Set 6: Inflation
Horizon Problem

- The horizon in a decelerating universe scales as $\eta \propto a^{(1+3w)/2}$, $w > -1/3$. For example in a matter dominated universe

  $$\eta \propto a^{1/2}$$

- CMB decoupled at $a_* = 10^{-3}$ so subtends an angle on the sky

  $$\frac{\eta_*}{\eta_0} = a_*^{1/2} \approx 0.03 \approx 2^\circ$$

- So why is the CMB sky isotropic to $10^{-5}$ in temperature if it is composed of $\sim 10^4$ causally disconnected regions

- If smooth by fiat, why are there $10^{-5}$ fluctuations correlated on superhorizon scales
Flatness & Relic Problems

- Flatness problem: why is the radius of curvature larger than the observable universe. (Before the CMB determinations, why is it at least comparable to observable universe $|\Omega_K| \lesssim \Omega_m$)

- Also phrased as a coincidence problem: since $\rho_K \propto a^{-2}$ and $\rho_m \propto a^{-3}$, why would they be comparable today – modern version is dark energy coincidence $\rho_\Lambda = \text{const.}$

- Relic problem – why don’t relics like monopoles dominate the energy density

- Inflation is a theory that solves all three problems at once and also supplies a source for density perturbations
Accelerating Expansion

- In a matter or radiation dominated universe, comoving Hubble length \( (1/aH) \) grows with \( a \) so that there’s no way to establish causal contact on larger scales, generally:

\[
\eta = \int d\ln a \frac{1}{aH(a)}
\]

- \( H^2 \propto \rho \propto a^{-3(1+w)} \), \( aH \propto a^{-(1+3w)/2} \), critical value of \( w = -1/3 \), the division between acceleration and deceleration determines whether as the universe expands comoving observers leave or come into causal contact

- Recall this is our fate in the current accelerating expansion – observers that were once in causal contact will no longer be able to communicate with each other due to the rapid expansion
Causal Contact

- True horizon always grows meaning that one always sees more and more of the universe. But the comoving Hubble length decreases: the difference in conformal time, the distance a photon can travel between two epochs denoted by the scale factor decreases. Regions that were in causal contact, leave causal contact.

- Horizon problem solved if the universe was in an acceleration phase up to $\eta_i$ and the conformal time since then is shorter than the total conformal age

$$\eta_0 \gg \eta_0 - \eta_i$$

- total distance $\gg$ distance traveled since inflation
- apparent horizon
Flatness & Relic

- Comoving radius of curvature is constant and can even be small compared to the full horizon $R \ll \eta_0$ yet still $\eta_0 \gg R \gg \eta_0 - \eta_i$

- In physical coordinates, the rapid expansion of the universe makes the current observable universe much smaller than the curvature scale

- Likewise, the number density of relics formed before the accelerating (or inflationary) epoch is diluted to make them rare in the current observable volume

- Common to place the zero point of (conformal) time at the end of inflation $\tilde{\eta} \equiv \eta - \eta_i$. Here conformal time is negative during inflation and size reflects the distance a photon can travel from that epoch to the end of inflation. To avoid confusion with the original zero point $\eta(a = 0) = 0$ let’s call this $\tilde{\eta}$. 
Sufficient Inflation

- If the accelerating component has equation of state $w = -1$, $\rho = \text{const.}$, $H = H_i \text{ const.}$ so that $a \propto \exp(Ht)$

$$
\tilde{\eta} = \int_{a_i}^{a} d \ln a \frac{1}{aH} = - \left. \frac{1}{aH_i} \right|_{a_i}^a \\
\approx - \frac{1}{aH_i} \quad (a_i \gg a)
$$

- In particular, the current horizon scale $H_0 \tilde{\eta}_0 \approx 1$ exited the horizon during inflation at

$$
\tilde{\eta}_0 \approx H_0^{-1} = \frac{1}{a_H H_i} \\
a_H = \frac{H_0}{H_i}
$$
Sufficient Inflation

- Given some energy scale for inflation that defines $H_i$, this tells us what the scale factor $a_H$ was when the current horizon left the horizon during inflation.

- If we knew what the scale factor $a_i$ was at the end of inflation, we could figure out the number of efolds $N = \ln(a_i/a_H)$ between these two epochs.

