Transfer Function

- Transfer function transfers the initial Newtonian curvature to its value today (linear response theory)

$$T(k) = \frac{\Phi(k, a=1)}{\Phi(k, a_{\rm init})} \frac{\Phi(k_{\rm norm}, a_{\rm init})}{\Phi(k_{\rm norm}, a=1)}$$

- Conservation of Bardeen curvature: Newtonian curvature is a constant when stress perturbations are negligible: above the horizon during radiation and dark energy domination, on all scales during matter domination
- When stress fluctuations dominate, perturbations are stabilized by the Jeans mechanism
- Hybrid Poisson equation: Newtonian curvature, comoving density perturbation $\Delta \equiv (\delta\rho/\rho)_{\rm com}$ implies Φ decays

$$(k^2 - 3K)\Phi = 4\pi G a^2 \rho \Delta \sim \eta^{-2} \Delta$$

Transfer Function

- Freezing of Δ stops at $\eta_{\rm eq}$

$$\Phi \sim (k\eta_{\rm eq})^{-2}\Delta_H \sim (k\eta_{\rm eq})^{-2}\Phi_{\rm init}$$

- Transfer function has a k^{-2} fall-off beyond $k_{\rm eq} \sim \eta_{\rm eq}^{-1}$

$$\eta_{\rm eq} = 15.7(\Omega_m h^2)^{-1}\left(\frac{T}{2.7K}\right)^2 \rm Mpc$$

- Small correction since growth with a smooth radiation component is logarithmic not frozen
- Transfer function is a direct output of an Einstein-Boltzmann code

Fitting Function

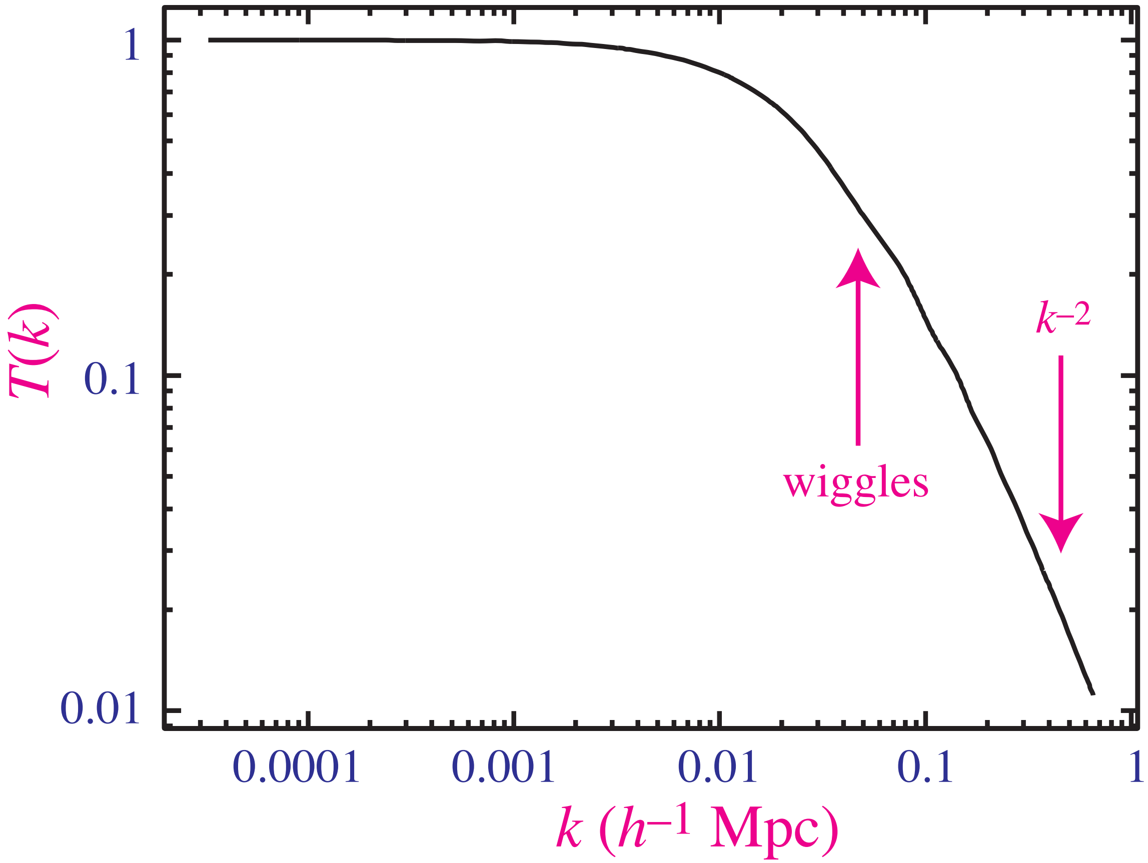
- Alternately accurate fitting formula exist, e.g. pure CDM form:

$$
\begin{aligned}
T(k(q)) &= \frac{L(q)}{L(q) + C(q)q^2} \\
L(q) &= \ln(e + 1.84q) \\
C(q) &= 14.4 + \frac{325}{1 + 60.5q^{1.11}} \\
q &= k/\Omega_m h^2 \mathrm{Mpc}^{-1} (T_{\rm CMB}/2.7K)^2
\end{aligned}
$$

- In h Mpc^{-1}, the critical scale depends on $\Gamma \equiv \Omega_m h$ also known as the shape parameter

