Set 8:
Nonlinear Structure
Nonlinear regime

- Inflationary initial perturbations provide density perturbations $\delta = \delta \rho / \rho$ that grow as $\delta \propto a$ in the linear regime

- $\Delta^2 = k^3 P(k) / 2\pi^2$
  
  contribution to variance $\langle \delta^2 \rangle$ per $d \ln k$

- Linear theory would predict that for $k > 0.1 h \text{Mpc}^{-1}$, $\langle \delta^2 \rangle > 1$

- Linear approximation breaks down at this point and we must follow the nonlinear equations

- Nonlinearities further enhance the formation of structure
Cosmological Simulations

- To evolve structure further requires cosmological simulations (see video)
- Many simulation results can be understood using simple analytic arguments
- High density fluctuations break away from the cosmological expansion and form bound objects called dark matter halos
- Halo formation can be understood in the spherical collapse model as an FRW background expansion in a slightly closed (positive curvature) universe
- If observable properties such as galaxies (stars) and gas can be associated with halos, then they can be modeled in a “halo model”
Newtonian Cosmology

- Recall that in the Newtonian interpretation of the expansion we can model the dynamics of a spherical volume of constant density using mass conservation.

- Energy conservation

\[
E = \frac{1}{2} mv^2 - \frac{GMm}{r} = \text{const}
\]

\[
\frac{1}{2} \left( \frac{dr}{dt} \right)^2 - \frac{GM}{r} = \text{const}
\]

\[
\frac{1}{2} \left( \frac{1}{r} \frac{dr}{dt} \right)^2 - \frac{GM}{r^3} = \frac{\text{const}}{r^2}
\]

\[
H^2 = \frac{8\pi G \rho}{3} - \frac{\text{const}}{a^2}
\]
Newtonian Energy Interpretation

- Constant determines whether the system recollapses or expands forever.
- These equations define the evolution of not just the homogeneous cosmology but also a spherically symmetric “top hat” or spatially constant density perturbation in a matter dominated universe.
- An overdense region will eventually collapse and form a “dark matter halo” just like a positive curvature universe.
Closed Universe

- **Friedmann equation** in a closed universe \( K > 0 \)

\[
H^2 + \frac{K}{a^2} = \frac{8\pi G}{3} \rho
\]

when \( K/a^2 = 8\pi G \rho/3 \) then \( H = 0 \) and the expansion of the universe turns around into recollapse. In cosmo parameters

\[
\frac{1}{a} \frac{da}{dt} = H_0 \left( \Omega_m a^{-3} - (\Omega_m - 1) a^{-2} \right)^{1/2}
\]

this would occur when \( a = \Omega_m/(\Omega_m - 1) \)

- Now consider an initially small local overdensity

\[
\rho(x, t) = \bar{\rho}(t)[1 + \delta(x, t)]
\]

in a globally \( \Omega_m = 1 \) universe- a local observer would see this as a slightly closed universe with an \( \Omega_m > 1 \)
Spherical Collapse

- Spherical collapse calculation makes use of this remapping by matching the initial density perturbations to the local expansion parameters and determining the epoch of collapse.
- Basic idea: mass $M$ enclosed by the region $r = ar_0$ remains constant so solve for $r(t)$ and match to the initial density perturbation $\delta_i$ at $a_i$.
- Completing this step (details below) we have the following parametric solution:

$$r(\theta) = \frac{3}{10} \frac{r_i}{\delta_i} (1 - \cos \theta)$$

$$t(\theta) = \frac{1}{2H_0 \Omega_m^{1/2}} \left( \frac{3}{5} \frac{a_i}{\delta_i} \right)^{3/2} (\theta - \sin \theta)$$

where $\theta$ is the development angle with $\theta = \pi$ the turnaround point, $\theta = 2\pi$ the collapse point.
Spherical Collapse Relations

• Scale factor $a \propto t^{2/3}$

$$a = \left( \frac{3}{4} \right)^{2/3} \left( \frac{3}{5} \frac{a_i}{\delta_i} \right) (\theta - \sin \theta)^{2/3}$$

