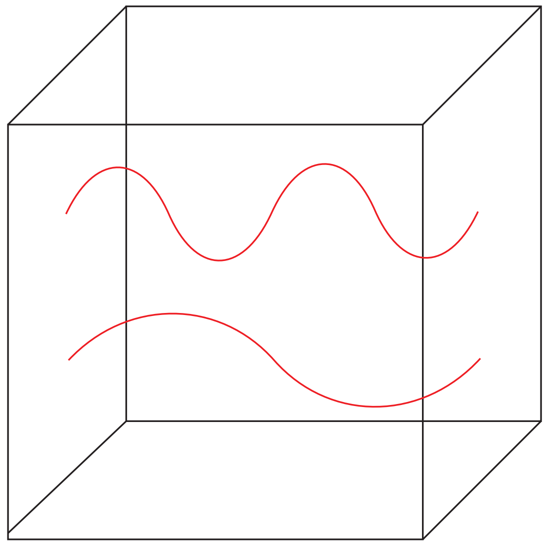
Supplement:

Statistical Physics

Fitting in a Box



- Counting momentum states with momentum q and de Broglie wavelength

$$\lambda = \frac{h}{q} = \frac{2\pi\hbar}{q}$$

- In a discrete volume L^3 there is a discrete set of states that satisfy periodic boundary conditions
- We will hereafter set $\hbar = c = 1$
- As in Fourier analysis

$$e^{2\pi i x/\lambda} = e^{iqx} = e^{iq(x+L)} \to e^{iqL} = 1$$

Fitting in a Box

- Periodicity yields a discrete set of allowed states

$$Lq = 2\pi m_i, \quad m_i = 1, 2, 3...$$
$$q_i = \frac{2\pi}{L} m_i$$

- In each of 3 directions

$$\sum_{m_{xi} m_{yj} m_{zk}} \to \int d^3 m$$

- The differential number of allowed momenta in the volume

$$d^3 m = \left(\frac{L}{2\pi} \right)^3 d^3 q$$

Density of States

- The total number of states allows for a number of internal degrees of freedom, e.g. spin, quantified by the degeneracy factor g
- Total density of states:

$$\frac{dN_s}{V} = \frac{g}{V} d^3 m = \frac{g}{(2\pi)^3} d^3 q$$

- If all states were occupied by a single particle, then particle density

$$n_s = \frac{N_s}{V} = \frac{1}{V} \int dN_s = \int \frac{g}{(2\pi)^3} d^3 q$$

Distribution Function

- The distribution function f quantifies the occupation of the allowed momentum states

$$n = \frac{N}{V} = \frac{1}{V}\int f dN_s = \int \frac{g}{(2\pi)^3} f d^3q$$

- f, aka phase space occupation number, also quantifies the density of particles per unit phase space $dN/(\Delta x)^3(\Delta q)^3$
- For photons, the spin degeneracy $g = 2$ accounting for the 2 polarization states
- Energy $E(q) = (q^2 + m^2)^{1/2}$
- Momentum $\rightarrow$ frequency $q = 2\pi/\lambda = 2\pi\nu = \omega = E$ (where $m = 0$ and $\lambda\nu = c = 1$)

Bulk Properties

- Integrals over the distribution function define the bulk properties of the collection of particles
- Number density