Transfer Function

- Numerical calculation

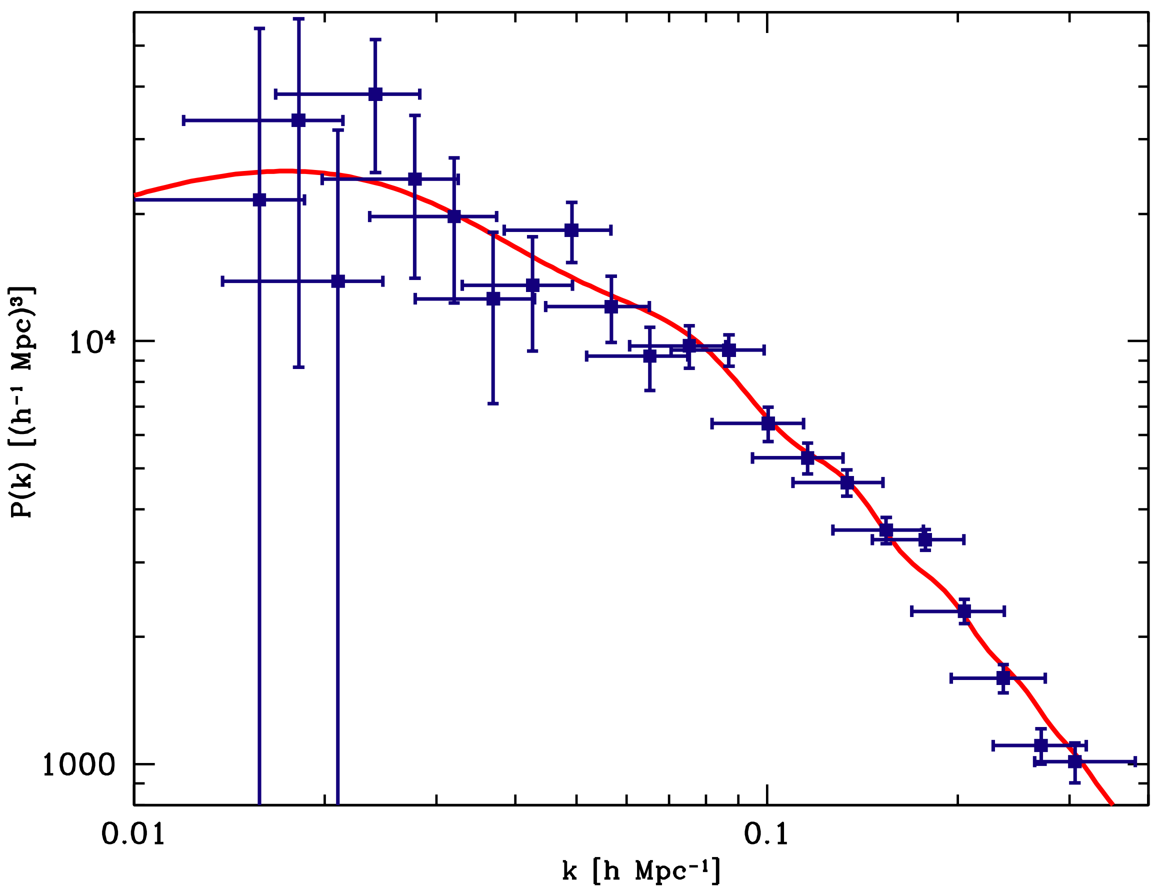


Baryon Acoustic Oscillations

- Baryons caught up in the acoustic oscillations of the CMB and impart acoustic oscillations to the transfer function. Density enhancements are produced kinematically through the continuity equation $\delta_b \sim (k\eta)v_b$ and hence are out of phase with CMB temperature peaks
- Dissipation of the acoustic oscillations eliminates both the CMB and baryon perturbations – known as Silk damping for the baryons. This suppression and the general fact that baryons are caught up with photons was one of the main arguments for CDM
- Detected first in the SDSS LRG survey.
- An excellent standard ruler for angular diameter distance $D_A(z)$ since it does not evolve in redshift in linear theory
- Radial extent of BAO gives $H(z)$

Power Spectrum

- SDSS data



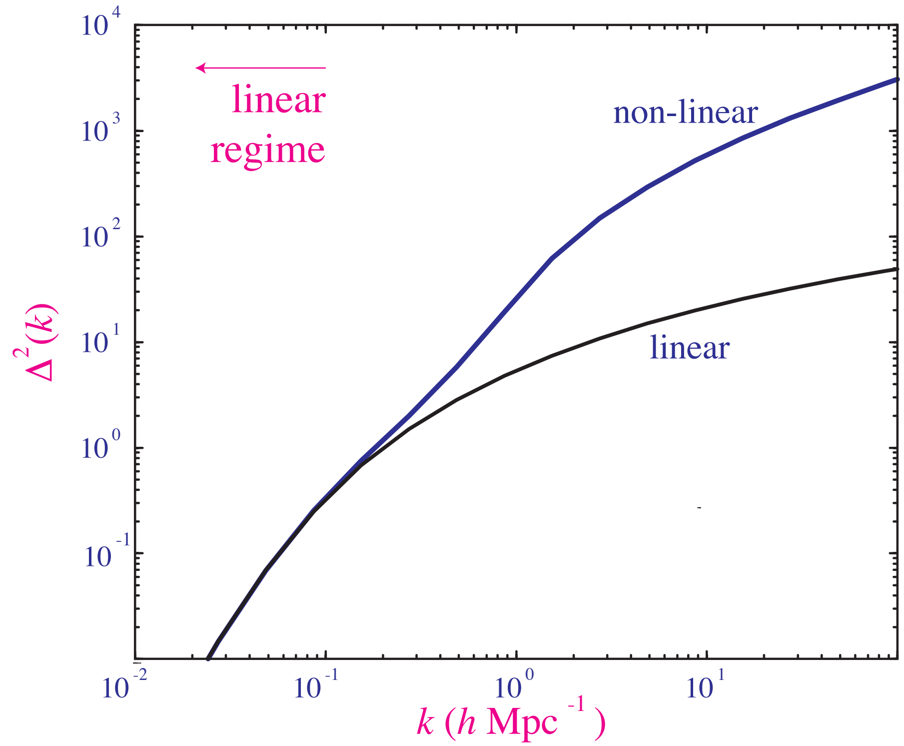
- Power spectrum defines large scale structure observables: galaxy clustering, velocity field, Lyα forest clustering, cosmic shear

Set 8:

Nonlinear Structure

Nonlinear regime

- Inflationary initial perturbations provide density perturbations $\delta = \delta\rho/\rho$ that grow as $\delta \propto a$ in the linear regime
- $\Delta^2 = k^3 P(k)/2\pi^2$ contribution to variance $\langle \delta^2 \rangle$ per $d\ln k$



- Linear theory would predict that for $k > 0.1 h\mathrm{Mpc}^{-1}$, $\langle \delta^2 \rangle > 1$.
- Linear approximation breaks down at this point and we must follow the nonlinear equations
- Nonlinearities further enhance the formation of structure

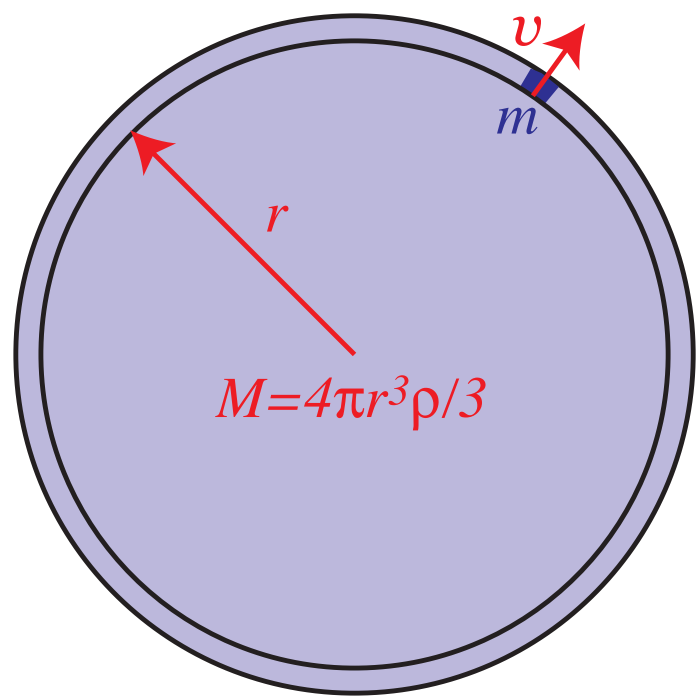
Cosmological Simulations

- To evolve structure further requires cosmological simulations (see video)
- Many simulation results can be understood using simple analytic arguments
- High density fluctuations break away from the cosmological expansion and form bound objects called dark matter halos
- Halo formation can be understood in the spherical collapse model as an FRW background expansion in a slightly closed (positive curvature) universe
- If observable properties such as galaxies (stars) and gas can be associated with halos, then they can be modeled in a "halo model"

Newtonian Cosmology

- Recall that in the Newtonian interpretation of the expansion we can model the dynamics of a spherical volume of constant density using mass conservation

- Energy conservation



$$E = \frac{1}{2}mv^2 - \frac{GMm}{r} = \text{const}$$

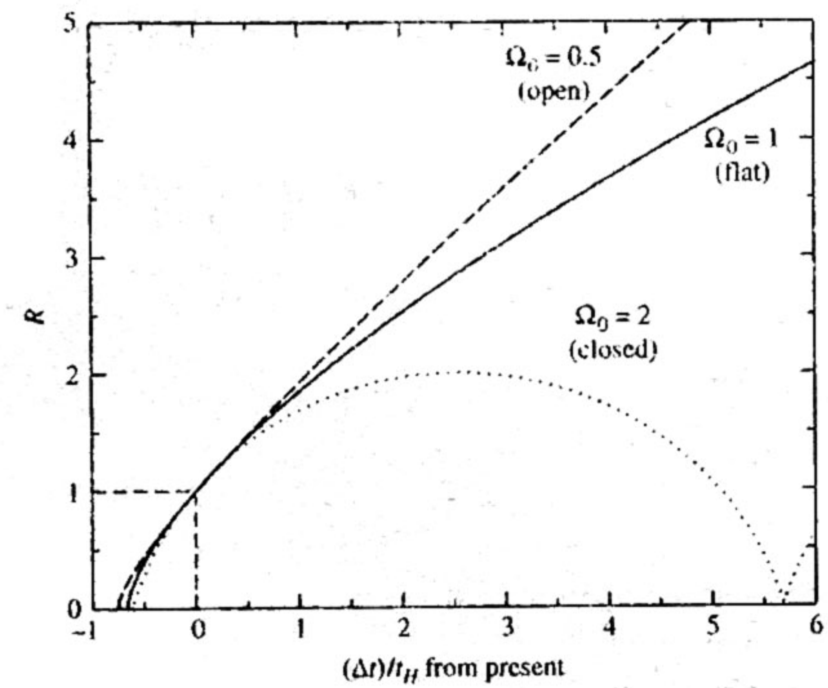
$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2 - \frac{GM}{r} = \text{const}$$

