Set 6:

Inflation

Horizon Problem

- The horizon in a decelerating universe scales as $\eta \propto a^{(1+3w)/2}$, $w > -1/3$. For example in a matter dominated universe

$$\eta \propto a^{1/2}$$

- CMB decoupled at $a_* = 10^{-3}$ so subtends an angle on the sky

$$\frac{\eta_*}{\eta_0} = a_*^{1/2} \approx 0.03 \approx 2^\circ$$

- So why is the CMB sky isotropic to 10^{-5} in temperature if it is composed of $\sim 10^4$ causally disconnected regions
- If smooth by fiat, why are there 10^{-5} fluctuations correlated on superhorizon scales

Flatness & Relic Problems

- Flatness problem: why is the radius of curvature larger than the observable universe. (Before the CMB determinations, why is it at least comparable to observable universe $|\Omega_K| \lesssim \Omega_m$)
- Also phrased as a coincidence problem: since $\rho_K \propto a^{-2}$ and $\rho_m \propto a^{-3}$, why would they be comparable today – modern version is dark energy coincidence $\rho_\Lambda = \text{const.}$
- Relic problem – why don't relics like monopoles dominate the energy density

- Inflation is a theory that solves all three problems at once and also supplies a source for density perturbations

Accelerating Expansion

- In a matter or radiation dominated universe, comoving Hubble length $(1/aH)$ grows with a a so that there's no way to establish causal contact on larger scales, generally:

$$\eta = \int d\ln a \frac{1}{aH(a)}$$

- $H^2 \propto \rho \propto a^{-3(1+w)}$, $aH \propto a^{-(1+3w)/2}$, critical value of $w = -1/3$, the division between acceleration and deceleration determines whether as the universe expands comoving observers leave or come into causal contact
- Recall this is our fate in the current accelerating expansion – observers that were once in causal contact will no longer be able to communicate with each other due to the rapid expansion

Causal Contact

- True horizon always grows meaning that one always sees more and more of the universe. But the comoving Hubble length decreases: the difference in conformal time, the distance a photon can travel between two epochs denoted by the scale factor decreases. Regions that were in causal contact, leave causal contact.
- Horizon problem solved if the universe was in an acceleration phase up to η_i and the conformal time since then is shorter than the total conformal age

$$\eta_0 \gg \eta_0 - \eta_i$$

$$\text{total distance} \gg \text{distance traveled since inflation}$$

$$\text{apparent horizon}$$

Flatness & Relic

- Comoving radius of curvature is constant and can even be small compared to the full horizon $R \ll \eta_0$ yet still $\eta_0 \gg R \gg \eta_0 - \eta_i$
- In physical coordinates, the rapid expansion of the universe makes the current observable universe much smaller than the curvature scale
- Likewise, the number density of relics formed before the accelerating (or inflationary) epoch is diluted to make them rare in the current observable volume
- Common to place the zero point of (conformal) time at the end of inflation $\tilde{\eta} \equiv \eta - \eta_i$. Here conformal time is negative during inflation and size reflects the distance a photon can travel from that epoch to the end of inflation. To avoid confusion with the original zero point $\eta(a = 0) = 0$ let's call this $\tilde{\eta}$.

Sufficient Inflation

- If the accelerating component has equation of state $w = -1$, $\rho =$ const., $H = H_i$ const. so that $a \propto \exp(Ht)$

$$
\begin{aligned}
\tilde{\eta} &= \int_{a_i}^{a} d\ln a \frac{1}{aH} = -\frac{1}{aH_i}\Big|_{a_i}^{a} \\
&\approx -\frac{1}{aH_i} \quad (a_i \gg a)
\end{aligned}
$$

- In particular, the current horizon scale $H_0\tilde{\eta}_0 \approx 1$ exited the horizon during inflation at

$$
\begin{aligned}
\tilde{\eta}_0 \approx H_0^{-1} &= \frac{1}{a_H H_i} \\
a_H &= \frac{H_0}{H_i}
\end{aligned}
$$