• At collapse $\theta = 2\pi$

$$a_{\text{col}} = \left( \frac{3}{4} \right)^{2/3} \left( \frac{3}{5} \frac{a_i}{\delta_i} \right) (2\pi)^{2/3} \approx 1.686 \frac{a_i}{\delta_i}$$

• In linear theory and a matter dominated universe $\delta \propto a$ so $\delta_{\text{linear}} = (a_{\text{col}}/a_i)\delta_i = 1.686$ – Rule of thumb to remember – perturbation collapses when linear theory predicts $\delta_c \equiv 1.686$

• Interpretation: when linear theory predicts an $O(1)$ density perturbation, in the real universe, that density perturbation has already collapsed to a nonlinear object
Derivation for advanced students: since $r = a r_0$ we can use the Friedmann solution for $a$ in a closed universe to find $r(t)$

In terms of development angle $\theta = H_0 \eta (\Omega_m - 1)^{1/2}$, scaled conformal time $\eta$

$$r(\theta) = A(1 - \cos \theta)$$
$$t(\theta) = B(\theta - \sin \theta)$$

where $A = r_0 \Omega_m / 2(\Omega_m - 1)$, $B = H_0^{-1} \Omega_m / 2(\Omega_m - 1)^{3/2}$.

Turn around at $\theta = \pi$, $r = 2A$, $t = B \pi$.

Collapse at $\theta = 2\pi$, $r \to 0$, $t = 2\pi B$.

Now we need to find the $A$ and $B$ constants given an initial density perturbation.
Spherical Collapse

- Parametric Solution:
Correspondence

- Eliminate cosmological correspondence in $A$ and $B$ in terms of enclosed mass $M$

$$M = \frac{4\pi}{3} r_0^3 \Omega_m \rho_c = \frac{4\pi}{3} r_0^3 \Omega_m \frac{3H_0^2}{8\pi G}$$

- Related as $A^3 = GM^2B^2$, and to initial perturbation

$$\lim_{\theta \to 0} r(\theta) = A \left( \frac{1}{2} \theta^2 - \frac{1}{24} \theta^4 \right)$$

$$\lim_{\theta \to 0} t(\theta) = B \left( \frac{1}{6} \theta^3 - \frac{1}{120} \theta^5 \right)$$

- Leading Order: $r = A\theta^2/2$, $t = B\theta^3/6$

$$r = \frac{A}{2} \left( \frac{6t}{B} \right)^{2/3}$$
Next Order

- Leading order is unperturbed matter dominated expansion
  \( r \propto a \propto t^{2/3} \)

- Iterate \( r \) and \( t \) solutions

\[
\lim_{\theta \to 0} t(\theta) = \frac{\theta^3}{6} B \left[ 1 - \frac{1}{20} \left( \frac{6t}{B} \right)^{2/3} \right]
\]

\[
\theta \approx \left( \frac{6t}{B} \right)^{1/3} \left[ 1 + \frac{1}{60} \left( \frac{6t}{B} \right)^{2/3} \right]
\]
Next Order

- Substitute back into \( r(\theta) \)

\[
r(\theta) = A \frac{\theta^2}{2} \left( 1 - \frac{\theta^2}{12} \right)
\]

\[
= \frac{A}{2} \left( \frac{6t}{B} \right)^{2/3} \left[ 1 - \frac{1}{20} \left( \frac{6t}{B} \right)^{2/3} \right]
\]

\[
= \frac{1}{2} (6t)^{2/3} (GM)^{1/3} \left[ 1 - \frac{1}{20} \left( \frac{6t}{B} \right)^{2/3} \right]
\]
Density Correspondence

- Density

\[ \rho_m = \frac{M}{\frac{4}{3} \pi r^3} = \frac{1}{6\pi t^2 G} \left[ 1 + \frac{3}{20} \left( \frac{6t}{B} \right)^{2/3} \right] \]

- Density perturbation

\[ \delta \equiv \frac{\rho_m - \bar{\rho}_m}{\bar{\rho}_m} \approx \frac{3}{20} \left( \frac{6t}{B} \right)^{2/3} \]
Density Correspondence

