so interpret as usual.
Hubble Parameter

- Useful to define the expansion rate or Hubble parameter

\[ H(t) \equiv \frac{1}{a} \frac{da}{dt} = \frac{d \ln a}{dt} \]

fractional change in the scale factor per unit time - \( \ln a = N \) is also known as the e-folds of the expansion

- Cosmic time becomes

\[ t = \int dt = \int \frac{d \ln a}{H(a)} \]

- Conformal time becomes

\[ \eta = \int \frac{dt}{a} = \int \frac{d \ln a}{aH(a)} \]

- Advantageous since conservation laws give matter evolution with \( a; a = (1 + z)^{-1} \) is a direct observable...
Redshift

- **Wavelength** of light “stretches” with the scale factor
- The physical wavelength $\lambda_{\text{emit}}$ associated with an observed wavelength today $\lambda_{\text{obs}}$ (or comoving=physical units today) is

$$\frac{\lambda_{\text{emit}}}{\lambda_{\text{obs}}} = \frac{a(t_{\text{emit}})}{a(t_{\text{obs}})} = a(t_{\text{emit}})$$

so that the redshift of spectral lines measures the scale factor of the universe at $t$, $1 + z = 1/a$.

- Interpreting the redshift as a **Doppler shift**, objects recede in an expanding universe
Distance-Redshift Relation

- Given atomically known rest wavelength $\lambda_{\text{emit}}$, redshift can be precisely measured from spectra
- Combined with a measure of distance, distance-redshift $D(z) \equiv D(z(a))$ can be inferred - given that photons travel $D = \Delta \eta$ this tells us how the scale factor of the universe evolves with time.
- Related to the expansion history as

$$D(a) = \int dD = \int_a^1 \frac{d \ln a'}{a' H(a')}\left[d \ln a' = -d \ln(1 + z) = -a' dz\right]$$

$$D(z) = -\int_z^0 \frac{dz'}{H(z')} = \int_0^z \frac{dz'}{H(z')}$$
Hubble Law

• Note limiting case is the Hubble law

\[
\lim_{z \to 0} D(z) = \frac{z}{H(z = 0)} \equiv \frac{z}{H_0}
\]

independently of the geometry and expansion dynamics

• Hubble constant usually quoted as as dimensionless \( h \)

\[
H_0 = 100h \, \text{km s}^{-1}\text{Mpc}^{-1}
\]

• Observationally \( h \sim 0.7 \) (see below)

• With \( c = 1 \), \( H_0^{-1} = 9.7778 (h^{-1} \, \text{Gyr}) \) defines the time scale (Hubble time, \( \sim \) age of the universe)

• As well as \( H_0^{-1} = 2997.9 (h^{-1} \, \text{Mpc}) \) a length scale (Hubble scale \( \sim \) Horizon scale)
Standard Ruler

- **Standard Ruler**: object of known physical size $\lambda$

- Let’s apply our rule of thumb: at emission the comoving size is $\Lambda$

$$\lambda = a(t)\Lambda$$

Now everything about light is normal: the object of comoving size $\Lambda$ subtends an observed angle $\alpha$ on the sky $\alpha$

$$\alpha = \frac{\Lambda}{D_A(z)}$$

- This is the easiest way of thinking about it. But we could also define an effective **physical** distance $d_A(z)$ which corresponds to what we would infer in a non expanding spacetime

$$\alpha \equiv \frac{\lambda}{d_A(z)} = \frac{a\Lambda}{d_A(z)} = \frac{\Lambda}{D_A(z)} \rightarrow d_A(z) = aD_A(z) = \frac{D_A(z)}{1 + z}$$
Standard Ruler

• Since \( D_A \to D_A(D_{\text{horizon}}) \) whereas \((1 + z)\) unbounded, angular size of a fixed physical scale at high redshift actually increases with radial distance.

• Paradox: the further away something is, the bigger it appears.
  – Easily resolved by thinking about comoving coordinates - a fixed physical scale \( \lambda \) as the universe shrinks and \( a \to 0 \) will eventually encompass the whole observable universe out to the horizon in comoving coordinates so of course it subtends a big angle on the sky!
  – But there are no such bound objects in the early universe - there is no causal way such bigger-than-the-horizon objects could form.

