

Astro 282

Lecture Notes: **Main Set**

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CMBology

- Universe is currently bathed in 2.725K blackbody radiation which composes the majority of the radiation density of the universe
mm-cm wavelength, 100 GHz photons near peak
400 photon cm^{-3}
- Radiation is extremely isotropic: aside from the 10^{-3} temperature variations due to the Doppler shift of our own motion, fluctuations in the temperature are at the 10^{-5} level.
- Fluctuations are the imprint of the origin of structure
- Fluctuations are polarized at the 10% level reflecting scattering processes by which they last interacted with matter
- Place CMB in cosmological context

Astro 282

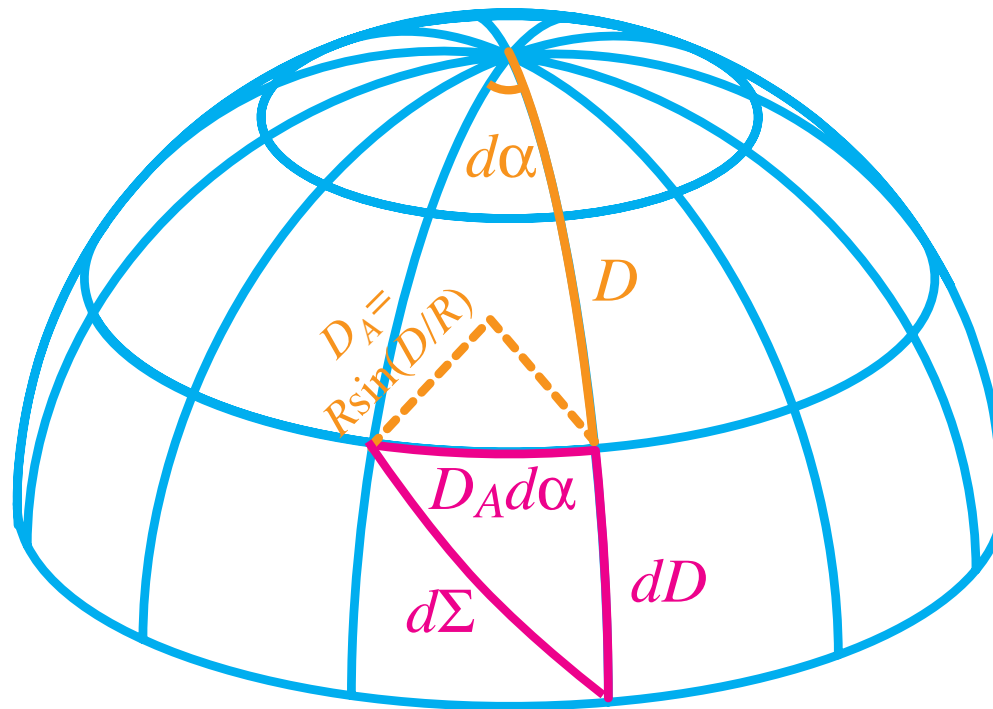
FRW Cosmology

FRW Cosmology

- FRW cosmology = homogeneous and isotropic on large scales
- Universe observed to be nearly isotropic (e.g. CMB, radio point sources, galaxy surveys)
- Copernican principle: must be isotropic to all observers (all locations)
- Implies homogeneity; also galaxy redshift surveys (LCRS, 2dF, SDSS) have seen the “end of greatness”, large scale homogeneity directly
- FRW cosmology (homogeneity, isotropy & Einstein equations) generically implies the expansion of the universe, except for special unstable cases

FRW Geometry

- Spatial geometry is that of a constant curvature (positive, negative, zero) surface
- Metric tells us how to measure distances on this surface
- Consider the closed geometry of a sphere of radius R and suppress one dimension



Angular Diameter Distance

- Spatial distance: restore 3rd dimension with the usual spherical polar angles

$$\begin{aligned}d\Sigma^2 &= dD^2 + D_A^2 d\alpha^2 \\ &= dD^2 + D_A^2 (d\theta^2 + \sin^2 \theta d\phi^2)\end{aligned}$$

- D_A is called the angular diameter distance since $D_A d\alpha$ corresponds to the transverse separation or size as opposed to the Euclidean $D d\alpha$, i.e. is the apparent distance to an object through the gravitational lens of the background geometry
- In a positively curved geometry $D_A < D$ and objects are further than they appear
- In a negatively curved universe R is imaginary and $R \sin(D/R) = i|R| \sin(D/i|R|) = |R| \sinh(D/|R|)$ – and $D_A > D$ objects are closer than they appear

Volume Element

- Metric also defines the volume element

$$\begin{aligned}dV &= (dD)(D_A d\theta)(D_A \sin \theta d\phi) \\ &= D_A^2 dD d\Omega\end{aligned}$$

- Most of classical cosmology boils down to these three quantities, (comoving) distance, (comoving) angular diameter distance, and volume element
- For example, distance to a high redshift supernova, angular size of the horizon at last scattering, number density of clusters...