- A rough way to characterize this is to quote it in terms of an effective temperature $T \propto T_{\text{CMB}} a^{-1}$ at the end of inflation.

$$\ln \frac{a_i}{a_H} = \ln \frac{T_{\text{CMB}} H_i}{T_i H_0} = 65 + 2 \ln \left( \frac{\rho_i^{1/4}}{10^{14}\text{GeV}} \right) - \ln \left( \frac{T_i}{10^{10}\text{GeV}} \right)$$

- So inflation lasted at least $\sim 60$ efolds - a more detailed calculation would involve the epoch of reheating and $g_*$ factors, so $T_i \neq T_{\text{reheat}}$.
Inflation: Acceleration from Scalar Field

- Unlike a true cosmological constant, the period of exponential expansion must end to produce the hot big bang phase.

- A cosmological constant is like potential energy - so imagine a ball rolling slowly in into a valley eventually converting potential into kinetic energy.

- Technically, this is a scalar field: where the position on the hill is $\phi$ and the height of the potential is $V(\phi)$.

- In spacetime $\phi(x, t)$ is a function of position: different spacetime points can be at different field positions.
Scalar Fields

- Inflation ends when the field rolls sufficiently down the potential that its kinetic energy becomes comparable to its potential energy.
- The field then oscillates at the bottom of the potential and small couplings to standard model particles “reheats” the universe converting the inflaton energy into particles.
- Due to the uncertainty principle in quantum mechanics, the field cannot remain perfectly unperturbed.
- The small field fluctuations mean that inflation ends at a slightly different time at different points in space - leaving fluctuations in the scale factor, which are curvature or gravitational potential fluctuations.
- Gravitational attraction into these potential wells forms all of the structure in the universe.
Scalar Fields

- Mathematically, the scalar field obeys the Klein-Gordon equation in an expanding universe

\[ \frac{d^2 \phi}{dt^2} + 3H \frac{d\phi}{dt} + V' = 0 \]

where \( V' = \frac{dV}{d\phi} \) is the slope of the potential - the first and third term look like the equations of motion of a ball rolling down a hill - acceleration = gradient of potential

- The second \( \frac{d\phi}{dt} \) term is a friction term provided by the expansion - “Hubble friction” - just like particle numbers and energy density dilute with the expansion, so too does the kinetic energy of the scalar field.
Scalar Fields

- Kinetic energy is

\[ \rho_{\text{kinetic}} = \frac{1}{2} \left( \frac{d\phi}{dt} \right)^2 \]

so, without the \( V' \) forcing term, how does the energy density decay?

- Solve

\[ \frac{d^2\phi}{dt^2} + 3H \frac{d\phi}{dt} = 0 \rightarrow \frac{d\phi}{dt} \propto a^{-3} \]

so kinetic energy would decay as \( \rho_{\text{kinetic}} \propto a^{-6} = a^{-3(1+w_{\text{kinetic}})} \), or \( w_{\text{kinetic}} = +1 \)

- Compare with the potential energy at fixed field position

\( w_{\text{potential}} = -1 \)
Scalar Fields

• As the field rolls it slowly loses total energy to friction, which defines the slow roll parameter

\[ \epsilon_H = -\frac{d \ln H}{d \ln a} = \frac{3}{2} (1 + w_\phi) \]

• Requirement that inflation last for the sufficient \( \sim 60 \) efolds requires that \( \epsilon_H \lesssim \frac{1}{60} \ll 1 \)

• This requirement also means that \( \epsilon_H \) must also be slowly varying so as not to grow much during these 60 efolds

\[ \delta_1 = \frac{1}{2} \frac{d \ln \epsilon_H}{d \ln a} - \epsilon_H \]

with \( |\delta_1| \ll 1 \) (advanced students: its defined this way since it also determines how close the roll is to friction dominated \( 3Hd\phi/dt \approx -V' \))
Perturbation Generation

- Horizon scale $\tilde{\eta}$ during inflation acts like an event horizon - things that are separated by more than this distance leave causal contact.

- Result of treating field fluctuations as a quantum simple harmonic oscillator (advanced students: see supplement) is that the uncertainty principle leads to inevitable fluctuations.