• Knowing \( \lambda \) or \( \Lambda \) and measuring \( \alpha \) and \( z \) allows us to infer the comoving angular diameter distance \( D_A(z) \).
**Standard Candle**

- **Standard Candle**: objects of same luminosity $L$, measured flux $F$

- Apply rules again: at emission in conformal-comoving coordinates
  - $L$ is the energy per unit time at emission
  - Since $E \propto \lambda^{-1}$ and comoving wavelength $\Lambda \propto \lambda/a$ so comoving energy $\mathcal{E} \propto \Lambda^{-1} \propto aE$
  - Per unit time at emission $\Delta t = a\Delta \eta$ in conformal time
  - So observed luminosity today is $\mathcal{L} = \mathcal{E}/\Delta \eta = a^2 L$
  - All photons must pass through the sphere at a given distance, so the comoving surface area is $4\pi D_A^2$

- Put this together to the observed flux at $a = 1$

\[
F = \frac{\mathcal{L}}{4\pi D_A^2} = \frac{L}{4\pi D_A^2} \frac{1}{(1 + z)^2}
\]

Notice the flux is diminished by two powers of $(1 + z)$
Luminosity Distance

- We can again define a physical “luminosity” distance that corresponds to our non-expanding spacetime intuition

\[ F \equiv \frac{L}{4\pi d_L^2} \]

- So luminosity distance

\[ d_L = (1 + z)D_A = (1 + z)^2 d_A \]

- As \( z \to 0 \), \( d_L = d_A = D_A \)

- But as \( z \to \infty \), \( d_L \gg d_A \) - key to understanding Olber’s paradox
Olber’s Paradox Redux

• Surface brightness - object of physical size $\lambda$

$$S = \frac{F}{\Delta \Omega} = \frac{L}{4\pi d_L^2} \frac{d_A^2}{\lambda^2}$$

• In a non-expanding geometry (regardless of curvature), surface brightness is conserved $d_A = d_L$

$$S = \text{const.}$$

– each site line in universe full of stars will eventually end on surface of star, night sky should be as bright as sun (not infinite)

• In an expanding universe

$$S \propto (1 + z)^{-4}$$
Olber’s Paradox Redux

- Second piece: age finite so even if stars exist in the early universe, not all site lines end on stars

- But even as age goes to infinity and the number of site lines goes to 100%, surface brightness of distant objects (of fixed physical size) goes to zero
  - Angular size increases
  - Redshift of “luminosity” i.e. energy and arrival time dilation
Measuring $D(z)$

- Astro units side: since flux ratios are very large in cosmology, it's more useful to take the log

$$m_1 - m_2 = -2.5 \log_{10}(F_1/F_2)$$

related to $d_L$ by definition by inverse square law

$$m_1 - m_2 = 5 \log_{10}[d_L(z_1)/d_L(z_2)]$$

- To quote in terms of a single object, introduce absolute magnitude as the magnitude that would be measured for the object at 10 pc

$$m - M = 5 \log_{10}[d_L(z)/10\text{pc}]$$

Knowing absolute magnitude is the same as knowing the absolute distance, otherwise distances are relative
Measuring $D(z)$

- If absolute magnitude unknown, then both standard candles and standard rulers measure relative sizes and fluxes – ironically this means that measuring the change in $H$ is easier than measuring $H_0$ – acceleration easier than rate!

For standard candle, e.g. compare magnitudes low $z_0$ to a high $z$ object - using the Hubble law $d_L(z_0) = z_0/H_0$ we have

$$\Delta m = m_z - m_{z_0} = 5 \log_{10} \frac{d_L(z)}{d_L(z_0)} = 5 \log_{10} \frac{H_0 d_L(z)}{z_0}$$

Likewise for a standard ruler comparison at the two redshifts

$$\frac{d_A(z)}{d_A(z_0)} = \frac{H_0 d_A(z)}{z_0}$$

- Distances are measured in units of $h^{-1} \text{ Mpc}$.
Measuring $D(z)$

- Since $z$ is a direct observable, in both cases $H_0D_A(z)$ is the measured quantity
- We can relate that back to $H_0D(z)$ recalling that

$$H_0D_A = H_0R \sin(H_0D/H_0R)$$

and use $h^{-1}$ Mpc as the unit for all lengths – furthermore, local observations are at distances much smaller than $R$ so $H_0D_A = H_0D$ is a good approximation
- Then since $D(z) = \int dz/H(z)$ we have

$$H_0D(z) = \int dz \frac{H_0}{H(z)}$$

- Fundamentally our low to high $z$ comparison tells us the change in expansion rate $H(z)/H_0$
Supernovae as Standard Candles

- Type 1A supernovae are white dwarfs that reach Chandrashekar mass where electron degeneracy pressure can no longer support the star, hence a very regular explosion.