Comoving Coordinates

- Remaining degree of freedom (preserving homogeneity and isotropy) is an overall scale factor that relates the geometry (fixed by the radius of curvature R) to physical coordinates – a function of time only

$$d\sigma^2 = a^2(t)d\Sigma^2$$

our conventions are that the scale factor today $a(t_0) \equiv 1$

- Similarly physical distances are given by $d(t) = a(t)D$,
 $d_A(t) = a(t)D_A$.
- Distances in capital case are *comoving* i.e. they comove with the expansion and do not change with time – simplest coordinates to work out geometrical effects

Redshift

- Wavelength of light “stretches” with the scale factor, so that it is convenient to define a shift-to-the-red or redshift as the scale factor increases

$$\lambda(a) = a(t)\Lambda$$

$$\frac{\lambda(1)}{\lambda(a)} = \frac{1}{a} \equiv (1 + z)$$

$$\frac{\delta\lambda}{\lambda} = -\frac{\delta\nu}{\nu} = z$$

- Given known frequency of emission $\nu(a)$, redshift can be precisely measured (modulo Doppler shifts from peculiar velocities) – interpreting the redshift as a Doppler shift, objects recede in an expanding universe - $v = zc$

Time and Conformal Time

- As in special relativity, time comes in with the opposite signature in measuring space-time separation
- Proper time

$$\begin{aligned}d\tau^2 &= dt^2 - d\sigma^2 \\ &= dt^2 - a^2(t)d\Sigma^2 \\ &\equiv a^2(t)(d\eta^2 - d\Sigma^2)\end{aligned}$$

- Special relativity: physics invariant under the set of linear coordinate transformations (Lorentz transformation) that preserve lengths ($d\tau^2$)
- General relativity: physics invariant under a general coordinate transformation that preserves lengths

A GR Aside

- We will generally skirt around General Relativity but knowledge of the language will be useful
- Proper time defines the metric $g_{\mu\nu}$

$$d\tau^2 \equiv g_{\mu\nu} dx^\mu dx^\nu$$

- Usually we will use comoving coordinates and conformal time as the “ x ” ’s unless otherwise specified – metric for other choices are related by $a(t)$ – e.g. in spherical coordinates $\mu \in \eta, \theta, \phi, D$

$$g_{\mu\nu} = a^2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -D_A^2 & 0 & 0 \\ 0 & 0 & -D_A^2 \sin^2 \theta & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Photon Cartography

- Classical cosmology is photon cartography – mapping out the expansion by tracking the distance a photon travels as a function of scale factor or redshift
- Taking out the scale factor in the time coordinate $d\eta = dt/a$ defines **conformal time** – useful in that photons travelling radially from observer then obey

$$\Delta D = \Delta\eta = \int \frac{dt}{a}$$

so that time and distance may be interchanged

Horizon

- Distance travelled by a photon in the whole lifetime of the universe defines the **horizon**
- Since $d\tau = 0$, the horizon is simply the conformal time elapsed

$$D_{\text{horizon}}(t) = \int_0^t \frac{dt'}{a} = \eta(t)$$

- Since the horizon always grows with time, there is always a point in time before which two observers separated by a distance D could not have been in causal contact
- Horizon problem: why is the universe homogeneous and isotropic on large scales, near the current horizon – problem deepens for objects seen at early times, e.g. CMB

Hubble Parameter

- Useful to define the expansion rate or Hubble parameter

$$H(t) \equiv \frac{1}{a} \frac{da}{dt}$$

since dynamics (Einstein equations) will give this directly as

$$H(a) \equiv H(t(a))$$

- Time becomes

$$t = \int dt = \int \frac{da}{aH(a)}$$

- Conformal time becomes

$$\eta = \int \frac{dt}{a} = \int \frac{da}{a^2 H(a)}$$

Distance-Redshift Relation

- All distance redshift relations based on comoving distance $D(z)$

$$D(a) = \int dD = \int_a^1 \frac{da'}{a^2 H(a)}$$
$$(da = -(1+z)^{-2} dz = -a^2 dz)$$

$$D(z) = - \int_z^0 \frac{dz'}{H(z')} = \int_0^z \frac{dz'}{H(z')}$$

- Note limiting case is the Hubble law

$$\lim_{z \rightarrow 0} D(z) = z/H(z=0) \equiv z/H_0$$

redshift (recession velocity) increases linearly with distance

- Hubble constant usually quoted as $H_0 = 100h \text{ km s}^{-1} \text{ Mpc}^{-1}$,
observationally $h \sim 0.7$; in natural units $H_0 = (2997.9)^{-1} h \text{ Mpc}^{-1}$
defines an inverse length scale