$$n(\mathbf{x}, t) \equiv \frac{N}{V} = g \int \frac{d^3q}{(2\pi)^3} f$$

- Energy density

$$\rho(\mathbf{x}, t) = g \int \frac{d^3q}{(2\pi)^3} E(q) f$$

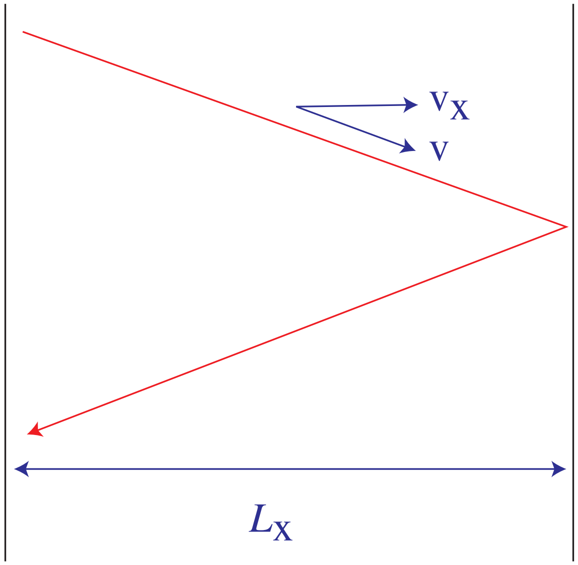
where $E^2 = q^2 + m^2$

- Momentum density

$$(\rho + p)\mathbf{v}(\mathbf{x}, t) = g \int \frac{d^3q}{(2\pi)^3} \mathbf{q} f$$

Bulk Properties

- Pressure: particles bouncing off a surface of area A in a volume spanned by L_x: per momentum state



$$p_q = \frac{F}{A} = \frac{N_{\rm part}}{A}\frac{\Delta q}{\Delta t}$$

$$(\Delta q = 2|q_x|, \quad \Delta t = 2L_x/v_x,)$$

$$= \frac{N_{\rm part}}{V}|q_x||v_x| = \frac{N_{\rm part}}{V}\frac{|q||v|}{3}$$

$$(v = \gamma m v/\gamma m = q/E)$$

$$= \frac{N_{\rm part}}{V}\frac{q^2}{3E}$$

Bulk Properties

- So that summed over occupied momenta states

$$p(\mathbf{x}, t) = g \int \frac{d^3q}{(2\pi)^3} \frac{|q|^2}{3E(q)} f$$

- Pressure is just one of the quadratic in q moments, in particular the isotropic one
- The remaining 5 components are the anisotropic stress (vanishes in the background)

$$\pi^i{}_j(\mathbf{x}, t) = g \int \frac{d^3q}{(2\pi)^3} \frac{3q^i q_j - q^2 \delta^i{}_j}{3E(q)} f$$

- We shall see that these are related to the 5 quadrupole moments of the angular distribution

Bulk Properties

- These are more generally the components of the stress-energy tensor

$$T^{\mu}{}_{\nu} = g \int \frac{d^3 q}{(2\pi)^3} \frac{q^{\mu} q_{\nu}}{E(q)} f$$

- 0-0: energy density
- 0-i: momentum density
- $i - i$: pressure
- $i \neq j$: anisotropic stress
- In the FRW background cosmology, isotropy requires that there be only a net energy density and pressure

Equilibrium

- Thermal physics describes the equilibrium distribution of particles for a medium at temperature T
- Expect that the typical energy of a particle by equipartition is $E \sim T$, so that $f_{\rm eq}(E/T, ?)$ in equilibrium
- Must be a second variable of import. Number density

$$n = g \int \frac{d^3 q}{(2\pi\hbar)^3} f_{\rm eq}(E/T) =? \quad n(T)$$

- If particles are conserved then n cannot simply be a function of temperature.
- The integration constant that concerns particle conservation is called the chemical potential. Relevant for photons when creation and annihilation processes are ineffective

Temperature and Chemical Potential

- Fundamental assumption of statistical mechanics is that all accessible states have an equal probability of being populated. The number of states G defines the entropy $S(U,N,V) = \ln G$ where U is the energy, N is the number of particles and V is the volume
- When two systems are placed in thermal contact they may exchange energy, particles, leading to a wider range of accessible states

$$G(U,N,V) = \sum_{U_1,N_1} G_1(U_1,N_1,V_1) G_2(U-U_1, N-N_1, V_2)$$

- The most likely distribution of U_1 and U_2 is given for the maximum $dG/dU_1 = 0$

$$\left(\frac{\partial G_1}{\partial U_1}\right)_{N_1,V_1} G_2 dU_1 + G_1 \left(\frac{\partial G_2}{\partial U_2}\right)_{N_2,V_2} dU_2 = 0 \qquad dU_1 + dU_2 = 0$$