- Moreover, the scatter in absolute magnitude is correlated with the shape of the light curve - the rate of decline from peak light, empirical “Phillips relation”

- Higher $^{56}N$, brighter SN, higher opacity, longer light curve duration
Beyond Hubble’s Law

- Type 1A are therefore “standardizable” candles leading to a very low scatter $\delta m \sim 0.15$ and visible out to high redshift $z \sim 1$

- Two groups in 1999 found that SN more distant at a given redshift than expected

- Cosmic acceleration
Acceleration of the Expansion

- Using SN as a relative indicator (independent of absolute magnitude), comparison of low and high $z$ gives

$$H_0 D(z) = \int dz \frac{H_0}{H}$$

more distant implies that $H(z)$ not increasing at expect rate, i.e. is more constant

- Take the limiting case where $H(z)$ is a constant (a.k.a. de Sitter expansion)

$$H = \frac{1}{a} \frac{da}{dt} = \text{const}$$

$$\frac{dH}{dt} = \frac{1}{a} \frac{d^2a}{dt^2} - H^2 = 0$$

$$\frac{1}{a} \frac{d^2a}{dt^2} = H^2 > 0$$
Acceleration of the Expansion

- Indicates that the expansion of the universe is accelerating
- Intuition tells us (FRW dynamics shows) ordinary matter decelerates expansion since gravity is attractive
- Ordinary expectation is that $H(z > 0) > H_0$
  
  so that the Hubble parameter is higher at high redshift
  
  Or equivalently that expansion rate decreases as it expands

- Notice that this a purely geometric inference and does not yet say anything about what causes acceleration – topic of next set of lectures on cosmic dynamics
Hubble Constant

- Getting $H_0$ itself is harder since we need to know the absolute distance $d_L$ to the objects: $H_0 = z_0 / d_L$
- Hubble actually inferred too large a Hubble constant of $H_0 \sim 500 \text{km/s/Mpc}$
- Miscalibration of the Cepheid distance scale - absolute measurement hard, checkered history
- Took 70 years to settle on this value with a factor of 2 discrepancy persisting until late 1990’s - which is after the projects which discovered acceleration were conceived!
- $H_0$ now measured as $74.03 \pm 1.42 \text{km/s/Mpc}$ by SH0ES calibrating off local, geometric absolute distances including AGN water maser
Hubble Constant History

- Difficult measurement since local galaxies have peculiar motions and so their velocity is not entirely due to the “Hubble flow”
- A “distance ladder” of cross calibrated measurements
- Primary distance indicators cepheids, novae planetary nebula, tip of red giant branch, or globular cluster luminosity function, AGN water maser
- Use more luminous secondary distance indications to go out in distance to Hubble flow
  - Tully-Fisher, fundamental plane, surface brightness fluctuations, Type 1A supernova
Modern Distance Ladder

- Geometry $\rightarrow$ Cepheids $\rightarrow$ SNIa
- Luminosity distance $d_L(m - M, z) \rightarrow H_0$
- SH0ES, Riess et al 2016
Maser-Cepheid-SN Distance Ladder

- Water maser around AGN, gas in Keplerian orbit
- Measure proper motion, radial velocity, acceleration of orbit
- Method 1: radial velocity plus orbit infer tangential velocity $= \text{distance} \times \text{angular proper motion}$

$$v_t = d_A \left( \frac{d\alpha}{dt} \right)$$

- Method 2: centripetal acceleration and radial velocity from line infer physical size

$$a = \frac{v^2}{R}, \quad R = d_A \theta$$
Maser-Cepheid-SN Distance Ladder

- Calibrate Cepheid period-luminosity relation in same galaxy
- SHOES project then calibrates SN distance in galaxies with Cepheids
  Also: consistent with recent HST parallax determinations of 10 galactic Cepheids (8% distance each) with ~ 20% larger $H_0$ error bars - normal metalicity as opposed to LMC Cepheids.
- Measure SN at even larger distances out into the Hubble flow
- Riess et al (2019) $H_0 = 74.03 \pm 1.42 \text{ km/s/Mpc}$ more precise (1.9%) than the HST Key Project calibration (11%).
- As of Spring 2020, this differs from the CMB/BAO distance ladder $H_0 = 67.66 \pm 0.42$ working from high redshifts at $> 4\sigma$...