Distance-Redshift Relation

- Example: object of known physical size $\lambda = a(t)\Lambda$ (“standard ruler”) subtending an (observed) angle on the sky α

$$\begin{aligned}\alpha &= \frac{\Lambda}{D_A(z)} = \frac{\lambda}{aR \sin(D(z)/R)} \\ &= \frac{\lambda}{R \sin(D(z)/R)} (1+z) \equiv \frac{\lambda}{d_A(z)}\end{aligned}$$

- Example: object of known luminosity L (“standard candle”) with a measured flux S . Comoving surface area $4\pi D_A^2$, frequency/energy $(1+z)$, time-dilation or arrival rate of photons (crests) $(1+z)$:

$$\begin{aligned}S &= \frac{L}{4\pi D_A^2} \frac{1}{(1+z)^2} \\ &\equiv \frac{L}{4\pi d_L^2} \quad (d_L = (1+z)D_A = (1+z)^2 d_A)\end{aligned}$$

Relative Measures

- If absolute calibration of standards unknown, then absolute distance (or Hubble constant) unknown

$$d_A(z) = \lambda/\alpha(z); d_L(z) = \sqrt{L/4\pi S(z)}$$

- Ratio at two different redshifts drops out the unknown standards λ , L and measures evolution of the distance-redshift relation $H_0 D(z)$:

$$\frac{d_{A,L}(z_2)}{d_{A,L}(z_1)} \approx \frac{H_0}{z_1} d_{A,L}(z_2) \quad [z_1 \ll 1]$$

- Alternately, distances & curvature are measured in units of h^{-1} Mpc.

Fundamental Observable

- Fundamental dependence (aside from $(1 + z)$ factors)

$$\begin{aligned} H_0 D_A(z) &= H_0 R \sin(D(z)/R) \\ &= \tilde{R} \sin(H_0 D(z)/\tilde{R}), \quad \tilde{R} = H_0 R \end{aligned}$$

$$H_0 D(z) = \int \frac{da}{a^2} \frac{H_0}{H(a)}$$

- Maps out the kinematics of the expansion
- Current best standard ruler: acoustic oscillations; current best standard candle supernovae type Ia
- Adding in the dynamics of the expansion, measurements of $D(z)$ indicate a flat universe whose expansion is accelerating

Evolution of Scale Factor

- FRW cosmology is fully specified if the function $a(t)$ is given
- General relativity relates the scale factor with the matter content of universe.
- Build the Einstein tensor G^μ_ν out of the metric and use Einstein equation

$$G^\mu_\nu = -8\pi G T^\mu_\nu$$

$$G^0_0 = -\frac{3}{a^2} \left[\left(\frac{\dot{a}}{a} \right)^2 + \frac{1}{R^2} \right]$$
$$G^i_j = -\frac{1}{a^2} \left[2\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a} \right)^2 + \frac{1}{R^2} \right] \delta^i_j$$

Einstein Equations

- Isotropy demands that the stress-energy tensor take the form

$$T^0_0 = \rho$$

$$T^i_j = -p\delta^i_j$$

where ρ is the energy density and p is the pressure

- So Einstein equations become

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{1}{R^2} = \frac{8\pi G}{3}a^2\rho$$

$$2\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 + \frac{1}{R^2} = -8\pi Ga^2p$$

$$\text{or } \frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 = -\frac{4\pi G}{3}a^2(\rho + 3p)$$

Friedman Equations

- More usual to see Einstein equations expressed in time not conformal time

$$\frac{\dot{a}}{a} = \frac{da}{d\eta} \frac{1}{a} = \frac{da}{dt} = aH(a)$$

$$\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 = \frac{d}{d\eta} \left(\frac{\dot{a}}{a}\right) = a \frac{d}{dt} \left(\frac{da}{dt}\right) = a \frac{d^2 a}{dt^2}$$

- Friedmann equations:

$$H^2(a) + \frac{1}{a^2 R^2} = \frac{8\pi G}{3} \rho$$
$$\frac{1}{a} \frac{d^2 a}{dt^2} = -\frac{4\pi G}{3} (\rho + 3p)$$

- Convenient fiction to describe curvature as an energy density component $\rho_K = -3/(8\pi G a^2 R^2) \propto a^{-2}$ that does not accelerate the expansion, $p_K = -\rho_K/3$