Temperature and Chemical Potential

- Or equilibrium requires

$$\left(\frac{\partial \ln G_1}{\partial U_1}\right)_{N_1,V_1} = \left(\frac{\partial \ln G_2}{\partial U_2}\right)_{N_2,V_2} \equiv \frac{1}{T}$$

which is the definition of the temperature (equal for systems in thermal contact)

- Likewise define a chemical potential μ for a system in diffusive equilibrium

$$\left(\frac{\partial \ln G_1}{\partial N_1}\right)_{U_1,V_1} = \left(\frac{\partial \ln G_2}{\partial N_2}\right)_{U_2,V_2} \equiv -\frac{\mu}{T}$$

defines the most likely distribution of particle numbers as a system with equal chemical potentials: generalize to multiple types of particles undergoing "chemical" reaction $\rightarrow$ law of mass action $\sum_i \mu_i dN_i = 0$

Temperature and Chemical Potential

- Equivalent definition: the chemical potential is the free energy cost associated with adding a particle at fixed temperature and volume

$$\mu = \frac{\partial F}{\partial N}\bigg|_{T,V}, \quad F = U - TS$$

  free energy: balance between minimizing energy and maximizing entropy S

- Temperature and chemical potential determine the probability of a state being occupied if the system is in thermal and diffusive contact with a large reservoir at temperature T

Gibbs or Boltzmann Factor

- Suppose the system has two states unoccupied $N_1 = 0, U_1 = 0$ and occupied $N_1 = 1, U_1 = E$ then the ratio of probabilities in the occupied to unoccupied states is given by

$$P = \frac{\exp[\ln G_{\text{res}}(U - E, N - 1, V)]}{\exp[\ln G_{\text{res}}(U, N, V)]}$$

- Taylor expand

$$\ln G_{\text{res}}(U - E, N - 1, V) \approx \ln G_{\text{res}}(U, N, V) - \frac{E}{T} + \frac{\mu}{T}$$

$$P \approx \exp[-(E - \mu)/T]$$

- This is the Gibbs factor.

Gibbs or Boltzmann Factor

- More generally the probability of a system being in a state of energy E_i and particle number N_i is given by the Gibbs factor

$$P(E_i, N_i) \propto \exp[-(E_i - \mu N_i)/T]$$

- Unlikely to be in an energy state $E_i \gg T$ mitigated by the number of particles
- Dropping the diffusive contact, this is the Boltzmann factor

Thermal & Diffusive Equilibrium

- A gas in thermal & diffusive contact with a reservoir at temperature T
- Probability of system being in state of energy E_i and number N_i (Gibbs Factor)

$$P(E_i, N_i) \propto \exp[-(E_i - \mu N_i)/T]$$

  where μ is the chemical potential (defines the free energy "cost" for adding a particle at fixed temperature and volume)

- Chemical potential appears when particles are conserved
- CMB photons can carry chemical potential if creation and annihilation processes inefficient, as they are after $t \sim 1$yr.

Distribution Function

- Mean occupation of the state in thermal equilibrium

$$f \equiv \frac{\sum N_i P(E_i, N_i)}{\sum P(E_i, N_i)}$$

where the total energy is related to the particle energy as $E_i = N_i E$ (ignoring zero pt)

- Density of (energy) states in phase space makes the net spatial density of particles

$$n = g \int \frac{d^3 p}{(2\pi)^3} f$$

where g is the number of spin states

Fermi-Dirac Distribution

- For fermions, the occupancy can only be $N_i = 0, 1$

$$f = \frac{P(E,1)}{P(0,0) + P(E,1)} = \frac{e^{-(E-\mu)/T}}{1 + e^{-(E-\mu)/T}} = \frac{1}{e^{(E-\mu)/T} + 1}$$

