Set 7:

CMB and Large Scale Structure
CMB Temperature Anisotropy

- Planck 2015 map of the temperature anisotropy (first discovered by COBE) from recombination:
CMB Temperature Anisotropy

- Power spectrum shows characteristic scales where the intensity of variations peak - reveals geometry and contents of the universe:
CMB Parameter Inferences

- Spectrum constrains the matter-energy contents of the universe
- Planck 2018 results [arXiv:1807.06209]

<table>
<thead>
<tr>
<th>Parameter</th>
<th>TT+lowE 68% limits</th>
<th>TE+lowE 68% limits</th>
<th>EE+lowE 68% limits</th>
<th>TT,TE,EE+lowE 68% limits</th>
<th>TT,TE,EE+lowE+lensing 68% limits</th>
<th>TT,TE,EE+lowE+lensing+BAO 68% limits</th>
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</thead>
<tbody>
<tr>
<td>$\Omega_{b}h^2$</td>
<td>0.02212 ± 0.00022</td>
<td>0.02249 ± 0.00025</td>
<td>0.0240 ± 0.0012</td>
<td>0.02236 ± 0.00015</td>
<td>0.02237 ± 0.00015</td>
<td>0.02242 ± 0.00014</td>
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<td>$\Omega_{c}h^2$</td>
<td>0.1206 ± 0.0021</td>
<td>0.1177 ± 0.0020</td>
<td>0.1158 ± 0.0046</td>
<td>0.1202 ± 0.0014</td>
<td>0.1200 ± 0.0012</td>
<td>0.11933 ± 0.00091</td>
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<td>$100\theta_{MC}$</td>
<td>1.04077 ± 0.00047</td>
<td>1.04139 ± 0.00049</td>
<td>1.03999 ± 0.00089</td>
<td>1.04090 ± 0.00031</td>
<td>1.04092 ± 0.00031</td>
<td>1.04101 ± 0.00029</td>
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<td>$\tau$</td>
<td>0.0522 ± 0.0080</td>
<td>0.0496 ± 0.0085</td>
<td>0.0527 ± 0.0090</td>
<td>0.0544+0.0070&lt;0.0081</td>
<td>0.0544 ± 0.0073</td>
<td>0.0561 ± 0.0071</td>
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<tr>
<td>$\ln(10^{10}A_s)$</td>
<td>3.040 ± 0.016</td>
<td>3.018+0.020&lt;0.018</td>
<td>3.052 ± 0.022</td>
<td>3.045 ± 0.016</td>
<td>3.044 ± 0.014</td>
<td>3.047 ± 0.014</td>
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<tr>
<td>$n_s$</td>
<td>0.9626 ± 0.0057</td>
<td>0.967 ± 0.011</td>
<td>0.980 ± 0.015</td>
<td>0.9649 ± 0.0044</td>
<td>0.9649 ± 0.0042</td>
<td>0.9665 ± 0.0038</td>
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<tr>
<td>$H_0$ [km s$^{-1}$ Mpc$^{-1}$]</td>
<td>66.88 ± 0.92</td>
<td>68.44 ± 0.91</td>
<td>69.9 ± 2.7</td>
<td>67.27 ± 0.60</td>
<td>67.36 ± 0.54</td>
<td>67.66 ± 0.42</td>
</tr>
</tbody>
</table>
Filtered Maps and Power Spectrum

- Take original $64^\circ \times 64^\circ$ map
- Band filter to a range of multipole moments
Power Spectrum

- Curvature power spectrum is scale invariant to the extent that $H$ and $\epsilon_H$ are constant

$$\Delta^2_R \propto H^2 / \epsilon_H$$

- Scalar spectral index

$$\frac{d \ln \Delta^2_R}{d \ln k} \equiv n_S - 1 = 2 \frac{d \ln H}{d \ln k} - \frac{d \ln \epsilon_H}{d \ln k}$$