Critical Density

- Friedmann equation for H then reads

$$H^2(a) = \frac{8\pi G}{3}(\rho + \rho_K) \equiv \frac{8\pi G}{3}\rho_c$$

defining a critical density today ρ_c in terms of the expansion rate

- In particular, its value today is given by the Hubble constant as

$$\rho_c(z = 0) = 3H_0^2/8\pi G = 1.8788 \times 10^{-29} h^2 \text{g cm}^{-3}$$

- Energy density today is given as a fraction of critical

$\Omega \equiv \rho/\rho_c|_{z=0}$. Radius of curvature then given by

$$R^{-2} = H_0^2(\Omega - 1)$$

- If $\Omega \approx 1$, $\rho \approx \rho_c$, then $\rho_K \ll \rho_c$ or $H_0 R \ll 1$, universe is flat across the Hubble distance. $\Omega < 1$ negatively curved; $\Omega > 1$ positively curved

Newtonian Interpretation

- Consider a test particle of mass m in expanding spherical region of radius r and total mass M . Energy conservation

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r} = \text{const}$$

$$\frac{1}{2} \left(\frac{dr}{dt} \right)^2 - \frac{GM}{r} = \text{const}$$

$$\frac{1}{2} \left(\frac{1}{r} \frac{dr}{dt} \right)^2 - \frac{GM}{r^3} = \frac{\text{const}}{r^2}$$

$$H^2 = \frac{8\pi G\rho}{3} - \frac{\text{const}}{a^2}$$

- Constant determines whether the system is bound and in the Friedmann equation is associated with curvature – not general since neglects pressure

Conservation Law

- Second Friedmann equation, or acceleration equation, simply expresses energy conservation (why: stress energy is automatically conserved in GR via Bianchi identity)

$$d\rho V + p dV = 0$$

$$d\rho a^3 + p da^3 = 0$$

$$\dot{\rho} a^3 + 3 \frac{\dot{a}}{a} \rho a^3 + 3 \frac{\dot{a}}{a} p a^3 = 0$$

$$\frac{\dot{\rho}}{\rho} = -3(1 + w) \frac{\dot{a}}{a} \quad w \equiv p/\rho$$

- If $w = \text{const.}$ then the energy density depends on the scale factor as $\rho \propto a^{-3(1+w)}$.

Multicomponent Universe

- The total energy density can be composed of a sum of components with differing equations of state

$$\rho(a) = \sum_i \rho_i(a) = \sum_i \rho_i(a=1) a^{-3(1+w_i)}, \quad \Omega_i \equiv \rho_i/\rho_c|_{a=1}$$

- Important cases: nonrelativistic matter $\rho_m = mn_m \propto a^{-3}$, $w_m = 0$; relativistic radiation $\rho_r = En_r \propto \nu n_r \propto a^{-4}$, $w_r = 1/3$; “curvature” $\rho_K \propto a^{-2}$, $w_K = -1/3$; constant energy density or cosmological constant $\rho_\Lambda \propto a^0$, $w_\Lambda = -1$
- Or generally with $w_c = p_c/\rho_c = (p + p_K)/(\rho + \rho_K)$

$$\rho_c(a) = \rho_c(a=1) e^{-\int d \ln a 3(1+w_c(a))}$$

$$H^2(a) = H_0^2 e^{-\int d \ln a 3(1+w_c(a))}$$

Acceleration Equation

- Time derivative of (first) Friedman equation

$$\begin{aligned}2 \frac{1}{a} \frac{da}{dt} \left[\frac{1}{a} \frac{d^2 a}{dt^2} - H^2(a) \right] &= \frac{8\pi G}{3} \frac{d\rho_c}{dt} \\ \left[\frac{1}{a} \frac{d^2 a}{dt^2} - \frac{8\pi G}{3} \rho_c \right] &= \frac{4\pi G}{3} [-3(1 + w_c)\rho_c] \\ \frac{1}{a} \frac{d^2 a}{dt^2} &= -\frac{4\pi G}{3} [(1 + 3w_c)\rho_c] \\ &= -\frac{4\pi G}{3} (\rho + \rho_K + 3p + 3p_K) \\ &= -\frac{4\pi G}{3} (1 + 3w)\rho\end{aligned}$$

- Acceleration equation says that universe decelerates if $w > -1/3$

Expansion Required

- Friedmann equations “predict” the expansion of the universe. Non-expanding conditions $da/dt = 0$ and $d^2a/dt^2 = 0$ require

$$\rho = -\rho_K \quad \rho = -3p$$

i.e. a positive curvature and a total equation of state

$$w \equiv p/\rho = -1/3$$

- Since matter is known to exist, one can in principle achieve this with

$$\rho = \rho_m + \rho_\Lambda = -\rho_K = -3p = 3\rho_\Lambda$$
$$\rho_\Lambda = -\frac{1}{3}\rho_K \quad \rho_m = -\frac{2}{3}\rho_K$$

Einstein introduced ρ_Λ for exactly this reason – “biggest blunder”; but note that this balance is unstable: ρ_m can be perturbed but ρ_Λ , a true constant cannot