$$\frac{1}{2}\left(\frac{1}{r}\frac{dr}{dt}\right)^2 - \frac{GM}{r^3} = \frac{\text{const}}{r^2}$$

$$H^2 = \frac{8\pi G\rho}{3} - \frac{\text{const}}{a^2}$$

Newtonian Energy Interpretation

- Constant determines whether the system recollapses or expands forever
- These equations define the evolution of not just the homogeneous cosmology but also a spherically symmetric "top hat" or spatially constant density perturbation in a matter dominated universe



- An overdense region will eventually collapse and form a "dark matter halo" just like a positive curvature universe

Closed Universe

- Friedmann equation in a closed universe $K > 0$

$$H^2 + \frac{K}{a^2} = \frac{8\pi G}{3}\rho$$

when $K/a^2 = 8\pi G\rho/3$ then $H = 0$ and the expansion of the universe turns around into recollapse. In cosmo parameters

$$\frac{1}{a}\frac{da}{dt} = H_0\left(\Omega_m a^{-3} - (\Omega_m - 1)a^{-2}\right)^{1/2}$$

this would occur when $a = \Omega_m/(\Omega_m - 1)$

- Now consider an initially small local overdensity

$$\rho(\mathbf{x}, \mathbf{t}) = \bar{\rho}(t)[1 + \delta(\mathbf{x}, t)]$$

in a globally $\Omega_m = 1$ universe- a local observer would see this as a slightly closed universe with an $\Omega_m > 1$

Spherical Collapse

- Spherical collapse calculation makes use of this remapping by matching the initial density perturbations to the local expansion parameters and determining the epoch of collapse
- Basic idea: mass M enclosed by the region $r = ar_0$ remains constant so solve for $r(t)$ and match to the initial density perturbation δ_i at a_i
- Completing this step (details below) we have the following parametric solution

$$
\begin{aligned}
r(\theta) &= \frac{3}{10}\frac{r_i}{\delta_i}(1-\cos\theta) \\
t(\theta) &= \frac{1}{2H_0\Omega_m^{1/2}}\left(\frac{3}{5}\frac{a_i}{\delta_i}\right)^{3/2}(\theta-\sin\theta)
\end{aligned}
$$

where θ is the development angle with $\theta = \pi$ the turnaround point, $\theta = 2\pi$ the collapse point

Spherical Collapse Relations

- Scale factor $a \propto t^{2/3}$

$$a = \left(\frac{3}{4}\right)^{2/3} \left(\frac{3}{5}\frac{a_i}{\delta_i}\right) (\theta - \sin\theta)^{2/3}$$

- At collapse $\theta = 2\pi$

$$a_{\rm col} = \left(\frac{3}{4}\right)^{2/3} \left(\frac{3}{5}\frac{a_i}{\delta_i}\right) (2\pi)^{2/3} \approx 1.686\frac{a_i}{\delta_i}$$

- In linear theory and a matter dominated universe $\delta \propto a$ so $\delta_{\rm linear} = (a_{\rm col}/a_i)\delta_i = 1.686$ – Rule of thumb to remember – perturbation collapses when linear theory predicts $\delta_c \equiv 1.686$
- Interpretation: when linear theory predicts an $\mathcal{O}(1)$ density perturbation, in the real universe, that density perturbation has already collapsed to a nonlinear object

Spherical Collapse

- Derivation for advanced students: since $r = ar_0$ we can use the Friedmann solution for a in a closed universe to find $r(t)$
- In terms of development angle $\theta = H_0\eta(\Omega_m - 1)^{1/2}$, scaled conformal time η

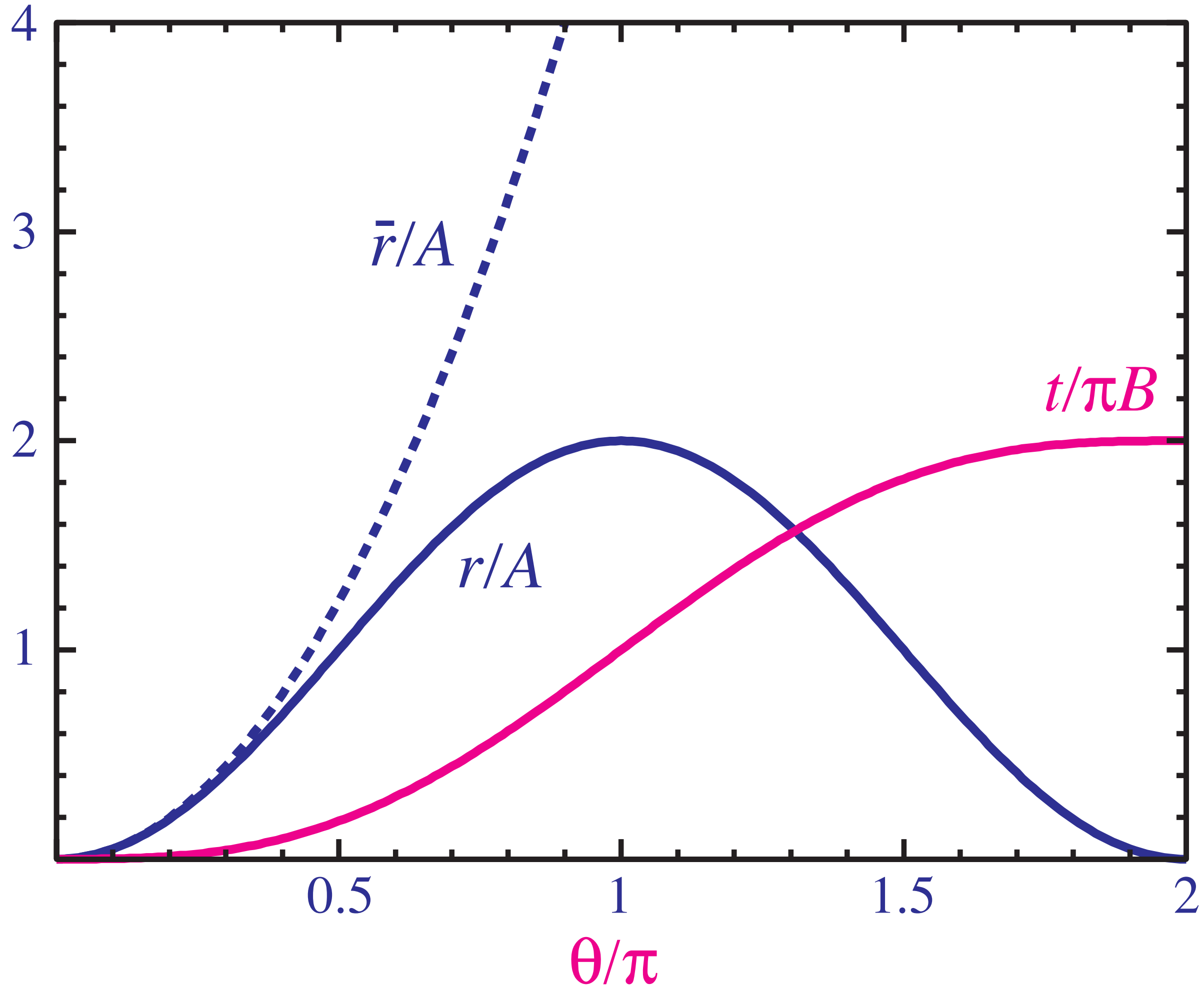
$$
\begin{aligned}
r(\theta) &= A(1 - \cos\theta) \\
t(\theta) &= B(\theta - \sin\theta)
\end{aligned}
$$

where $A = r_0\Omega_m/2(\Omega_m - 1)$, $B = H_0^{-1}\Omega_m/2(\Omega_m - 1)^{3/2}$.