- Evaluate at horizon crossing where fluctuation freezes $k = aH$

$$\frac{d \ln H}{d \ln k} \approx \frac{d \ln H}{d \ln a} = -\epsilon_H$$

$$\frac{d \ln \epsilon}{d \ln k} \approx \frac{d \ln \epsilon}{d \ln a} = 2(\delta_1 + \epsilon_H)$$
Power Spectrum: $A_S, n_S$

- Tilt in the slow-roll approximation

$$n_S - 1 = -4\epsilon_H - 2\delta_1$$

- Power spectrum parameters:

$$\Delta_R^2 = A_S \left( \frac{k}{0.05\text{Mpc}^{-1}} \right)^{n_S-1}$$

with pivot scale 0.05 Mpc$^{-1}$ chosen to be approximately where the data constrains inflation
Gravitational Waves

- Gravitational wave power $\propto H^2 \propto V \propto E_i^4$ where $E_i$ is the energy scale of inflation
- Tensor-scalar ratio is therefore generally small

$$r \equiv 4\frac{\Delta^2}{\Delta^2_{R}} = 16\epsilon_H$$

- Gravitational waves from inflation can be measured via its imprint on the polarization of the CMB
CMB Power Spectra

- Power spectra of CMB
  - temperature
  - polarization
  - lensing
CMB Lensing

- Temperature fluctuations experience magnification and shear allowing mass reconstruction
- CMB lensing by an unrealistically large lens
Galaxy Redshift Surveys

- Galaxy redshift surveys (e.g. 2dF and SDSS) measure the three-dimensional distribution of galaxies today:

![Diagram showing the distribution of galaxies in redshift space with NEAR, MID, and FAR subsamples indicated.](image)
Gravitational Lensing

- Gravitational Lensing measures projected mass
- Planck CMB lensing map
Matter Power Spectrum

- Compilation of Redshift Surveys, Lensing, CMB
Structure Formation

- Small perturbations from inflation over the course of the 14Gyr life of the universe are gravitationally enhanced into all of the structure seen today.

- Cosmic microwave background shows a snapshot at a few hundred thousand years old at recombination.

- Discovery in 1992 of cosmic microwave background anisotropy provided the observational breakthrough - convincing support for adiabatic initial density fluctuations of amplitude $10^{-5}$.

- Combine with galaxy clustering - large scale structure seen in galaxy surveys - right amplitude given cold dark matter.
- Take apart features in the power spectrum

\[ \Delta (\mu K) \]

\[ l \quad l_{eq} \quad l_A \quad l_d \]
Fluid Approximation

- Thomson scattering of photons and free electrons before recombination is sufficiently rapid that the bayrons and photons are in equilibrium and hence move together.
- Mean free path of the photons for $z \approx 10^3$ and $\Omega_b h^2 \approx 0.02$

\[ \lambda_C \equiv \frac{1}{n_e \sigma_T a} \sim 2.5 \text{Mpc} \]

small by cosmological standards!

- On scales $\lambda \gg \lambda_C$ photons are tightly coupled to the electrons by Thomson scattering which in turn are tightly coupled to the baryons by Coulomb interactions.

- Specifically, their bulk velocities are defined by a single fluid velocity $v_\gamma = v_b$ and the photons carry no anisotropy in the rest frame of the baryons.
Zeroth Order Approximation

- **Momentum density** of a fluid is \((\rho + p)v\), where \(p\) is the pressure

- **Neglect** the momentum density of the baryons

\[
R \equiv \frac{(\rho_b + p_b)v_b}{(\rho_\gamma + p_\gamma)v_\gamma} = \frac{\rho_b + p_b}{\rho_\gamma + p_\gamma} = \frac{3\rho_b}{4\rho_\gamma}
\]

\[
\approx 0.6 \left( \frac{\Omega_b h^2}{0.02} \right) \left( \frac{a}{10^{-3}} \right)
\]

since \(\rho_\gamma \propto T^4\) is fixed by the CMB temperature \(T = 2.73(1 + z)K\)

- **OK substantially before recombination**

- **Neglect radiation in the expansion**

\[
\frac{\rho_m}{\rho_r} = 3.6 \left( \frac{\Omega_m h^2}{0.15} \right) \left( \frac{a}{10^{-3}} \right)
\]