Dark Energy

- Distance redshift relation depends on energy density components

$$\begin{aligned} H_0 D(z) &= \int \frac{da}{a^2} \frac{H_0}{H(a)} \\ &= \int \frac{da}{a^2} e^{\int d \ln a \frac{3}{2}(1+w_c(a))} \end{aligned}$$

- Distant supernova Ia as standard candles imply that $w_c < -1/3$ so that the expansion is accelerating
- Consistent with a cosmological constant that is $\Omega_\Lambda = \rho_\Lambda / \rho_{\text{crit}} = 2/3$ of the total energy density
- Coincidence problem: different components of matter scale differently with a . Why are (at least) two components comparable today? – Anthropic?

Dark Matter

- Since Zwicky in the 1930's non-luminous or dark matter has been known to dominate over luminous matter in stars (and hot gas)
- Arguments are basically from a balance of gravitational force against “pressure” from internal motions: rotation velocity in galactic disks, velocity dispersion of galaxies in clusters, gas pressure in clusters, radiation pressure in CMB
- Assuming that the object is somehow typical in its non-luminous to luminous density, these measures are converted to an overall dark matter density through a “mass-to-light ratio”
- From galaxy surveys the luminosity density in solar units is

$$\rho_L = 2 \pm 0.7 \times 10^8 h L_{\odot} \text{Mpc}^{-3}$$

(h 's: distances in h^{-1} Mpc; luminosity inferred from flux
 $L \propto Sd^2 \propto h^{-2}$; inverse volume $\propto h^3$)

Dark Matter

- Critical density in solar units is $\rho_c = 2.7754 \times 10^{11} h^2 M_\odot \text{Mpc}^{-3}$ so that the critical mass-to-light ratio in solar units is

$$\left(\frac{M}{L}\right) \approx 1400h$$

- Flat rotation curves: $GM/r^2 \approx v^2/r \rightarrow M \approx v^2 r/G$, so the observed flat rotation curve implies $M \propto r$ out to $30h^{-1}$ kpc, beyond the light. Implies $M/L > 30h$ and perhaps more – closure if flat out to ~ 1 Mpc.
- Similar argument holds in clusters of galaxies where velocity dispersion replaces circular velocity and centripetal force is replaced by a “pressure gradient” $T/m = \sigma^2$ and $p = \rho T/m = \rho \sigma^2$ – generalization of hydrostatic equilibrium: Zwicky got $M/L \approx 300h$.

Hydrostatic Equilibrium

- Evidence for dark matter in X -ray clusters also comes from direct hydrostatic equilibrium inference from the gas: balance radial pressure gradient with gravitational potential gradient
- Infinitesimal volume of area dA and thickness dr at radius r and interior mass $M(r)$: pressure difference supports the gas

$$[p_g(r) - p_g(r + dr)]dA = \frac{GmM}{r^2} = \frac{G\rho_g M}{r^2}dV$$
$$\frac{dp_g}{dr} = -\rho_g \frac{d\Phi}{dr}$$

with $p_g = \rho_g T_g / m$ becomes

$$\frac{GM}{r} = -\frac{T_g}{m} \left(\frac{d \ln \rho_g}{d \ln r} + \frac{d \ln T_g}{d \ln r} \right)$$

- ρ_g from X-ray luminosity; T_g sometimes taken as isothermal

Gravitational Lensing

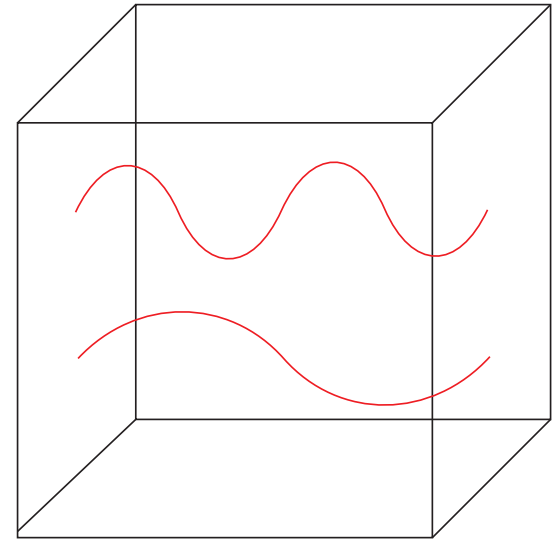
- Mass can be directly measured in the gravitational lensing of sources behind the cluster
- Strong lensing (giant arcs) probes central region of clusters
- Weak lensing (1-10%) elliptical distortion to galaxy image probes outer regions of cluster and total mass
- All techniques agree on the necessity of dark matter and are roughly consistent with a dark matter density $\Omega_m \sim 0.2 - 0.4$
- $\Omega_m + \Omega_\Lambda \approx 1$ from matter density + dark energy
- CMB provides a test of $D_A \neq D$ through the standard rulers of the acoustic peaks and shows that the universe is close to flat $\Omega \approx 1$
- Consistency has lead to the standard model for the cosmological matter budget

Astro 282 Supplement
Statistical Mechanics

How Many Particles Fit in a Box?