- Time $\rightarrow$ scale factor

$$t = \frac{2}{3H_0 \Omega_m^{1/2}} a^{3/2}$$

$$\delta = \frac{3}{20} a \left( \frac{4}{BH_0 \Omega_m^{1/2}} \right)^{2/3}$$

- $A$ and $B$ constants $\rightarrow$ initial cond.

$$B = \frac{1}{2H_0 \Omega_m^{1/2}} \left( \frac{3 a_i}{5 \delta_i} \right)^{3/2}$$

$$A = \frac{3 r_i}{10 \delta_i}$$

- End derivation for advanced student
Virialization

- A real density perturbation is neither spherical nor homogeneous
- **Shell crossing** if \( \delta_i \) doesn’t monotonically decrease
- Collapse does not proceed to a point but reaches *virial equilibrium*

\[
U = -2K, \quad E = U + K = U(r_{\text{max}}) = \frac{1}{2}U(r_{\text{vir}})
\]

\[
r_{\text{vir}} = \frac{1}{2}r_{\text{max}} \text{ since } U \propto r^{-1}. \text{ Thus } \theta_{\text{vir}} = \frac{3}{2}\pi
\]

- **Overdensity** at virialization

\[
\frac{\rho_m(\theta = 3\pi/2)}{\bar{\rho}_m(\theta = 2\pi)} = 18\pi^2 \approx 178
\]

- Threshold \( \Delta_v = 178 \) often used to define a collapsed object
- Equivalently relation between virial mass, radius, overdensity:

\[
M_v = \frac{4\pi}{3} r_v^3 \rho_m \Delta_v
\]
Virialization

- Schematic Picture:
The Mass Function

- **Spherical collapse** predicts the end state as virialized halos given an initial density perturbation.
- Allows us to predict, from a linear analysis, the outcome of a nonlinear simulation, e.g. the abundance of halos.

- Initial density perturbation is a **Gaussian random field**.
- Compare the variance in the linear density field to threshold $\delta_c = 1.686$ to determine collapse fraction.
- Combine to form the **mass function**, the number density of halos in a range $dM$ around $M$.
- Halo density defined entirely by linear theory.
Press-Schechter Formalism

- **Smooth** linear density field on mass scale $M$ with tophat

$$R = \left( \frac{3M}{4\pi} \right)^{1/3}$$

- Result is a Gaussian random field with variance $\sigma^2(M)$

- Fluctuations above the threshold $\delta_c$ correspond to **collapsed** regions. The fraction in halos $> M$ becomes

$$\frac{1}{\sqrt{2\pi\sigma(M)}} \int_{\delta_c}^{\infty} d\delta \exp \left( -\frac{\delta^2}{2\sigma^2(M)} \right) = \frac{1}{2} \text{erfc} \left( \frac{\nu}{\sqrt{2}} \right)$$

where $\nu \equiv \delta_c / \sigma(M)$

- **Problem**: even as $\sigma(M) \to \infty$, $\nu \to 0$, collapse fraction $\to 1/2$ – only **overdense regions** participate in spherical collapse.

- **Multiply by 2!** Justified by underdensity within overdensity
Press-Schechter Mass Function

- **Differentiate** in $M$ to find fraction in range $dM$ and multiply by $\rho_m/M$ the number density of halos if all of the mass were composed of such halos $\rightarrow$ **differential number density** of halos

$$\frac{dn}{d \ln M} = \frac{\rho_m}{M} \frac{d}{d \ln M} \text{erfc} \left( \frac{\nu}{\sqrt{2}} \right)$$

$$= \sqrt{\frac{2}{\pi}} \frac{\rho_m}{M} \frac{d \ln \sigma^{-1}}{d \ln M} \nu \exp(-\nu^2/2)$$

- **High mass**: exponential cut off above $M_*$ where $\sigma(M_*) = \delta_c$

$$M_* \sim 10^{13} h^{-1} M_\odot \quad \text{today}$$

- **Low mass divergence**: (too many for the observations?)