- **Neglect gravity**
Fluid Equations

- **Density** \( n_\gamma \propto T^3 \) so define **temperature fluctuation** \( \Theta \)

\[
\delta n_\gamma = 3 \frac{\delta T}{T} n_\gamma \equiv 3 \Theta n_\gamma
\]

- **Real space** continuity eqn.: the local number or energy density of photons changes if there is a divergence of the velocity field - a flow inwards or outwards or a change in the volume

- We know in the background expansion \( n_\gamma \propto a^{-3} \) so continuity:

\[
[a^3 \delta n_\gamma] \dot{} = -a^3 n_\gamma \nabla \cdot \mathbf{v}_\gamma
\]

which we transform to Fourier space

\[
\nabla (e^{i\mathbf{k} \cdot \mathbf{x}}) \rightarrow i\mathbf{k} (e^{i\mathbf{k} \cdot \mathbf{x}})
\]

\[
\dot{\Theta} = -\frac{1}{3} k v_\gamma
\]
Fluid Equations

- Euler equation (neglecting gravity for now): momentum conservation says that pressure gradients generate changes in momentum density $k \delta p_\gamma = kc_s^2 \delta \rho_\gamma$

$$\dot{\gamma} = \frac{k c_s^2}{1 + w_\gamma} \delta_\gamma$$

$$= kc_s^2 \frac{3}{4} \delta_\gamma = 3 c_s^2 k \Theta$$

where the sound speed $c_s^2 = \delta p / \delta \rho$ is the pressure response to a density fluctuation

- So if you squeeze the photon gas to raise its density, its going to respond with a restoring force by raising the pressure and resisting compression $\rightarrow$ acoustic oscillations
Oscillator: Take One

• Combine these to form the simple harmonic oscillator equation

\[ \ddot{\Theta} + c_s^2 k^2 \Theta = 0 \]

where the sound speed is adiabatic

\[ c_s^2 = \frac{\delta p}{\delta \rho} = \frac{\dot{p}_\gamma}{\dot{\rho}_\gamma} \]

here \( c_s^2 = 1/3 \) since we are photon-dominated

• General solution:

\[ \Theta(\eta) = \Theta(0) \cos(k_s \eta) + \frac{\dot{\Theta}(0)}{kc_s} \sin(k_s \eta) \]

where the sound horizon is defined as \( s \equiv \int c_s d\eta \)
Harmonic Extrema

- All modes begin at end of inflation and are frozen in at recombination (denoted with a subscript $*$).
- Temperature perturbations of different amplitude for different modes.
- For the adiabatic (curvature mode) initial conditions

$$\dot{\Theta}(0) = 0$$

- So solution

$$\Theta(\eta_*) = \Theta(0) \cos(ks_*)$$
Harmonic Extrema

- Modes caught in the **extrema** of their oscillation will have enhanced fluctuations

\[ k_n s_* = n\pi \]

yielding a **fundamental scale** or frequency, related to the inverse sound horizon

\[ k_A = \pi / s_* \]

and a **harmonic relationship** to the other extrema as 1 : 2 : 3...
Temperature Anisotropy

- Spatial oscillations frozen at recombination; photons then stream
- Viewed at distance $D_*$ as angular anisotropy $L \approx kD_*$
Peak Location

- The fundamental **physical scale** is translated into a fundamental **angular scale** by simple projection according to the angular diameter distance $D_A$

\[
\theta_A = \frac{\lambda_A}{D_A} \\
\ell_A = k_A D_A
\]

- In a flat universe, the distance is simply $D_A = D \equiv \eta_0 - \eta_* \approx \eta_0$, the horizon distance, and $k_A = \frac{\pi}{s_*} = \sqrt{3\pi}/\eta_*$ so

\[
\theta_A \approx \frac{\eta_*}{\eta_0}
\]

- In a **matter-dominated** universe $\eta \propto a^{1/2}$ so $\theta_A \approx 1/30 \approx 2^\circ$ or

\[
\ell_A \approx 200
\]
In a curved universe, the apparent or angular diameter distance is no longer the arc distance $D_A = R \sin(D/R) \neq D$.