- In the non-relativistic, non-degenerate limit

$$E = (q^2 + m^2)^{1/2} \approx m + \frac{1}{2}\frac{q^2}{m}$$

and $m \gg T$ so the distribution is Maxwell-Boltzmann

$$f = e^{-(m-\mu)/T} e^{-q^2/2mT} = e^{-(m-\mu)/T} e^{-mv^2/2T}$$

Bose-Einstein Distribution

- For bosons each state can have multiple occupation,

$$f = \frac{\frac{d}{d\mu/T}\sum_{N=0}^{\infty}(e^{-(E-\mu)/T})^N}{\sum_{N=0}^{\infty}(e^{-(E-\mu)/T})^N} \qquad \text{with} \sum_{N=0}^{\infty} x^N = \frac{1}{1-x}$$

$$= \frac{1}{e^{(E-\mu)/T} - 1}$$

- Again, non relativistic distribution is Maxwell-Boltzmann

$$f = e^{-(m-\mu)/T} e^{-q^2/2mT} = e^{-(m-\mu)/T} e^{-mv^2/2T}$$

with a spatial number density

$$n = g e^{-(m-\mu)/T} \int \frac{d^3q}{(2\pi)^3} e^{-q^2/2mT}$$

$$= g e^{-(m-\mu)/T} \left(\frac{mT}{2\pi}\right)^{3/2}$$

Ultra-Relativistic Bulk Properties

- Chemical potential $\mu = 0$, $\zeta(3) \approx 1.202$
- Number density

$$n_{\text{boson}} = gT^3 \frac{\zeta(3)}{\pi^2} \qquad \zeta(n+1) \equiv \frac{1}{n!}\int_0^\infty dx \frac{x^n}{e^x - 1}$$

$$n_{\text{fermion}} = \frac{3}{4} gT^3 \frac{\zeta(3)}{\pi^2}$$

- Energy density

$$\rho_{\text{boson}} = gT^4 \frac{3}{\pi^2}\zeta(4) = gT^4 \frac{\pi^2}{30}$$

$$\rho_{\text{fermion}} = \frac{7}{8} gT^4 \frac{3}{\pi^2}\zeta(4) = \frac{7}{8} gT^4 \frac{\pi^2}{30}$$

- Pressure $q^2/3E = E/3 \rightarrow p = \rho/3$, $w_r = 1/3$

Boltzmann Equation

- Interactions or “collisions” between particles drive the various distributions to equilibrium through the Boltzmann equation
- Boltzmann equation is also known as the particle transport or radiative transfer equation
- Composed of two parts: the free propagation or Liouville equation and the collisions

Liouville Equation

- Liouville theorem: phase space distribution function is conserved along a trajectory in the absence of particle interactions

$$\frac{Df}{Dt} = \left[\frac{\partial}{\partial t} + \frac{d\mathbf{q}}{dt}\frac{\partial}{\partial \mathbf{q}} + \frac{d\mathbf{x}}{dt}\frac{\partial}{\partial \mathbf{x}}\right] f = 0$$

Expanding universe: de Broglie wavelength of particles “stretches”

$$q \propto a^{-1}$$

- Homogeneous and isotropic limit

$$\frac{\partial f}{\partial t} + \frac{dq}{dt}\frac{\partial f}{\partial q} = \frac{\partial f}{\partial t} - H(a)\frac{\partial f}{\partial \ln q} = 0$$

- Implies energy conservation: $d\rho/dt = -3H(\rho + p)$

Boltzmann Equation

- Boltzmann equation says that Liouville theorem must be modified to account for collisions

$$\frac{Df}{Dt} = C[f]$$

- Heuristically

$$C[f] = \text{particle sources - sinks}$$

- Collision term: integrate over phase space of incoming particles, connect to outgoing state with some interaction strength

Boltzmann Equation

- Form:

$$C[f] = \frac{1}{E} \int d(\text{phase space})[\text{energy-momentum conservation}]$$
$$\times |M|^2[\text{emission} - \text{absorption}]$$