- Counting momentum states due to the wave nature of particles with momentum q and de Broglie wavelength (in this supplement we retain \hbar and c to be explicit)

$$\lambda = \frac{h}{q} = \frac{2\pi\hbar}{q}$$



- In a discrete volume L^3 there is a discrete set of states that satisfy periodic boundary conditions

How Many Particles Fit in a Box?

- As in Fourier analysis: $e^{2\pi i x/\lambda} = e^{i(q/\hbar)x} = e^{i(q/\hbar)(x+L)}$ yields a discrete set of allowed states

$$\lambda_i = \frac{L}{m_i} = \frac{2\pi\hbar}{q_i}, \quad m_i = 1, 2, 3\dots$$
$$q_i = m_i \frac{2\pi\hbar}{L}$$

- In each of 3 directions: $\sum_{m_{xi} m_{yj} m_{zk}} \rightarrow \int d^3 m$
- The differential number of allowed momenta in the volume

$$d^3 m = \left(\frac{L}{2\pi\hbar} \right)^3 d^3 q$$

Density of States

- The total number of states allows for a number of internal degrees of freedom, e.g. spin, quantified by the degeneracy factor g : total density of states:

$$\frac{dN_s}{V} = \frac{g}{V} d^3 m = \frac{g}{(2\pi\hbar)^3} d^3 q$$

- If all states were occupied by a single particle, then the particle density

$$n_s = \frac{N_s}{V} = \frac{1}{V} \int dN_s = \int \frac{g}{(2\pi\hbar)^3} d^3 q$$

Distribution Function

- The distribution function f quantifies the occupation of the allowed momentum states

$$n = \frac{N}{V} = \frac{1}{V} \int f dN_s = \int \frac{g}{(2\pi\hbar)^3} f d^3q$$

- f , aka phase space occupation number, also quantifies the density of particles per unit phase space $dN/(\Delta x)^3(\Delta q)^3$
- For photons, the spin degeneracy $g = 2$ accounting for the 2 polarization states
- Energy $E(q) = (q^2c^2 + m^2c^4)^{1/2}$
- Momentum \rightarrow frequency $q = h/\lambda = h\nu/c = E/c$ (where $m = 0$ and $\lambda\nu = c$)

Number Density

- Momentum state defines the direction of the radiation

$$\begin{aligned}n &= g \int \frac{d^3 q}{(2\pi\hbar)^3} f \\&= 2 \int \frac{d\Omega q^2 dq}{(2\pi\hbar)^3} f \\&= 2 \int d\Omega \left(\frac{h}{c}\right)^3 \frac{1}{h^3} \int \nu^2 d\nu f \\&= 2 \int d\Omega \frac{1}{c^3} \int \nu^2 d\nu f\end{aligned}$$

- Gives number density in a given direction and frequency band

Energy Density

- In general the energy density is

$$\rho = g \int \frac{d^3 q}{(2\pi\hbar)^3} E(q) f$$

- For radiation

$$\rho = g \int \frac{d^3 q}{(2\pi\hbar)^3} E(q) f = 2 \int d\Omega \frac{1}{c^3} \int \nu^2 d\nu h\nu f$$

- So specific energy density

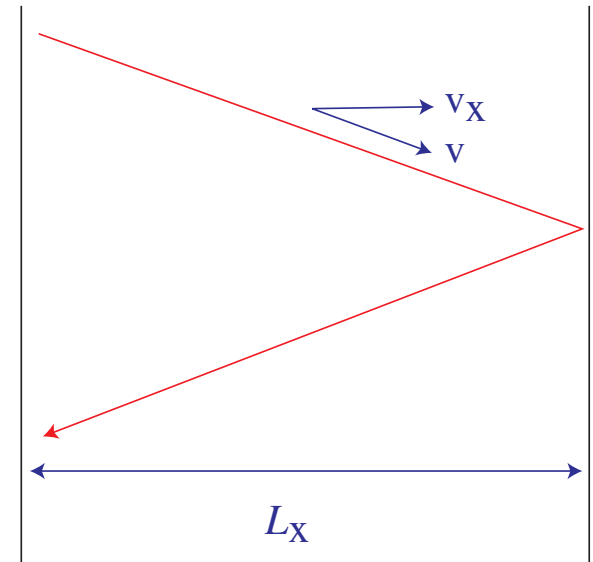
$$\rho_\nu(\Omega) = \frac{d^2 \rho}{d\Omega d\nu} = \frac{2\nu^3 h}{c^3} f$$