- Objects in a closed universe are further than they appear! gravitational lensing of the background...

- Curvature scale of the universe must be substantially larger than current horizon.
Curvature in the CMB

- Curvature and $\Lambda$ – consistent with flat $\Lambda$CDM
Restoring Gravity

- Take a simple photon dominated system with gravity

- **Continuity** altered since a gravitational potential represents a stretching of the spatial fabric that dilutes number densities – formally a spatial curvature perturbation

- Think of this as a perturbation to the scale factor $a \rightarrow a(1 + \Phi)$ so that the cosmogical redshift is generalized to

\[
\frac{\dot{a}}{a} \rightarrow \frac{\dot{a}}{a} + \dot{\Phi}
\]

so that the **continuity equation** becomes

\[
\dot{\Theta} = -\frac{1}{3} kv_\gamma - \dot{\Phi}
\]
Restoring Gravity

- Gravitational force in momentum conservation $\mathbf{F} = -m \nabla \Psi$ generalized to momentum density modifies the Euler equation to

$$\dot{v}_\gamma = k (\Theta + \Psi)$$

- General relativity says that $\Phi$ and $\Psi$ are the relativistic analogues of the Newtonian potential and that $\Phi \approx -\Psi$.

- In our matter-dominated approximation, $\Phi$ represents matter density fluctuations through the cosmological Poisson equation

$$k^2 \Phi = 4\pi G a^2 \rho_m \Delta_m$$

where the difference comes from the use of comoving coordinates for $k$ ($a^2$ factor), the removal of the background density into the background expansion ($\rho \Delta_m$) and finally a coordinate subtlety that enters into the definition of $\Delta_m$. 
Constant Potentials

- In the matter dominated epoch potentials are constant because infall generates velocities as $v_m \sim k\eta \Psi$

- Velocity divergence generates density perturbations as $\Delta_m \sim -k\eta v_m \sim -(k\eta)^2 \Psi$

- And density perturbations generate potential fluctuations

$$\Phi = \frac{4\pi G a^2 \rho \Delta}{k^2} \approx \frac{3}{2} \frac{H^2 a^2}{k^2} \Delta \sim \frac{\Delta}{(k\eta)^2} \sim -\Psi$$

keeping them constant. Note that because of the expansion, density perturbations must grow to keep potentials constant.
Constant Potentials

- More generally, if stress perturbations are negligible compared with density perturbations ($\delta p \ll \delta \rho$) then potential will remain roughly constant

- More specifically a variant called the Bardeen or comoving curvature is strictly constant

$$R = \text{const} \approx \frac{5 + 3w}{3 + 3w} \Phi$$

where the approximation holds when $w \approx \text{const}$. 
Oscillator: Take Two

- Combine these to form the simple harmonic oscillator equation

\[ \ddot{\Theta} + c_s^2 k^2 \Theta = -\frac{k^2}{3} \Psi - \dot{\Phi} \]

- In a CDM dominated expansion \( \dot{\Phi} = \dot{\Psi} = 0 \). Also for photon domination \( c_s^2 = 1/3 \) so the oscillator equation becomes

\[ \ddot{\Theta} + \ddot{\Psi} + c_s^2 k^2 (\Theta + \Psi) = 0 \]

- Solution is just an offset version of the original

\[ [\Theta + \Psi](\eta) = [\Theta + \Psi](0) \cos(ks) \]

- \( \Theta + \Psi \) is also the observed temperature fluctuation since photons lose energy climbing out of gravitational potentials at recombination
Effective Temperature

- Photons climb out of potential wells at last scattering
- Lose energy to gravitational redshifts
- Observed or effective temperature
  \[ \Theta + \Psi \]
- Effective temperature oscillates around zero with amplitude given by the initial conditions
- Note: initial conditions are set when the perturbation is outside of horizon, need inflation or other modification to matter-radiation FRW universe.
- GR says that initial temperature is given by initial potential
Sachs-Wolfe Effect and the Magic 1/3