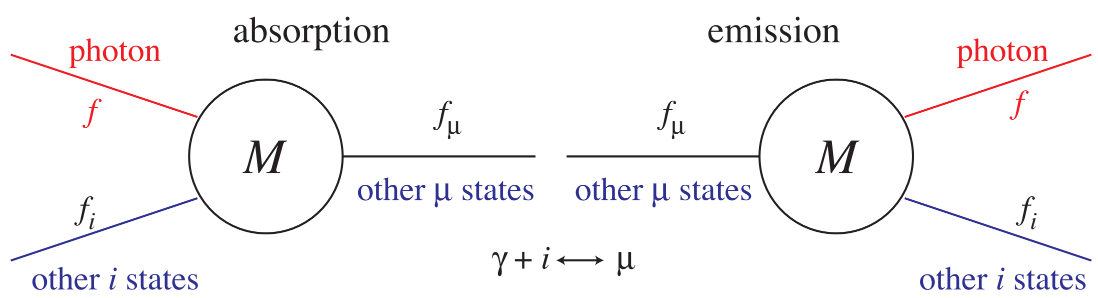
- Matrix element M, assumed T [or CP] invariant
- (Lorentz invariant) phase space element

$$\int d(\text{phase space}) = \Pi_i \frac{g_i}{(2\pi)^3} \int \frac{d^3 q_i}{2E_i}$$

- Energy conservation: $(2\pi)^4 \delta^{(4)}(q_1 + q_2 + ...)$

Boltzmann Equation

- Emission - absorption term involves the particle occupation of the various states
- For concreteness: take f to be the photon distribution function
- Interaction ($\gamma + \sum i \leftrightarrow \sum \mu$); sums are over all incoming and outgoing other particles



- [emission-absorption] $+ =$ boson; $- =$ fermion

$$\Pi_i \Pi_\mu f_\mu (1 \pm f_i)(1 \pm f) - \Pi_i \Pi_\mu (1 \pm f_\mu) f_i f$$

Boltzmann Equation

- Photon Emission: $f_\mu(1 \pm f_i)(1 + f)$

  f_μ: proportional to number of emitters

  $(1 \pm f_i)$: if final state is occupied and a fermion, process blocked; if boson the process enhanced

  $(1 + f)$: final state factor for photons: "1": spontaneous emission (remains if $f = 0$); "$+f$": stimulated and proportional to the occupation of final photon

- Photon Absorption: $-(1 \pm f_\mu) f_i f$

  $(1 \pm f_\mu)$: if final state is occupied and fermion, process blocked; if boson the process enhanced

  f_i: proportional to number of absorbers

  f: proportional to incoming photons

Boltzmann Equation

- If interactions are rapid they will establish an equilibrium distribution where the distribution functions no longer change $C[f_{\rm eq}] = 0$
- Solve by inspection

$$\Pi_i \Pi_\mu f_\mu (1 \pm f_i)(1 \pm f) - \Pi_i \Pi_\mu (1 \pm f_\mu) f_i f = 0$$

- Try $f_a = (e^{E_a/T} \mp 1)^{-1}$ so that $(1 \pm f_a) = e^{-E_a/T}(e^{E_a/T} \mp 1)^{-1}$

$$e^{-\sum (E_i + E)/T} - e^{-\sum E_\mu /T} = 0$$

and energy conservation says $E + \sum E_i = \sum E_\mu$, so identity is satisfied if the constant T is the same for all species, i.e. are in thermal equilibrium

Boltzmann Equation

- If the interaction does not create or destroy particles then the distribution

$$f_{\rm eq} = (e^{(E-\mu)/T} \mp 1)^{-1}$$

also solves the equilibrium equation: e.g. a scattering type reaction

$$\gamma_E + i \to \gamma_{E'} + j$$

where i and j represent the same collection of particles but with different energies after the scattering

$$\sum (E_i - \mu_i) + (E - \mu) = \sum (E_j - \mu_j) + (E' - \mu)$$

since $\mu_i = \mu_j$ for each particle

- Not surprisingly, this is the Fermi-Dirac distribution for fermions and the Bose-Einstein distribution for bosons