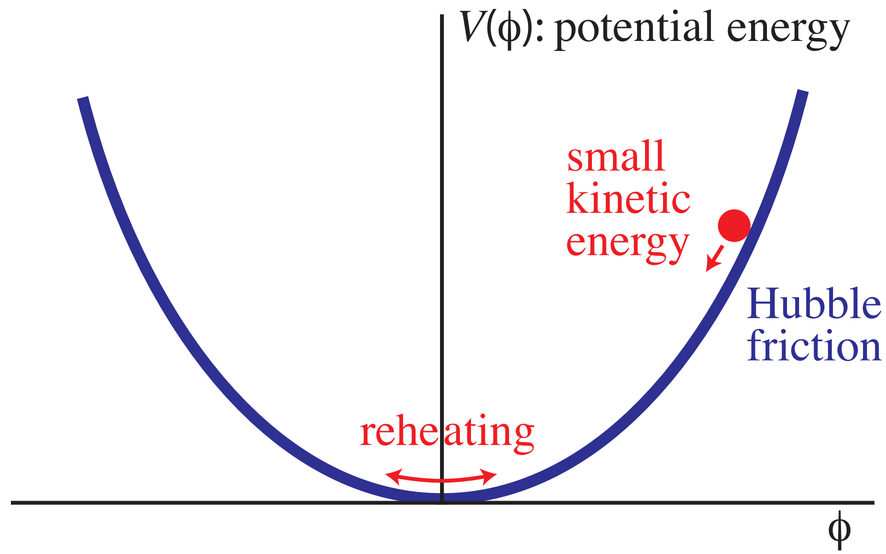
Sufficient Inflation

- Given some energy scale for inflation that defines H_i, this tells us what the scale factor a_H was when the current horizon left the horizon during inflation
- If we knew what the scale factor a_i was at the end of inflation, we could figure out the number of efolds $N = \ln(a_i/a_H)$ between these two epochs
- A rough way to characterize this is to quote it in terms of an effective temperature $T \propto T_{\rm CMB} a^{-1}$ at the end of inflation

$$\ln \frac{a_i}{a_H} = \ln \frac{T_{\rm CMB}}{T_i} \frac{H_i}{H_0} = 65 + 2 \ln \left(\frac{\rho_i^{1/4}}{10^{14}{\rm GeV}} \right) - \ln \left(\frac{T_i}{10^{10}{\rm GeV}} \right)$$

- So inflation lasted at least ~ 60efolds - a more detailed calculation would involve the epoch of reheating and g_* factors, so $T_i \neq T_{\rm reheat}$

Inflation: Acceleration from Scalar Field



- Unlike a true cosmological constant, the period of exponential expansion must end to produce the hot big bang phase
- A cosmological constant is like potential energy - so imagine a ball rolling slowly in into a valley eventually converting potential into kinetic energy
- Technically, this is a scalar field: where the position on the hill is ϕ and the height of the potential is $V(\phi)$
- In spacetime $\phi(\mathbf{x}, t)$ is a function of position: different spacetime points can be at different field positions

Scalar Fields

- Inflation ends when the field rolls sufficiently down the potential that its kinetic energy becomes comparable to its potential energy
- The field then oscillates at the bottom of the potential and small couplings to standard model particles "reheats" the universe converting the inflaton energy into particles
- Due to the uncertainty principle in quantum mechanics, the field cannot remain perfectly unperturbed
- The small field fluctuations mean that inflation ends at a slightly different time at different points in space - leaving fluctuations in the scale factor, which are curvature or gravitational potential fluctuations
- Gravitational attraction into these potential wells forms all of the structure in the universe

Scalar Fields

- Mathematically, the scalar field obeys the Klein-Gordon equation in an expanding universe

$$\frac{d^2\phi}{dt^2} + 3H\frac{d\phi}{dt} + V' = 0$$

where $V' = dV/d\phi$ is the slope of the potential - the first and third term look like the equations of motion of a ball rolling down a hill - acceleration = gradient of potential

- The second $d\phi/dt$ term is a friction term provided by the expansion - "Hubble friction" - just like particle numbers and energy density dilute with the expansion, so too does the kinetic energy of the scalar field.

Scalar Fields

- Kinetic energy is

$$\rho_{\rm kinetic} = \frac{1}{2}\left(\frac{d\phi}{dt}\right)^2$$

  so, without the V' forcing term, how does the energy density decay?