- A gravitational potential is a perturbation to the temporal coordinate [formally a gauge transformation]

$$\frac{\delta t}{t} = \Psi$$

- Convert this to a perturbation in the scale factor, in a matter dominated expansion $a \propto t^{2/3}$ so

$$\frac{\delta a}{a} = \frac{2}{3} \frac{\delta t}{t}$$

- CMB temperature is cooling as $T \propto a^{-1}$ so

$$\Theta + \Psi \equiv \frac{\delta T}{T} + \Psi = -\frac{\delta a}{a} + \Psi = \frac{1}{3} \Psi$$
Sachs-Wolfe Normalization

- Use measurements of $\Delta T/T \approx 10^{-5}$ in the Sachs-Wolfe effect to infer $\Delta_R^2$

- Recall in matter domination $\Psi = -3\mathcal{R}/5$ and so $\Delta T/T = -\mathcal{R}/5$

- So that the amplitude of initial curvature fluctuations is $\Delta_R \approx 5 \times 10^{-5}$

- This then determines the amplitude of the inflationary power spectrum $A_S = \Delta_R^2$ in the previous lecture set
Baryon Loading

- Baryons add extra mass to the photon-baryon fluid
- Controlling parameter is the momentum density ratio:
  \[
  R \equiv \frac{p_b + \rho_b}{p_\gamma + \rho_\gamma} \approx 30 \Omega_b h^2 \left( \frac{a}{10^{-3}} \right)
  \]
of order unity at recombination
- Momentum density of the joint system is conserved
  \[
  (\rho_\gamma + p_\gamma)v_\gamma + (\rho_b + p_b)v_b \approx (p_\gamma + p_\gamma + \rho_b + \rho_\gamma)v_\gamma = (1 + R)(\rho_\gamma + p_\gamma)v_\gamma
  \]
- Momentum density ratio enters as
  \[
  [(1 + R)v_\gamma] = k\Theta + (1 + R)k\Psi
  \]
- Oscillations around hydrostatic equilibrium point:
  \[
  \Theta + (1 + R)\Psi = 0 - \text{like clusters, measurement of dark matter}
New Euler Equation

- Photon continuity remains the same

\[ \dot{\Theta} = -\frac{k}{3} v_\gamma - \dot{\Phi} \]

- Modification of oscillator equation

\[ \frac{d}{d\eta}[(1 + R)\dot{\Theta}] + \frac{1}{3} k^2 \Theta = -\frac{1}{3} k^2 (1 + R)\Psi - \frac{d}{d\eta}[(1 + R)\dot{\Phi}] \]

- In a CDM dominated expansion \( \dot{\Phi} = \dot{\Psi} = 0 \) and the adiabatic approximation where the sound speed evolves slowly

\[ c_s = \sqrt{\frac{1}{3} \frac{1}{1 + R}} \]

\[ [\Theta + (1 + R)\Psi](\eta) = [\Theta + (1 + R)\Psi](0) \cos(k_s) \]
Baryons in the CMB

- Modulation, amplitude, sound horizon scale

![Graph showing modulation, amplitude, and sound horizon scale in the CMB spectrum.](image-url)
Baryon Peak Phenomenology

- Photon-baryon ratio enters in three ways
- Overall larger amplitude:
  \[ [\Theta + (1 + R)\Psi](0) = \frac{1}{3}(1 + 3R)\Psi(0) \]
- Even-odd peak modulation of effective temperature
  \[ [\Theta + \Psi]_{\text{peaks}} = [\pm(1 + 3R) - 3R] \frac{1}{3}\Psi(0) \]
  \[ [\Theta + \Psi]_1 - [\Theta + \Psi]_2 = [-6R] \frac{1}{3}\Psi(0) \]
- Shifting of the sound horizon down or \( \ell_A \) up
  \[ \ell_A \propto \sqrt{1 + R} \]
Baryons in the Power Spectrum