- Turn around at $\theta = \pi$, $r = 2A$, $t = B\pi$.
- Collapse at $\theta = 2\pi$, $r \to 0$, $t = 2\pi B$
- Now we need to find the A and B constants given an initial density perturbation

Spherical Collapse

- Parametric Solution:



Correspondence

- Eliminate cosmological correspondence in A and B in terms of enclosed mass M

$$M = \frac{4\pi}{3} r_0^3 \Omega_m \rho_c = \frac{4\pi}{3} r_0^3 \Omega_m \frac{3H_0^2}{8\pi G}$$

- Related as $A^3 = GMB^2$, and to initial perturbation

$$\lim_{\theta\to 0} r(\theta) = A\left(\frac{1}{2}\theta^2 - \frac{1}{24}\theta^4\right)$$

$$\lim_{\theta\to 0} t(\theta) = B\left(\frac{1}{6}\theta^3 - \frac{1}{120}\theta^5\right)$$

- Leading Order: $r = A\theta^2/2$, $t = B\theta^3/6$

$$r = \frac{A}{2}\left(\frac{6t}{B}\right)^{2/3}$$

Next Order

- Leading order is unperturbed matter dominated expansion
  $r \propto a \propto t^{2/3}$
- Iterate r and t solutions

$$\lim_{\theta \to 0} t(\theta) = \frac{\theta^3}{6} B \left[1 - \frac{1}{20} \left(\frac{6t}{B} \right)^{2/3} \right]$$

$$\theta \approx \left(\frac{6t}{B} \right)^{1/3} \left[1 + \frac{1}{60} \left(\frac{6t}{B} \right)^{2/3} \right]$$

Next Order

- Substitute back into $r(\theta)$

$$
\begin{aligned}
r(\theta) &= A\frac{\theta^2}{2}\left(1-\frac{\theta^2}{12}\right) \\
&= \frac{A}{2}\left(\frac{6t}{B}\right)^{2/3}\left[1-\frac{1}{20}\left(\frac{6t}{B}\right)^{2/3}\right] \\
&= \frac{1}{2}(6t)^{2/3}(GM)^{1/3}\left[1-\frac{1}{20}\left(\frac{6t}{B}\right)^{2/3}\right]
\end{aligned}
$$

Density Correspondence

- Density

$$
\begin{aligned}
\rho_m &= \frac{M}{\frac{4}{3}\pi r^3} \\
&= \frac{1}{6\pi t^2 G}\left[1 + \frac{3}{20}\left(\frac{6t}{B}\right)^{2/3}\right]
\end{aligned}
$$

- Density perturbation

$$
\delta \equiv \frac{\rho_m - \bar{\rho}_m}{\bar{\rho}_m} \approx \frac{3}{20}\left(\frac{6t}{B}\right)^{2/3}
$$

Density Correspondence

- Time $\rightarrow$ scale factor

$$
\begin{aligned}
t &= \frac{2}{3H_0\Omega_m^{1/2}} a^{3/2} \\
\delta &= \frac{3}{20} a \left(4/B H_0 \Omega_m^{1/2}\right)^{2/3}
\end{aligned}
$$

- A and B constants $\rightarrow$ initial cond.

$$
\begin{aligned}
B &= \frac{1}{2H_0\Omega_m^{1/2}} \left(\frac{3}{5}\frac{a_i}{\delta_i}\right)^{3/2} \\
A &= \frac{3}{10}\frac{r_i}{\delta_i}
\end{aligned}
$$

- End derivation for advanced student

Virialization

- A real density perturbation is neither spherical nor homogeneous
- Shell crossing if δ_i doesn't monotonically decrease
- Collapse does not proceed to a point but reaches virial equilibrium

$$U = -2K, \qquad E = U + K = \frac{1}{2}U(r_{\rm vir})[= U(r_{\rm max})]$$

conserving E so $r_{\rm vir} = \frac{1}{2}r_{\rm max}$ since $U \propto r^{-1}$. Thus $\theta_{\rm vir} = \frac{3}{2}\pi$

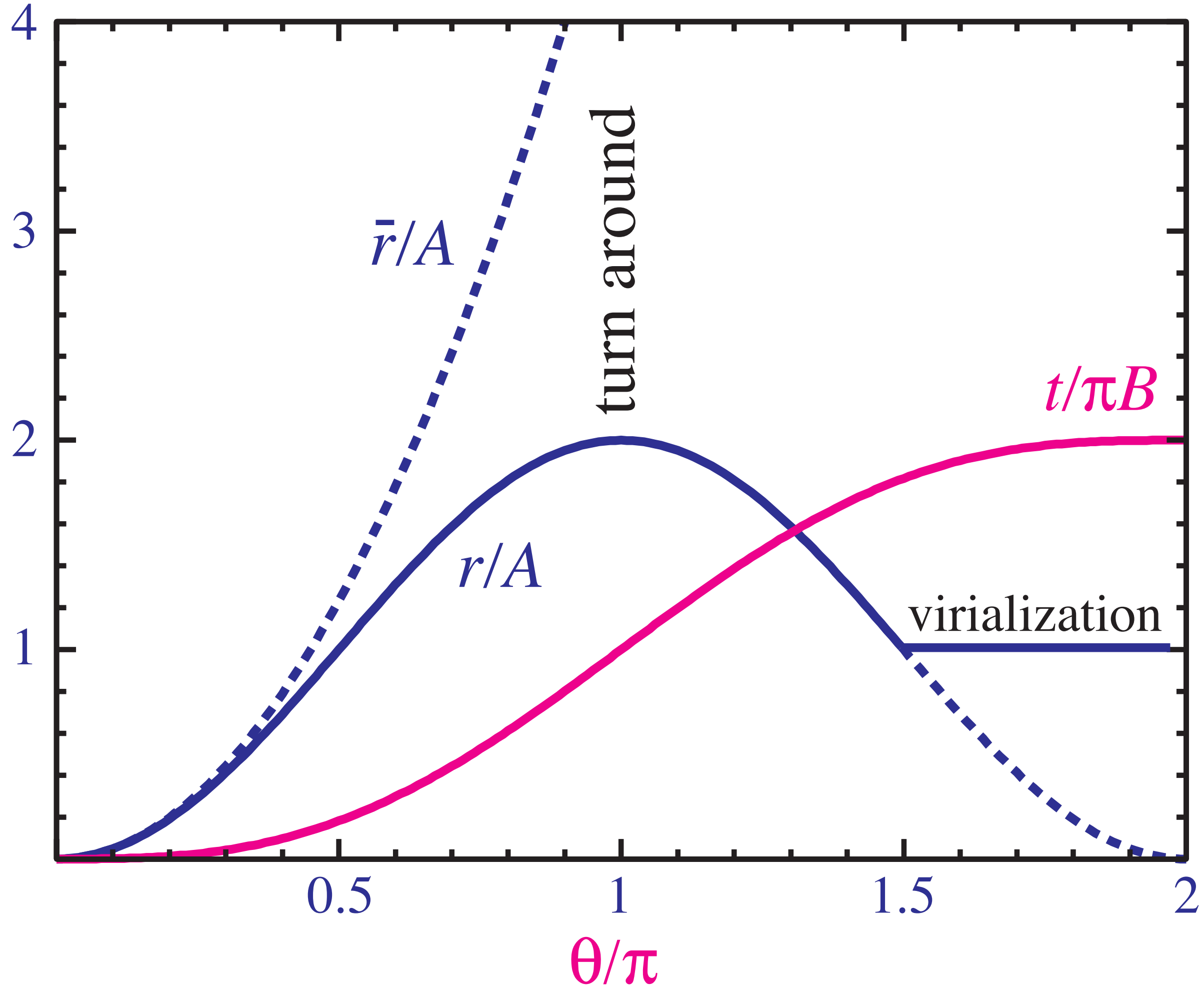
- Overdensity at virialization

$$\frac{\rho_m(\theta = 3\pi/2)}{\bar{\rho}_m(\theta = 2\pi)} = 18\pi^2 \approx 178$$