$$\frac{dn}{d \ln M} \propto \sim M^{-1}$$
**Numerical Mass Function**

- Fit cosmological simulations to Press Schechter motivated form

\[
\frac{dN}{dM} = \frac{\Omega_m}{\Gamma} \left( \frac{M}{M_*} \right)^{\gamma} \exp \left( - \frac{\Delta^2}{2\sigma^2} \right)
\]

\[M_* = \frac{M}{h} \]
Halo Bias

- If halos are formed without regard to the underlying density fluctuation and move under the gravitational field then their number density is an unbiased tracer of the dark matter density fluctuation

\[
\left( \frac{\delta n}{n} \right)_{\text{halo}} = \left( \frac{\delta \rho}{\rho} \right)
\]

- However spherical collapse says the probability of forming a halo depends on the initial density field

- Large scale density field acts as “background” enhancement of probability of forming a halo or “peak”

- Peak-Background Split (Efstathiou 1998; Cole & Kaiser 1989; Mo & White 1997)
Peak-Background Split

- Schematic Picture:

![Graph showing peak-background split with labeled axes and annotations. The graph illustrates a large-scale "background" and enhanced "peaks" with a threshold at \( \delta_c \).]
Perturbed Mass Function

- Density fluctuation split

\[ \delta = \delta_b + \delta_p \]

- Lowers the threshold for collapse

\[ \delta_{cp} = \delta_c - \delta_b \]

so that \( \nu = \delta_{cp}/\sigma \)

- Taylor expand number density \( n_M \equiv dn/d\ln M \)

\[ n_M + \frac{dn_M}{d\nu} \frac{d\nu}{d\delta_b} \delta_b \ldots = n_M \left[ 1 + \frac{(\nu^2 - 1)}{\sigma \nu} \delta_b \right] \]

if mass function is given by Press-Schechter

\[ n_M \propto \nu \exp(-\nu^2/2) \]
Halo Bias

- Halos are biased tracers of the “background” dark matter field with a bias \( b(M) \) that is given by spherical collapse and the form of the mass function

- Combine the enhancement with the original unbiased expectation

\[
\frac{\delta n_M}{n_M} = b(M) \delta_b
\]

- For Press-Schechter

\[
b(M) = 1 + \frac{\nu^2 - 1}{\delta_c}
\]

- Improved by the Sheth-Torman mass function

\[
b(M) = 1 + \frac{a\nu^2 - 1}{\delta_c} + \frac{2p}{\delta_c[1 + (a\nu^2)^p]}
\]

with \( a = 0.75 \) and \( p = 0.3 \) to match simulations.
Numerical Bias

- Example of halo bias from a simulation

$\langle b(M) \rangle = \frac{\xi_{hm}/\xi_{mm}}{x}$
NFW Profile

- Density profile well-described by (Navarro, Frenk & White 1997)

\[ \rho(r) = \frac{\rho_s}{(r/r_s)(1 + r/r_s)^2} \]
The Halo Model

- NFW halos, of abundance $n_M$ given by mass function, clustered according to the halo bias $b(M)$ and the linear theory $P(k)$

- Power spectrum example:
Galaxy Power Spectrum

- For galaxies, one defines a halo occupation distribution which determines the number of galaxies (satisfying a certain observational criteria) that can occupy a halo of mass $M$.
- Galaxy clustering or power spectrum composed from the distribution of galaxies in halos and the clustering of halos in large scale structure.
- Generalize to clustering statistic of any observable that is associated with dark matter haloes, e.g. gas, gravitational lensing etc.
Hierarchical Structure Formation

- With the density power spectrum from linear theory implying that the variance increases as $R$ decreases and time increases, small halos form first.

- Exponential suppression of high mass halos – evolution starts with low masses and progresses to higher masses as halos merge (see video).

- Small halos are not hot enough to stimulate atomic line transitions and then cool and fragment by radiating.

- Stars form in halos only at $z < 10$ and radiation from stars can then reionize the plasma.
Galaxy Formation

- Merging proto-Galactic objects of $10^6 - 10^8 M_\odot$ can eventually assemble the galaxies of $10^{12} M_\odot$ we see today. Both lower and upper range determined by cooling.

- Proto-galactic objects can form if cooling is sufficiently rapid that the heating of the gas during collapse (which would prevent collapse due to pressure, internal motions) can be overcome.