Boltzmann Equation

- More generally, equilibrium is satisfied if the sum of the chemical potentials on both sides of the interaction are equal, $\gamma + i \leftrightarrow \nu$

$$\sum \mu_i + \mu = \sum \mu_\nu$$

  i.e. the law of mass action is satisfied

- If interactions that create or destroy particles are in equilibrium then this law says that the chemical potential will vanish: e.g. $\gamma + e^- \rightarrow 2\gamma + e^-$

$$\mu_e + \mu = \mu_e + 2\mu \rightarrow \mu = 0$$

  so that the chemical potential is driven to zero if particle number is not conserved in interaction

Maxwell Boltzmann Distribution

- For the nonrelativistic limit $E = m + \frac{1}{2}q^2/m$, nondegenerate limit $(E - \mu)/T \gg 1$ so both distributions go to the Maxwell-Boltzmann distribution

$$f_{\rm eq} = \exp[-(m-\mu)/T]\exp(-q^2/2mT)$$

- Here it is even clearer that the chemical potential μ is the normalization parameter for the number density of particles whose number is conserved.
- μ and n can be used interchangably

Poor Man's Boltzmann Equation

- Non expanding medium

$$\frac{\partial f}{\partial t} = \Gamma\,(f - f_{\rm eq})$$

where Γ is some rate for collisions

- Add in expansion in a homogeneous medium

$$\frac{\partial f}{\partial t} + \frac{dq}{dt}\frac{\partial f}{\partial q} = \Gamma\,(f - f_{\rm eq})$$

$$(q \propto a^{-1} \rightarrow \frac{1}{q}\frac{dq}{dt} = -\frac{1}{a}\frac{da}{dt} = H)$$

$$\frac{\partial f}{\partial t} - H\frac{\partial f}{\partial \ln q} = \Gamma\,(f - f_{\rm eq})$$

- So equilibrium will be maintained if collision rate exceeds expansion rate $\Gamma > H$

Non-Relativistic Bulk Properties

- Number density

$$n = ge^{-(m-\mu)/T}\frac{4\pi}{(2\pi)^3}\int_0^\infty q^2 dq \exp(-q^2/2mT)$$

$$= ge^{-(m-\mu)/T}\frac{2^{3/2}}{2\pi^2}(mT)^{3/2}\int_0^\infty x^2 dx \exp(-x^2)$$

$$= g(\frac{mT}{2\pi})^{3/2}e^{-(m-\mu)/T}$$

- Energy density $E = m \to \rho = mn$
- Pressure $q^2/3E = q^2/3m \to p = nT$, ideal gas law

Ultra-Relativistic Bulk Properties

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- Number density

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$$n_{\text{fermion}} = \frac{3}{4} gT^3 \frac{\zeta(3)}{\pi^2}$$

- Energy density

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- Pressure $q^2/3E = E/3 \rightarrow p = \rho/3$, $w_r = 1/3$

Entropy Density

- First law of thermodynamics

$$dS = \frac{1}{T}(d\rho(T)V + p(T)dV)$$

so that

$$\left.\frac{\partial S}{\partial V}\right|_T = \frac{1}{T}[\rho(T) + p(T)]$$

$$\left.\frac{\partial S}{\partial T}\right|_V = \frac{V}{T}\frac{d\rho}{dT}$$

- Since $S(V,T) \propto V$ is extensive

$$S = \frac{V}{T}[\rho(T) + p(T)] \quad \sigma = S/V = \frac{1}{T}[\rho(T) + p(T)]$$

Entropy Density

- Integrability condition $dS/dV dT = dS/dT dV$ relates the evolution of entropy density

$$\frac{d\sigma}{dT} = \frac{1}{T}\frac{d\rho}{dT}$$

$$\frac{d\sigma}{dt} = \frac{1}{T}\frac{d\rho}{dt} = \frac{1}{T}[-3(\rho+p)]\frac{d\ln a}{dt}$$

$$\frac{d\ln\sigma}{dt} = -3\frac{d\ln a}{dt} \qquad \sigma \propto a^{-3}$$

comoving entropy density is conserved in thermal equilibrium

- For ultra relativisitic bosons $s_{\rm boson} = 3.602 n_{\rm boson}$; for fermions factor of $7/8$ from energy density.

$$g_* = \sum_{\rm bosons} g_b + \frac{7}{8}\sum g_f$$