- Threshold $\Delta_v = 178$ often used to define a collapsed object
- Equivalently relation between virial mass, radius, overdensity: $M_{\rm vir} = \frac{4\pi}{3} r_{\rm vir}^3 \rho_m \Delta_v$

Virialization

- Schematic Picture:



The Mass Function

- Spherical collapse predicts the end state as virialized halos given an initial density perturbation
- Allows us to predict, from a linear analysis, the outcome of a nonlinear simulation, e.g. the abundance of halos

- Initial density perturbation is a Gaussian random field
- Compare the variance in the linear density field to threshold $\delta_c = 1.686$ to determine collapse fraction
- Combine to form the mass function, the number density of halos in a range dM around M.
- Halo density defined entirely by linear theory

Press-Schechter Formalism

- Smooth linear density density field on mass scale M with tophat

$$R = \left(\frac{3M}{4\pi}\right)^{1/3}$$

- Result is a Gaussian random field with variance $\sigma^2(M)$
- Fluctuations above the threshold δ_c correspond to collapsed regions. The fraction in halos $> M$ becomes

$$\frac{1}{\sqrt{2\pi}\sigma(M)} \int_{\delta_c}^{\infty} d\delta \exp\left(-\frac{\delta^2}{2\sigma^2(M)}\right) = \frac{1}{2}\mathrm{erfc}\left(\frac{\nu}{\sqrt{2}}\right)$$

where $\nu \equiv \delta_c/\sigma(M)$

- Problem: even as $\sigma(M) \to \infty$, $\nu \to 0$, collapse fraction $\to 1/2$ – only overdense regions participate in spherical collapse.
- Multiply by 2! Justified by underdensity within overdensity

Press-Schechter Mass Function

- Differentiate in M to find fraction in range dM and multiply by ρ_m/M the number density of halos if all of the mass were composed of such halos $\rightarrow$ differential number density of halos

$$\frac{dn}{d\ln M} = \frac{\rho_m}{M}\frac{d}{d\ln M}\mathrm{erfc}\left(\frac{\nu}{\sqrt{2}}\right)$$
$$= \sqrt{\frac{2}{\pi}}\frac{\rho_m}{M}\frac{d\ln\sigma^{-1}}{d\ln M}\nu\exp(-\nu^2/2)$$

- High mass: exponential cut off above M_* where $\sigma(M_*) = \delta_c$

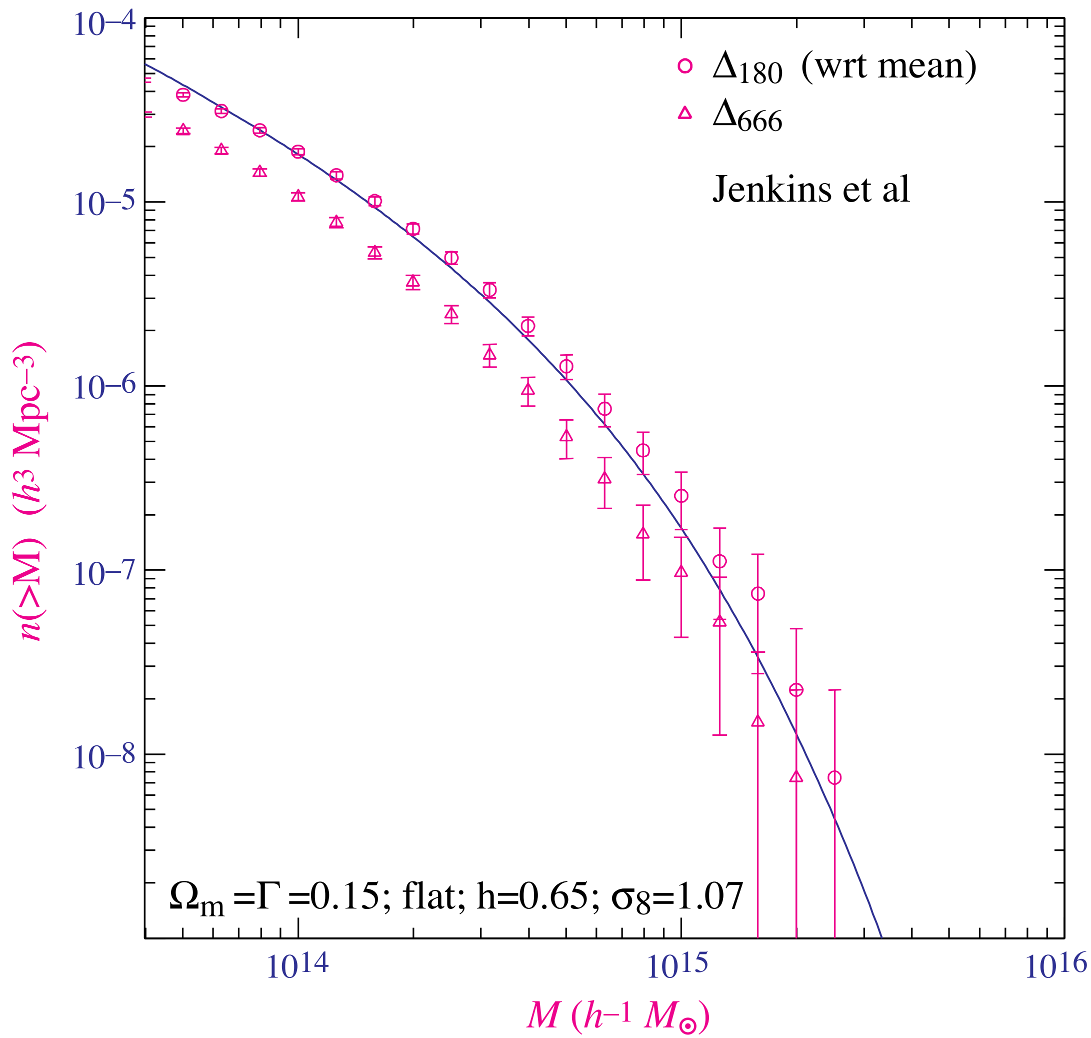
$$M_* \sim 10^{13} h^{-1} M_\odot \quad \text{today}$$

- Low mass divergence: (too many for the observations?)