- And specific intensity

$$I_\nu(\Omega) = \rho_\nu(\Omega) c = \frac{2\nu^3 h}{c^2} f$$

Pressure

- Pressure: particles bouncing off a surface of area A in a volume spanned by L_x : per momentum state



$$p_q = \frac{F}{A} = \frac{N_{\text{part}}}{A} \frac{\Delta q_x}{\Delta t}$$
$$(\Delta q_x = 2|q_x|, \quad \Delta t = 2L_x/v_x, \quad q/E = v/c^2)$$
$$= \frac{N_{\text{part}}}{V} |q_x| |v_x| = f \frac{|q| |v|}{3} = f \frac{q^2 c^2}{3E}$$

(\cos^2 term in radiative pressure calc.)

Moments

- So that summed over states

$$p = g \int \frac{d^3q}{(2\pi\hbar)^3} \frac{|q|^2 c^2}{3E(q)} f$$

- Radiation

$$p = g \int \frac{d^3q}{(2\pi\hbar)^3} \frac{E(q)}{3} f = \frac{1}{3}\rho$$

- Energy and pressure are part of the angular moments of the distribution function – the isotropic ones
- First order anisotropy is the bulk momentum density or dipole of the distribution:

$$(u + p)\mathbf{v}/c = g \int \frac{d^3q}{(2\pi\hbar)^3} \mathbf{q} c f$$

Fluid Approximation Redux

- Continue with the second moments: radiative viscosity or anisotropic stress

$$\pi_{ij} = g \int \frac{d^3q}{(2\pi\hbar)^3} \frac{3q_i q_j - q^2 \delta_{ij}}{3E(q)} f$$

- Fluid approximation is that all the higher order moments from the radiative viscosity onward vanishes - due isotropization from a high collision rate
- Since particle kinetics must obey energy and momentum conservation, in the fluid limit there are two equations of motion: continuity and Euler equations
- Three quantities of interest: energy density, pressure, bulk velocity means that a third relation is needed: $p(\rho)$ the equation of state

Equilibrium

- Thermal physics describes the equilibrium distribution of particles for a medium at temperature T
- Expect that the typical energy of a particle by equipartition is $E \sim kT$, so that $f(E/kT, ?)$ in equilibrium
- Must be a second variable of import. Number density

$$n = g \int \frac{d^3q}{(2\pi\hbar)^3} f(E/kT) =? \quad n(T)$$

- If particles are conserved then n cannot simply be a function of temperature.
- The integration constant that concerns particle conservation is called the chemical potential. Relevant for photons when creation and annihilation processes are ineffective

Temperature and Chemical Potential

- Fundamental assumption of statistical mechanics is that all accessible states have an equal probability of being populated. The number of states G defines the entropy $S(U, N, V) = k \ln G$ where U is the energy, N is the number of particles and V is the volume
- When two systems are placed in thermal contact they may exchange energy, leading to a wider range of accessible states

$$G(U, N, V) = \sum_{U_1} G_1(U_1, N_1, V_1) G_2(U - U_1, N - N_1, V - V_1)$$

- The most likely distribution of U_1 and U_2 is given for the maximum $dG/dU_1 = 0$

$$\left(\frac{\partial G_1}{\partial U_1} \right)_{N_1, V_1} G_2 dU_1 + G_1 \left(\frac{\partial G_2}{\partial U_2} \right)_{N_2, V_2} dU_2 = 0 \quad dU_1 + dU_2 = 0$$

Temperature and Chemical Potential

- Or equilibrium requires

$$\left(\frac{\partial \ln G_1}{\partial U_1} \right)_{N_1, V_1} = \left(\frac{\partial \ln G_2}{\partial U_2} \right)_{N_2, V_2} \equiv \frac{1}{kT}$$

which is the definition of the temperature (equal for systems in thermal contact)

- Likewise define a chemical potential μ for a system in diffusive equilibrium

$$\left(\frac{\partial \ln G_1}{\partial N_1} \right)_{U_1, V_1} = \left(\frac{\partial \ln G_2}{\partial N_2} \right)_{U_2, V_2} \equiv -\frac{\mu}{kT}$$

defines the most likely distribution of particle numbers as a system with equal chemical potentials: generalize to multiple types of particles undergoing “chemical” reaction \rightarrow law of mass action

$$\sum_i \mu_i dN_i = 0$$

Temperature and Chemical Potential

- Equivalent definition: the chemical potential is the free energy cost associated with adding a particle at fixed temperature and volume

$$\mu = \left. \frac{\partial F}{\partial N} \right|_{T,V}, \quad F = U - TS$$

free energy: balance between minimizing energy and maximizing entropy S

- Temperature and chemical potential determine the probability of a state being occupied if the system is in thermal and diffusive contact with a large reservoir at temperature T