- Fluctuations freeze in when the comoving wavelength $\lambda = 2\pi/k$ becomes larger than the comoving horizon $1/\alpha H$, so that parts of the fluctuation are no longer in causal contact with itself, i.e. when $k \approx \alpha H$.

$$\delta \phi \approx \frac{H}{2\pi}$$

- We can also view this as an “origins” problem. Quantum fluctuations behave as a simple harmonic oscillator with frequency or rate $\omega \approx k/\alpha$ and freezeout occurs when $\omega = H$, so $k/\alpha = H$. 
Perturbation Generation

- Interpretation: universe is expanding quickly enough that various parts of the wave cannot “find” each other to maintain “equilibrium” (continue oscillating)

- Can heuristically understand the freezout value in the same way as Hawking radiation from a black hole - virtual particles become real when separated by the horizon

- Here $H$ defines the horizon area (or in black hole language the Hawking temperature) and dimensional analysis says the field fluctuation must scale with $H$, the only dimensionful quantity

- Because $H$ remains roughly constant during inflation the result is a scale invariant spectrum of fluctuations
Curvature Fluctuation

- Field fluctuations change the scale factor at which inflation ends

\[ R = -\delta \ln a = - \frac{d \ln a}{dt} \frac{d \delta \phi}{d \phi} = - \frac{H^2}{2\pi} \frac{dt}{d\phi} \]

- Using the equation of state of \( \phi \) we can convert \( d\phi/dt \) to \( \epsilon_H \)

\[
\begin{align*}
w_\phi &= \frac{p_\phi}{\rho_\phi} \\
&= \frac{(d\phi/dt)^2/2 - V}{(d\phi/dt)^2/2 + V} \\
&\approx \frac{(d\phi/dt)^2}{V} - 1
\end{align*}
\]

and \( H^2 \approx 8\pi G V/3 \) from Friedmann
Curvature Fluctuation

• So

\[ \epsilon_H \approx \frac{3}{2} \frac{(d\phi/dt)^2}{V} \approx 4\pi G \frac{(d\phi/dt)^2}{H^2} \]

and the variance of fluctuations per log wavenumber \( d \ln k \)

\[ \Delta_R^2 \equiv \langle R^2 \rangle \approx \frac{H^4}{4\pi^2} \frac{4\pi G}{H^2 \epsilon_H} \approx \frac{G}{\pi} \frac{H^2}{\epsilon_H} \]

• Remember this: \( \Delta_R^2 \propto H^2/\epsilon_H \)!
Tilt

- Curvature power spectrum is scale invariant to the extent that $H$ and $\epsilon_H$ are constant
- Scalar spectral index

$$\frac{d \ln \Delta_R^2}{d \ln k} \equiv n_S - 1 = 2 \frac{d \ln H}{d \ln k} - \frac{d \ln \epsilon_H}{d \ln k}$$

- Evaluate at horizon crossing where fluctuation freezes $k = aH$

$$\frac{d \ln H}{d \ln k} \approx \frac{d \ln H}{d \ln a} = -\epsilon_H$$

$$\frac{d \ln \epsilon}{d \ln k} \approx \frac{d \ln \epsilon}{d \ln a} = 2(\delta_1 + \epsilon_H)$$

- Tilt in the slow-roll approximation

$$n_S - 1 = -4\epsilon_H - 2\delta_1$$
Gravitational Waves

- Gravitational wave amplitude satisfies Klein-Gordon equation $(K = 0)$, same as scalar field
  
  $$\frac{d^2 h_{+,\times}}{dt^2} + 3H \frac{dh_{+,\times}}{dt} + \frac{k^2}{a^2} h_{+,\times} = 0.$$  

- Acquires quantum fluctuations in same manner as $\phi$. Canonical normalization (Lagrangian) sets the normalization

- Scale-invariant gravitational wave amplitude

  $$\Delta_{+,,\times}^2 = 16\pi G \frac{H^2}{(2\pi)^2}.$$  

- Gravitational wave power $\propto H^2 \propto V \propto E_i^4$ where $E_i$ is the energy scale of inflation
Gravitational Waves

- Tensor-scalar ratio is therefore generally small

\[ r \equiv 4 \frac{\Delta_+^2}{\Delta_R^2} = 16\epsilon_H \]

- Tensor tilt:

\[ \frac{d \ln \Delta_+^2}{d \ln k} \equiv n_T = 2 \frac{d \ln H}{d \ln k} = -2\epsilon_H \]

- Consistency relation between tensor-scalar ratio and tensor tilt

\[ r = 16\epsilon = -8n_T \]

- Measurement of scalar tilt and gravitational wave amplitude constrains inflationary model in the slow roll context

- Comparison of tensor-scalar ratio and tensor tilt tests the idea of slow roll itself
Observability

- Gravitational waves from inflation can be measured via its imprint on the polarization of the CMB...