- Solve

$$\frac{d^2\phi}{dt^2} + 3H\frac{d\phi}{dt} = 0 \to \frac{d\phi}{dt} \propto a^{-3}$$

  so kinetic energy would decay as $\rho_{\rm kinetic} \propto a^{-6} = a^{-3(1+w_{\rm kinetic})}$, or $w_{\rm kinetic} = +1$

- Compare with the potential energy at fixed field position $w_{\rm potential} = -1$

Scalar Fields

- As the field rolls it slowly loses total energy to friction, which defines the slow roll parameter

$$\epsilon_H = -\frac{d\ln H}{d\ln a} = \frac{3}{2}(1 + w_\phi)$$

- Requirement that inflation last for the sufficient $\sim$60 efolds requires that $\epsilon_H \lesssim 1/60 \ll 1$
- This requirement also means that ϵ_H must also be slowly varying so as not to grow much during these 60 efolds

$$\delta_1 = \frac{1}{2}\frac{d\ln \epsilon_H}{d\ln a} - \epsilon_H$$

with $|\delta_1| \ll 1$ (advanced students: its defined this way since it also determines how close the roll is to friction dominated $3H d\phi/dt \approx -V'$)

Perturbation Generation

- Horizon scale $\tilde{\eta}$ during inflation acts like an event horizon - things that are separated by more than this distance leave causal contact
- Result of treating field fluctuations as a quantum simple harmonic oscillator (advanced students: see supplement) is that the uncertainty principle leads to inevitable fluctuations
- Fluctuations freeze in when the comoving wavelength $\lambda = 2\pi/k$ becomes larger than the comoving horizon $1/aH$, so that parts of the fluctuation are no longer in causal contact with itself, i.e. when $k \approx aH$

$$\delta\phi \approx \frac{H}{2\pi}$$

- We can also view this as an "origins" problem. Quantum fluctuations behave as a simple harmonic oscillator with frequency or rate $\omega \approx k/a$ and freezeout occurs when $\omega = H$, so $k/a = H$

Perturbation Generation

- Interpretation: universe is expanding quickly enough that various parts of the wave cannot "find" each other to maintain "equilibrium" (continue oscillating)
- Can heuristically understand the freezout value in the same way as Hawking radiation from a black hole - virtual particles become real when separated by the horizon
- Here H defines the horizon area (or in black hole language the Hawking temperature) and dimensional analysis says the field fluctuation must scale with H, the only dimensionful quantity
- Because H remains roughly constant during inflation the result is a scale invariant spectrum of fluctuations

Curvature Fluctuation

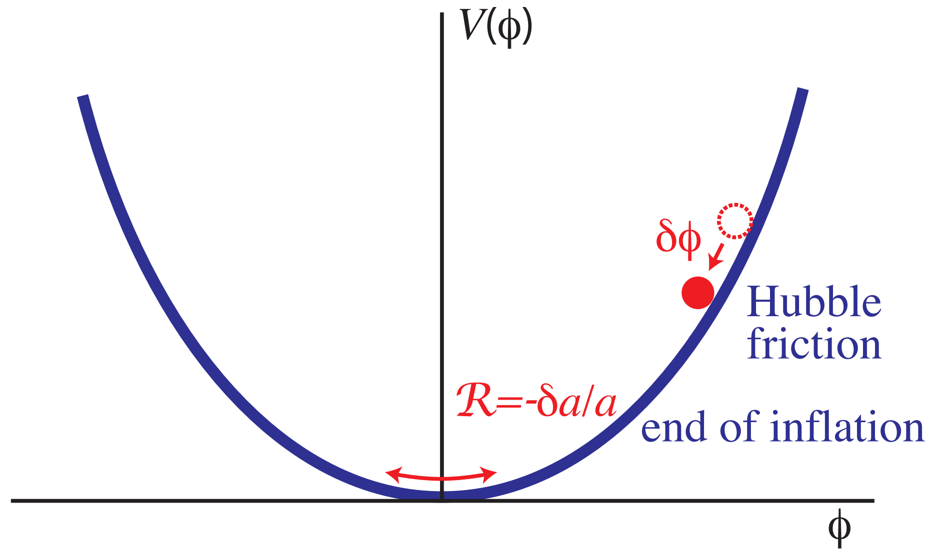
- Field fluctuations change the scale factor at which inflation ends

$$\mathcal{R} = -\delta \ln a = -\frac{d \ln a}{dt}\frac{dt}{d\phi}\delta\phi = -\frac{H^2}{2\pi}\frac{dt}{d\phi}$$