$$\frac{dn}{d\ln M} \propto\sim M^{-1}$$

Numerical Mass Function

- Fit cosmological simulations to Press Schechter motivated form



Halo Bias

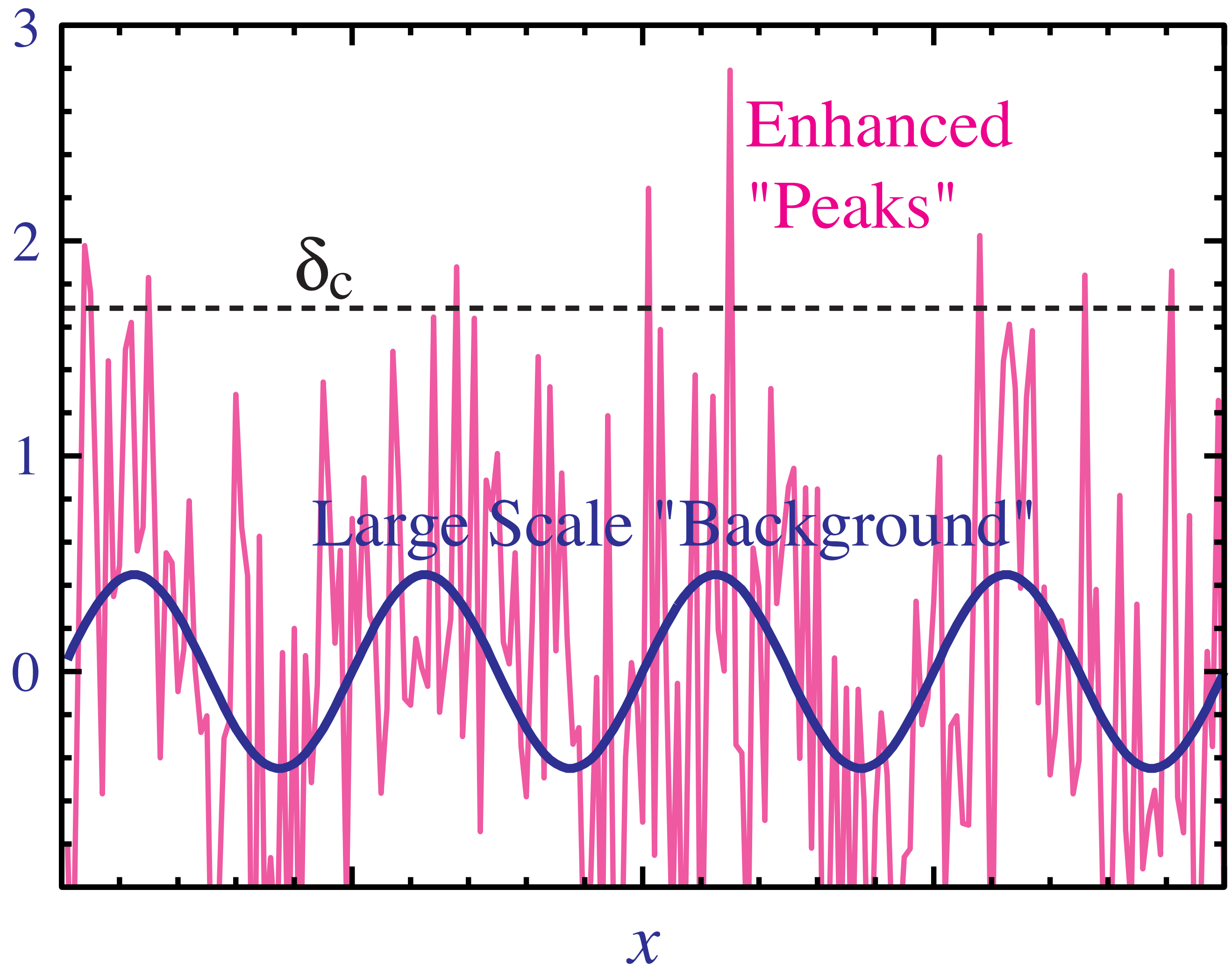
- If halos are formed without regard to the underlying density fluctuation and move under the gravitational field then their number density is an unbiased tracer of the dark matter density fluctuation

$$\left(\frac{\delta n}{n}\right)_{\rm halo} = \left(\frac{\delta\rho}{\rho}\right)$$

- However spherical collapse says the probability of forming a halo depends on the initial density field
- Large scale density field acts as "background" enhancement of probability of forming a halo or "peak"
- Peak-Background Split (Efstathiou 1998; Cole & Kaiser 1989; Mo & White 1997)

Peak-Background Split

- Schematic Picture:



Perturbed Mass Function

- Density fluctuation split

$$\delta = \delta_b + \delta_p$$

- Lowers the threshold for collapse

$$\delta_{cp} = \delta_c - \delta_b$$

so that $\nu = \delta_{cp}/\sigma$

- Taylor expand number density $n_M \equiv dn/d\ln M$

$$n_M + \frac{dn_M}{d\nu}\frac{d\nu}{d\delta_b}\delta_b \ldots = n_M \left[1 + \frac{(\nu^2-1)}{\sigma\nu}\delta_b\right]$$

if mass function is given by Press-Schechter

$$n_M \propto \nu \exp(-\nu^2/2)$$

Halo Bias

- Halos are biased tracers of the "background" dark matter field with a bias $b(M)$ that is given by spherical collapse and the form of the mass function
- Combine the enhancement with the original unbiased expectation

$$\frac{\delta n_M}{n_M} = b(M)\delta_b$$

- For Press-Schechter

$$b(M) = 1 + \frac{\nu^2 - 1}{\delta_c}$$

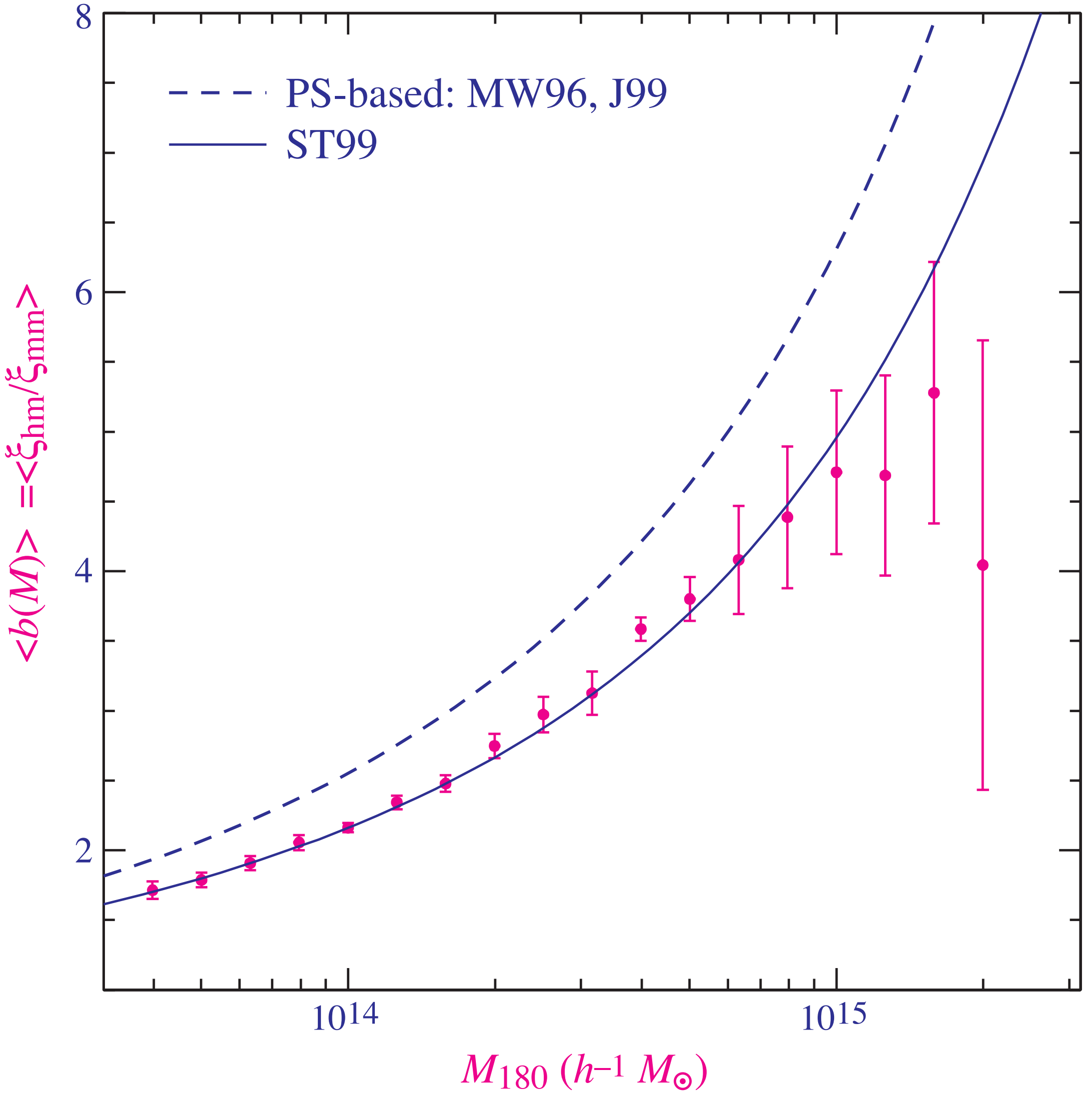
- Improved by the Sheth-Torman mass function

$$b(M) = 1 + \frac{a\nu^2 - 1}{\delta_c} + \frac{2p}{\delta_c[1 + (a\nu^2)^p]}$$

with $a = 0.75$ and $p = 0.3$ to match simulations.