- Recall virial theorem supplies estimate of thermal kinetic energy

$$-2\langle K \rangle = \langle U \rangle$$

$$-2N \frac{1}{2} \mu m_H \sigma^2 = -\frac{3}{5} \frac{GMN\mu m_H}{R}$$

where $\mu m_H$ is the average mass of particles in the gas, $M$ is the total mass and $\sigma$ is the rms velocity.
Galaxy Formation

• Solve for velocity dispersion for a self gravitating system

\[ \sigma = \left( \frac{3 \, GM}{5 \, R} \right)^{1/2} \]

• Associate the average kinetic energy with a temperature, called the virial temperature

\[ \frac{1}{2} \mu m_H \sigma^2 = \frac{3}{2} k T_{\text{virial}} \]

where \( \mu \) is the mean molecular weight. Solve for virial temperature

\[ T_{\text{virial}} = \frac{\mu m_H \sigma^2}{3k} = \frac{\mu m_H}{5k} \frac{GM}{R} \approx \frac{\mu m_H}{5k} GM^{2/3} \left( \frac{4\pi \rho}{3} \right)^{1/3} \]

• Cooling is a function of the gas temperature through the cooling function.
Galaxy Formation

- Cooling rate (luminosity) per volume

\[ r_{\text{cool}} = n^2 \Lambda(T) \]

- \( n^2 \) (number density squared) since cooling is usually a 2 body process – for \( T > 10^6 \text{K} \) thermal bremsstrahlung and Compton scattering, for \( T \sim 10^4 - 10^5 \text{K} \) from the collisional excitation of atomic lines of hydrogen and helium

- Galaxy formation only starts when dark matter mass makes the virial temperature exceed \( T \sim 10^4 \text{K} \) when cooling becomes efficient \( M \sim 10^8 M_\odot \) - first objects and current dwarf ellipticals

- At \( z < 10 \), these halos abundant enough for UV light from their stars to reionize \( H \) – final ΛCDM parameter \( \tau \), optical depth to Thomson scattering
Galaxy Formation

- Cooling time is the time required to radiate away all of the thermal energy of the gas

\[ r_{cool} V t_{cool} = \frac{3}{2} N k T_{\text{virial}} \]

\[ t_{cool} = \frac{3}{2} \frac{k T_{\text{virial}}}{n \Lambda} \]

- Compared with the free fall time - from our dimensional relation

\[ GM \sim R v^2 \sim R \left( R^2 / t_{ff}^2 \right), \quad M \propto \rho R^3 \]

we get \( t_{ff} \propto (G \rho)^{-1/2} \) with the proportionality given for the time of collapse for a homogeneous sphere of initial density \( \rho \)

\[ t_{ff} = \left( \frac{3 \pi}{32} \frac{1}{G \rho} \right)^{1/2} \]
Galaxy Formation

• If $t_{\text{cool}} < t_{\text{ff}}$ then the object will collapse essentially in free fall - fragment and form stars. If opposite, then gravitational potential energy heats the gas making it stabilized by pressure establishing virial equilibrium

\[
\left( \frac{t_{\text{ff}}}{t_{\text{cool}}} \right) > \left( \frac{3\pi}{32} \frac{1}{G\rho} \right)^{1/2} \frac{2}{3} \frac{n\Lambda}{kT_{\text{virial}}}
\]

• Taking typical numbers $T \sim 10^6 \text{K}$ and $n \sim 5 \times 10^4 \text{m}^{-3}$ and with the density of the collapsing medium being associated with the gas $\rho = \mu m_H n$ gives an upper limit on the gas mass that can cool of $10^{12} M_\odot$ comparable to a large galaxy.
Disk Formation

- Proto-galactic gas fragment and collide retaining initial angular momentum provided from torques from other proto-galactic systems

- Rotationally supported gas disk, cooling in dense regions until HI clouds form from which star formation occurs - thick disk

- Cool molecular gas settles to midplane of thick disk efficiently forming stars - thinness is self regulating - if disk continued to get thinner then density and star formation goes up heating the material and re-puffing out the disk