- Using the equation of state of ϕ we can convert $d\phi/dt$ to ϵ_H

$$\begin{aligned} w_\phi &= \frac{p_\phi}{\rho_\phi} \\ &= \frac{(d\phi/dt)^2/2 - V}{(d\phi/dt)^2/2 + V} \\ &\approx \frac{(d\phi/dt)^2}{V} - 1 \end{aligned}$$

and $H^2 \approx 8\pi GV/3$ from Friedmann



Curvature Fluctuation

- So

$$\epsilon_H \approx \frac{3}{2}\frac{(d\phi/dt)^2}{V} \approx 4\pi G \frac{(d\phi/dt)^2}{H^2}$$

and the variance of fluctuations per log wavenumber $d\ln k$

$$\Delta^2_{\mathcal{R}} \equiv \langle \mathcal{R}^2 \rangle \approx \frac{H^4}{4\pi^2}\frac{4\pi G}{H^2 \epsilon_H} \approx \frac{G}{\pi}\frac{H^2}{\epsilon_H}$$

- Remember this: $\Delta^2_{\mathcal{R}} \propto H^2/\epsilon_H$!

Tilt

- Curvature power spectrum is scale invariant to the extent that H and ϵ_H are constant

$$\Delta^2_{\mathcal{R}} \propto H^2/\epsilon_H \approx \text{const}$$

- But with a small tilt that indicates inflation must end in ~60 efolts

$$\frac{d\ln \Delta^2_{\mathcal{R}}}{d\ln k} \equiv n_S - 1 = 2\frac{d\ln H}{d\ln k} - \frac{d\ln \epsilon_H}{d\ln k}$$

- Evaluate at horizon crossing where fluctuation freezes $k = aH$

$$\frac{d\ln H}{d\ln k} \approx \frac{d\ln H}{d\ln a} = -\epsilon_H$$

$$\frac{d\ln \epsilon}{d\ln k} \approx \frac{d\ln \epsilon}{d\ln a} = 2(\delta_1 + \epsilon_H)$$

Power Spectrum: A_S, n_S

- Tilt in the slow-roll approximation

$$n_S - 1 = -4\epsilon_H - 2\delta_1$$

- Power spectrum parameters:

$$\Delta^2_{\mathcal{R}} = A_S \left(\frac{k}{0.05\mathrm{Mpc}^{-1}} \right)^{n_S - 1}$$

with pivot scale 0.05 Mpc^{-1} chosen to be approximately where the data constrains inflation

Gravitational Waves

- Gravitational wave amplitude satisfies Klein-Gordon equation $(K = 0)$, same as scalar field

$$\frac{d^2 h_{+,\times}}{dt^2} + 3H \frac{dh_{+,\times}}{dt} + \frac{k^2}{a^2} h_{+,\times} = 0\,.$$

- Acquires quantum fluctuations in same manner as ϕ. Canonical normalization (Lagrangian) sets the normalization
- Scale-invariant gravitational wave amplitude

$$\Delta^2_{+,\times} = 16\pi G \frac{H^2}{(2\pi)^2}$$

- Gravitational wave power $\propto H^2 \propto V \propto E_i^4$ where E_i is the energy scale of inflation

Gravitational Waves

- Tensor-scalar ratio is therefore generally small

$$r \equiv 4\frac{\Delta_+^2}{\Delta_{\mathcal{R}}^2} = 16\epsilon_H$$

- Tensor tilt:

$$\frac{d\ln\Delta_+^2}{d\ln k} \equiv n_T = 2\frac{d\ln H}{d\ln k} = -2\epsilon_H$$

- Consistency relation between tensor-scalar ratio and tensor tilt

$$r = 16\epsilon = -8n_T$$

- Measurement of scalar tilt and gravitational wave amplitude constrains inflationary model in the slow roll context
- Comparision of tensor-scalar ratio and tensor tilt tests the idea of slow roll itself

Observability

- Gravitational waves from inflation can be measured via its imprint on the polarization of the CMB…