Numerical Bias

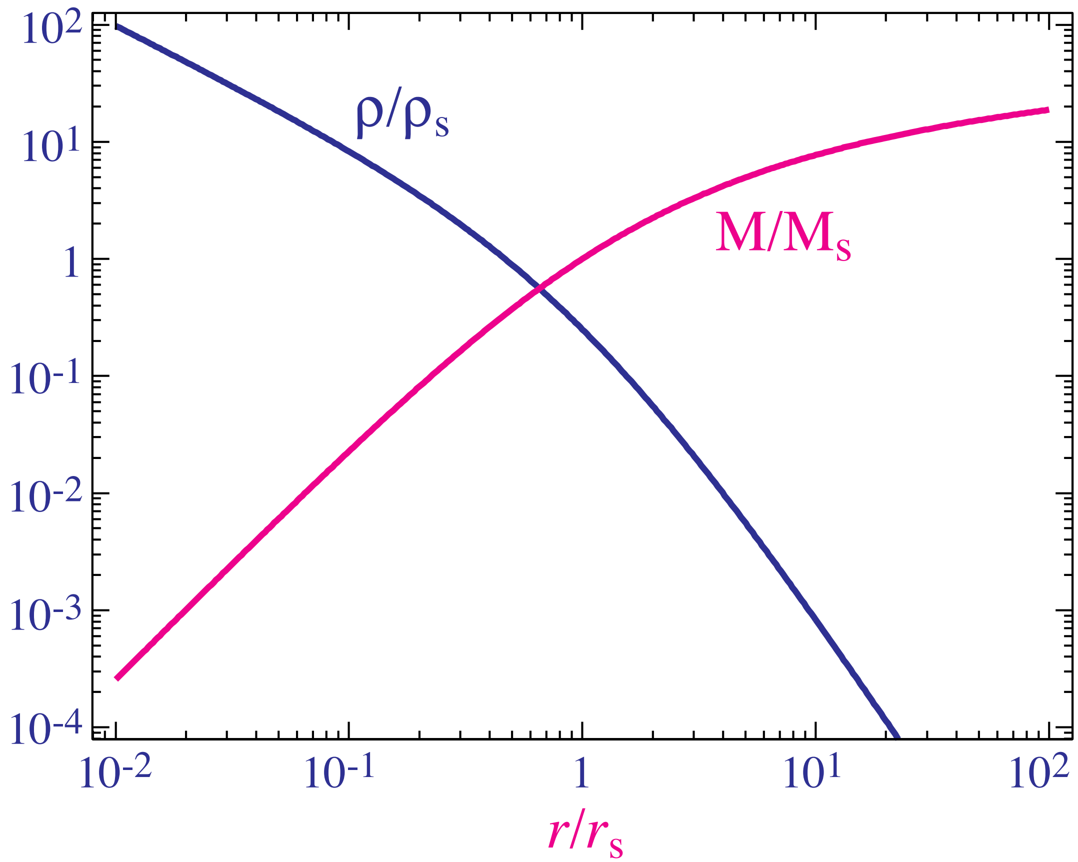
- Example of halo bias from a simulation



NFW Profile

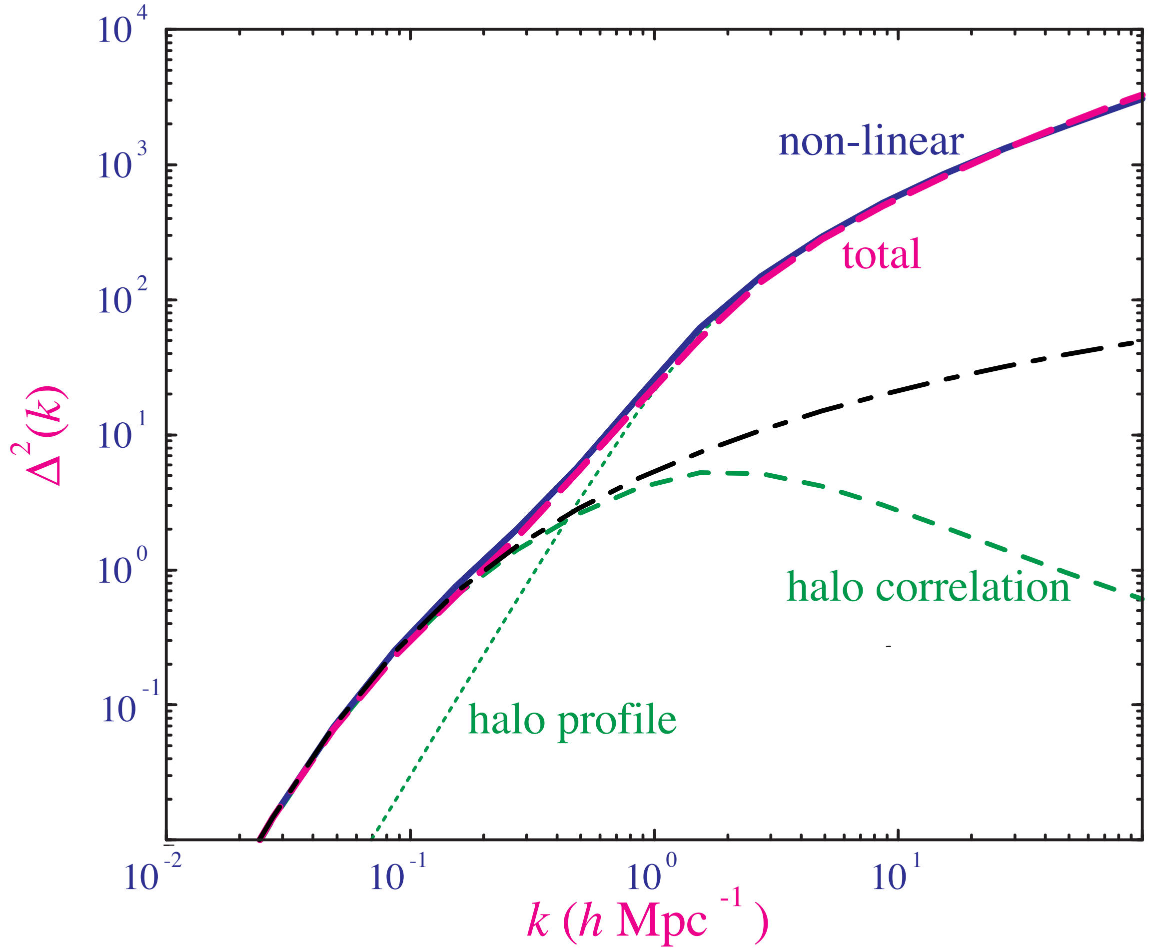
- Density profile well-described by (Navarro, Frenk & White 1997)