- Relative heights of peaks

\[
\Omega_b h^2
\]

\[
\Delta T (\mu K)
\]

\[
l
\]
Oscillator: Take Three and a Half

- The not-quite-so simple harmonic oscillator equation is a forced harmonic oscillator

\[(c_s^{-2} \dot{\Theta})' + k^2 \Theta = -\frac{k^2}{3} c_s^{-2} \Psi - (c_s^{-2} \dot{\Phi})'.\]

changes in the gravitational potentials alter the form of the acoustic oscillations

- If the forcing term has a temporal structure that is related to the frequency of the oscillation, this becomes a driven harmonic oscillator

- Term involving \(\Psi\) is the ordinary gravitational force

- Term involving \(\Phi\) involves the \(\dot{\Phi}\) term in the continuity equation as a (curvature) perturbation to the scale factor
Potential Decay

- Matter-to-radiation ratio

\[
\frac{\rho_m}{\rho_r} \approx 24 \Omega_m h^2 \left( \frac{a}{10^{-3}} \right)
\]

of order unity at recombination in a low \( \Omega_m \) universe

- Radiation is not stress free and so impedes the growth of structure

\[
k^2 \Phi = 4\pi G a^2 \rho_r \Delta_r
\]

\( \Delta_r \sim 4\Theta \) oscillates around a constant value, \( \rho_r \propto a^{-4} \) so the Newtonian curvature decays.

- General rule: potential decays if the dominant energy component has substantial stress fluctuations, i.e. below the generalized sound horizon or Jeans scale
Radiation Driving

- Decay is timed precisely to drive the oscillator - close to fully coherent

\[ |[\Theta + \Psi](\eta)| = |[\Theta + \Psi](0) + \Delta\Psi - \Delta\Phi| \]

\[ = |\frac{1}{3}\Psi(0) - 2\Psi(0)| = \frac{5}{3}|\Psi(0)| \]

- \(5\times\) the amplitude of the Sachs-Wolfe effect!
Radiation Driving

- Cartoon version (doubled by local scale factor Φ effect):

Decay and Gravitational Driving
Cold Dark Matter in the CMB

- Hydrostatic equilibrium, oscillation forcing, damping
Matter-Radiation in the Power Spectrum

- Coherent approximation is exact for a photon-baryon fluid but reality is reduced to $\sim 4 \times$ because neutrino contribution is free streaming not fluid like

- Neutrinos drive the oscillator less efficiently and also slightly change the phase of the oscillation

- Actual initial conditions are $\Theta + \Psi = \Psi/2$ for radiation domination but comparison to matter dominated SW correct

- With 3 peaks, it is possible to solve for both the baryons and dark matter densities, providing a calibration for the sound horizon

- Higher peaks check consistency with assumptions: e.g. extra relativistic d.o.f.s
Schematic CMB Spectrum

- Take apart features in the power spectrum
Damping

- Tight coupling equations assume a perfect fluid: no viscosity, no heat conduction
- Fluid imperfections are related to the mean free path of the photons in the baryons
  \[ \lambda_C = \frac{1}{\dot{\tau}} \quad \text{where} \quad \dot{\tau} = n_e \sigma T a \]
  \( \lambda_C \) is the conformal opacity to Thomson scattering
- Dissipation is related to the diffusion length: random walk approximation
  \[ \lambda_D = \sqrt{\lambda_C} = \sqrt{\frac{\eta}{\lambda_C}} \lambda_C = \sqrt{\eta \lambda_C} \]
  the geometric mean between the horizon and mean free path
- \( \lambda_D/\eta_* \sim \text{few \%} \), so expect the peaks \( \gg 3 \) to be affected by dissipation
Near Perfection in 6 Numbers

- All this precision data described by 6 $\Lambda$CDM parameters
  - $\Omega_c h^2$: CDM
  - $\Omega_b h^2$: baryons
  - $\theta_s$: sound scale
  - $A_s$: amplitude
  - $n_s$: tilt
  - $\tau$: reionization

- Measured to sub percent precision (except $\tau$)
Predictive Power

- Predicts all other observables, which direct measurements test

- Good agreement, even weak lensing, clusters, and yes $H_0 (< 10\%)$
Polarization and Gravitational Waves

- Thomson scattering generates linear polarization
Polarization and Gravitational Waves

- An isotropic medium by symmetry leads to no net polarization
Polarization and Gravitational Waves

- Quadrupole anisotropy provides polarization source
Gravitational Wave Quadrupoles

- Changing transverse-traceless distortion of space creates a quadrupole CMB anisotropy much like the distortion of test ring of particles.

- As the tensor mode enters the horizon it imprints a quadrupole temperature $\ell = 2, m = \pm 2$ in plane wave coordinates $k \parallel z$.

- Modes that cross before recombination: effect erased by rescattering $e^{-\tau}$ due to its isotropizing effect.

- Modes that cross after recombination: integrate contributions along the line of sight - tensor ISW effect.
Observability

- Gravitational waves from inflation yet to be detected but is the goal of many current and future CMB experiments
Transfer Function

- Transfer function transfers the initial Newtonian curvature to its value today (linear response theory)

\[ T(k) = \frac{\Phi(k, a = 1)}{\Phi(k, a_{\text{init}})} \cdot \frac{\Phi(k_{\text{norm}}, a_{\text{init}})}{\Phi(k_{\text{norm}}, a = 1)} \]

- Conservation of Bardeen curvature: Newtonian curvature is a constant when stress perturbations are negligible: above the horizon during radiation and dark energy domination, on all scales during matter domination

- When stress fluctuations dominate, perturbations are stabilized by the Jeans mechanism

- Hybrid Poisson equation: Newtonian curvature, comoving density perturbation \( \Delta \equiv (\delta \rho/\rho)_{\text{com}} \) implies \( \Phi \) decays

\[ (k^2 - 3K)\Phi = 4\pi G a^2 \rho \Delta \sim \eta^{-2} \Delta \]
Transfer Function

- Freezing of $\Delta$ stops at $\eta_{eq}$

$$\Phi \sim (k\eta_{eq})^{-2} \Delta_H \sim (k\eta_{eq})^{-2} \Phi_{init}$$

- Transfer function has a $k^{-2}$ fall-off beyond $k_{eq} \sim \eta_{eq}^{-1}$

$$\eta_{eq} = 15.7(\Omega_m h^2)^{-1} \left(\frac{T}{2.7 K}\right)^2 \text{Mpc}$$

- Small correction since growth with a smooth radiation component is logarithmic not frozen

- Transfer function is a direct output of an Einstein-Boltzmann code
Fitting Function

• Alternately accurate fitting formula exist, e.g. pure CDM form:

\[
T(k(q)) = \frac{L(q)}{L(q) + C(q)q^2}
\]

\[
L(q) = \ln(e + 1.84q)
\]

\[
C(q) = 14.4 + \frac{325}{1 + 60.5q^{1.11}}
\]

\[
q = \frac{k}{\Omega_m h^2 \text{Mpc}^{-1}} \left(\frac{T_{\text{CMB}}}{2.7K}\right)^2
\]

• In \(h \text{ Mpc}^{-1}\), the critical scale depends on \(\Gamma \equiv \Omega_m h\) also known as the shape parameter
Transfer Function

- Numerical calculation

The diagram shows the transfer function $T(k)$ as a function of $k (h^{-1} \text{Mpc})$, where $k$ is the wave number. The plot indicates a power-law behavior with $k^{-2}$, and there are wiggles in the data.
Baryon Acoustic Oscillations

- Baryons caught up in the acoustic oscillations of the CMB and impart acoustic oscillations to the transfer function. Density enhancements are produced kinematically through the continuity equation $\delta_b \sim (k\eta)v_b$ and hence are out of phase with CMB temperature peaks.

- Dissipation of the acoustic oscillations eliminates both the CMB and baryon perturbations – known as Silk damping for the baryons. This suppression and the general fact that baryons are caught up with photons was one of the main arguments for CDM.

- Detected first in the SDSS LRG survey.

- An excellent standard ruler for angular diameter distance $D_A(z)$ since it does not evolve in redshift in linear theory.

- Radial extent of BAO gives $H(z)$. 

Power Spectrum

- Large scale structure: galaxy clustering, Ly$\alpha$ forest, cosmic shear