$$\rho(r) = \frac{\rho_s}{(r/r_s)(1 + r/r_s)^2}$$



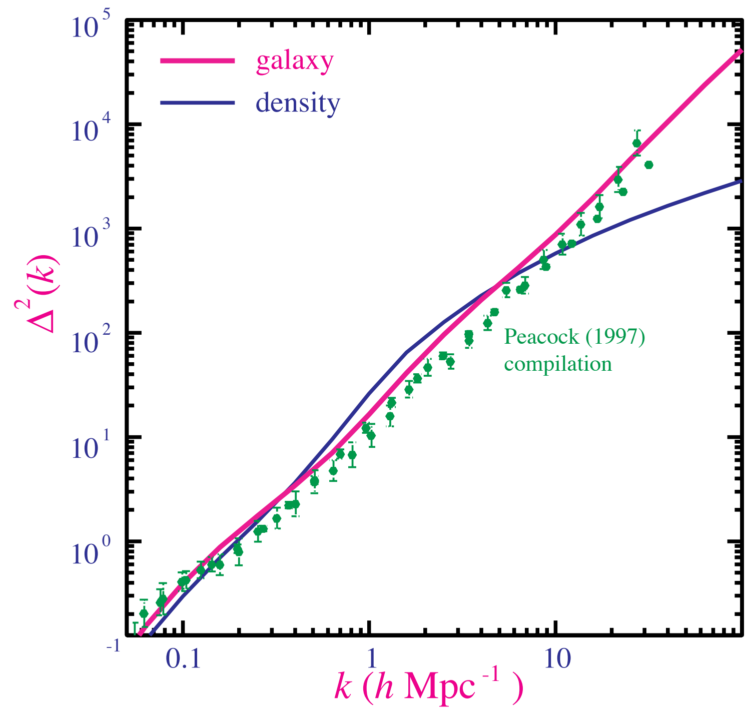
The Halo Model

- NFW halos, of abundance n_M given by mass function, clustered according to the halo bias $b(M)$ and the linear theory $P(k)$
- Power spectrum example:



Galaxy Power Spectrum

- For galaxies, one defines a halo occupation distribution which determines the number of galaxies (satisfying a certain observational criteria) that can occupy a halo of mass M



- Galaxy clustering or power spectrum composed from the distribution of galaxies in halos and the clustering of halos in large scale structure
- Generalize to clustering statistic of any observable that is associated with dark matter haloes, e.g. gas, gravitational lensing etc.

Hierarchical Structure Formation

- With the density power spectrum from linear theory implying that the variance increases as R decreases and time increases, small halos form first
- Exponential suppression of high mass halos – evolution starts with low masses and progresses to higher masses as halos merge (see video)
- Small halos are not hot enough to stimulate atomic line transitions and then cool and fragment by radiating
- Stars form in halos only at $z < 10$ and radiation from stars can then reionize the plasma

Galaxy Formation

- Merging proto-Galactic objects of $10^6 - 10^8 M_\odot$ can eventually assemble the galaxies of $10^{12} M_\odot$ we see today. Both lower and upper range determined by cooling.
- Proto-galactic objects can form if cooling is sufficiently rapid that the heating of the gas during collapse (which would prevent collapse due to pressure, internal motions) can be overcome
- Recall virial theorem supplies estimate of thermal kinetic energy

$$-2\langle K\rangle = \langle U\rangle$$

$$-2N\frac{1}{2}\mu m_H \sigma^2 = -\frac{3}{5}\frac{GMN\mu m_H}{R}$$

where μm_H is the average mass of particles in the gas, M is the total mass and σ is the rms velocity

Galaxy Formation

- Solve for velocity dispersion for a self gravitating system

$$\sigma = \left(\frac{3}{5}\frac{GM}{R}\right)^{1/2}$$

- Associate the average kinetic energy with a temperature, called the virial temperature

$$\frac{1}{2}\mu m_H \sigma^2 = \frac{3}{2}kT_{\rm virial}$$

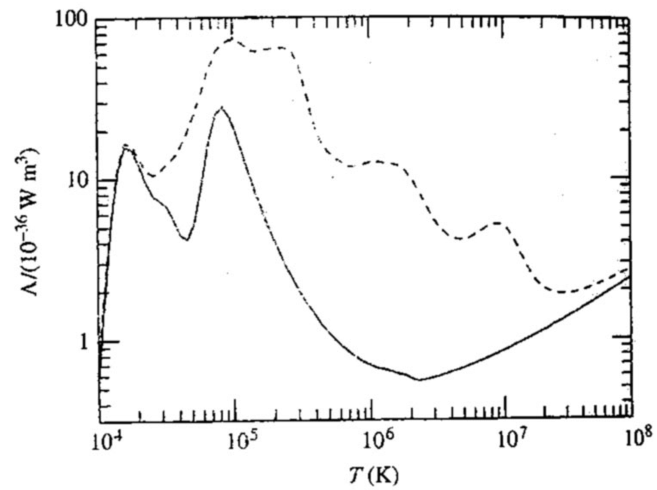
where μ is the mean molecular weight. Solve for virial temperature

$$T_{\rm virial} = \frac{\mu m_H \sigma^2}{3k} = \frac{\mu m_H}{5k}\frac{GM}{R} \approx \frac{\mu m_H}{5k} GM^{2/3}\left(\frac{4\pi\rho}{3}\right)^{1/3}$$

- Cooling is a function of the gas temperature through the cooling function.

Galaxy Formation

- Cooling rate (luminosity) per volume



$$r_{\rm cool} = n^2 \Lambda(T)$$

n^2 (number density squared) since cooling is usually a 2 body process – for $T > 10^6$K thermal bremsstrahlung and Compton scattering, for $T \sim 10^4 - 10^5$K from the collisional excitation of atomic lines of hydrogen and helium

- Galaxy formation only starts when dark matter mass makes the virial temperture exceed $T \sim 10^4$K when cooling becomes efficient $M \sim 10^8 M_\odot$ -first objects and current dwarf ellipticals
- At $z < 10$, these halos abundant enough for UV light from their stars to reionize H – final ΛCDM parameter τ, optical depth to Thomson scattering

Galaxy Formation

- Cooling time is the time required to radiate away all of the thermal energy of the gas

$$r_{\rm cool} V t_{\rm cool} = \frac{3}{2} N k T_{\rm virial}$$

$$t_{\rm cool} = \frac{3}{2} \frac{k T_{\rm virial}}{n\Lambda}$$

- Compared with the free fall time - from our dimensional relation

$$GM \sim Rv^2 \sim R(R^2/t_{\rm ff}^2), \quad M \propto \rho R^3$$

we get $t_{\rm ff} \propto (G\rho)^{-1/2}$ with the proportionality given for the time of collapse for a homogeneous sphere of initial density ρ

$$t_{\rm ff} = \left(\frac{3\pi}{32} \frac{1}{G\rho} \right)^{1/2}$$

Galaxy Formation

- If $t_{\rm cool} < t_{\rm ff}$ then the object will collapse essentially in free fall - fragment and form stars. If opposite, then gravitational potential energy heats the gas making it stabilized by pressure establishing virial equilibrium

$$\left(\frac{t_{\rm ff}}{t_{\rm cool}}\right) > \left(\frac{3\pi}{32}\frac{1}{G\rho}\right)^{1/2} \frac{2}{3}\frac{n\Lambda}{kT_{\rm virial}}$$

- Taking typical numbers $T \sim 10^6$K and $n \sim 5 \times 10^4 {\rm m}^{-3}$ and with the density of the collapsing medium being associated with the gas $\rho = \mu m_H n$ gives an upper limit on the gas mass that can cool of $10^{12} M_\odot$ comparable to a large galaxy.

Disk Formation

- Proto-galactic gas fragment and collide retaining initial angular momentum provided from torques from other proto-galactic systems
- Rotationally supported gas disk, cooling in dense regions until HI clouds form from which star formation occurs - thick disk
- Cool molecular gas settles to midplane of thick disk efficiently forming stars - thinness is self regulating - if disk continued to get thinner then density and star formation goes up heating the material and re-puffing out the disk