Gibbs or Boltzmann Factor

- Suppose the system has two states unoccupied $N_1 = 0, U_1 = 0$ and occupied $N_1 = 1, U_1 = E$ then the ratio of probabilities in the occupied to unoccupied states is given by

$$P = \frac{\exp[\ln G_{\text{res}}(U - E, N - 1, V)]}{\exp[\ln G_{\text{res}}(U, N, V)]}$$

- Taylor expand

$$\ln G_{\text{res}}(U - E, N - 1, V) \approx \ln G_{\text{res}}(U, N, V) - E \frac{1}{kT} + \frac{\mu}{kT}$$

$$P \approx \exp[-(E - \mu)/kT]$$

- This is the Gibbs factor.

Gibbs or Boltzmann Factor

- More generally the probability of a system being in a state of energy E_i and particle number N_i is given by the Gibbs factor

$$P(E_i, N_i) \propto \exp[-(E_i - \mu N_i)/kT]$$

- Unlikely to be in an energy state $E_i \gg kT$ mitigated by the number of particles
- Dropping the diffusive contact, this is the Boltzmann factor

Mean Occupation

- Mean occupation in thermal equilibrium

$$f = \langle N \rangle = \frac{\sum_i N_i P(E_i, N_i)}{\sum P(E_i, N_i)}$$

- Take $E_i = N_i E$ where E is the particle energy (zero point drops out)
- For fermions: occupancy $N_i = 0, 1$

$$\begin{aligned} f &= \frac{P(E, 1)}{P(0, 0) + P(E, 1)} = \frac{\exp[-(E - \mu)/kT]}{1 + \exp[-(E - \mu)/kT]} \\ &= \frac{1}{\exp[(E - \mu)/kT] + 1} \quad \text{Fermi-Dirac Distribution} \end{aligned}$$

- $T \rightarrow 0, f \rightarrow [e^{\pm\infty} + 1]^{-1}$ ($E > \mu, f = 0$); ($E < \mu, f = 1$), occupied out to a sharp energy or Fermi surface with $\delta E = kT$

Bose-Einstein Distribution

- For bosons:

$$\begin{aligned}\sum_i P[E_i, N_i] &= \sum_{N_i=0}^{\infty} \exp[-N_i(E - \mu)/kT] = \sum_{N_i=0}^{\infty} [e^{-(E-\mu)/kT}]^{N_i} \\ &= \frac{1}{1 - e^{-(E-\mu)/kT}}\end{aligned}$$

$$\begin{aligned}\sum_i N_i P[E_i, N_i] &= \sum_{N_i=0}^{\infty} N_i \exp[-N_i(E - \mu)/kT] \\ &= \frac{\partial}{\partial \mu/kT} \sum_{N_i=0}^{\infty} [e^{-(E-\mu)/kT}]^{N_i} \\ &= \frac{\partial}{\partial \mu/kT} \left(\frac{1}{1 - e^{-(E-\mu)/kT}} \right) = \frac{e^{-(E-\mu)/kT}}{(1 - e^{-(E-\mu)/kT})^2}\end{aligned}$$

Bose-Einstein Distribution

- Bose Einstein distribution:

$$f = \frac{\sum_i N_i P[E_i, N_i]}{\sum_i P[E_i, N_i]} = \frac{1}{e^{(E-\mu)/kT} - 1}$$

For $E - \mu \gg kT$, $f \rightarrow 0$. For $E - \mu < kT \ln 2$, $f > 1$, high occupation (Bose-Einstein condensate).

- General equilibrium distribution

$$f = \frac{1}{e^{(E-\mu)/kT} \pm 1}$$

+ = fermions, - = bosons

- μ alters the number of particles at temperature T

Maxwell Boltzmann Distribution

- In both cases, if $(E - \mu) \gg kT$ (including rest mass energy), then

$$f = e^{-(E-\mu)/kT}$$

- For non relativistic particles

$$E = (q^2 c^2 + m^2 c^4)^{1/2} = mc^2 (1 + q^2 / m^2 c^2)^{1/2}$$

$$\approx mc^2 (1 + q^2 / 2m^2 c^2) = mc^2 + \frac{1}{2}mv^2$$

$$f = e^{-(mc^2 - \mu)/kT} e^{-mv^2/2kT}$$

Planck (Black Body) Distribution

- When particles can be freely created and destroyed $\mu \rightarrow 0$ and for bosons this is the black body distribution

$$f = \frac{1}{e^{E/kT} - 1}$$

- Specific intensity

$$I_\nu = B_\nu = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1}$$

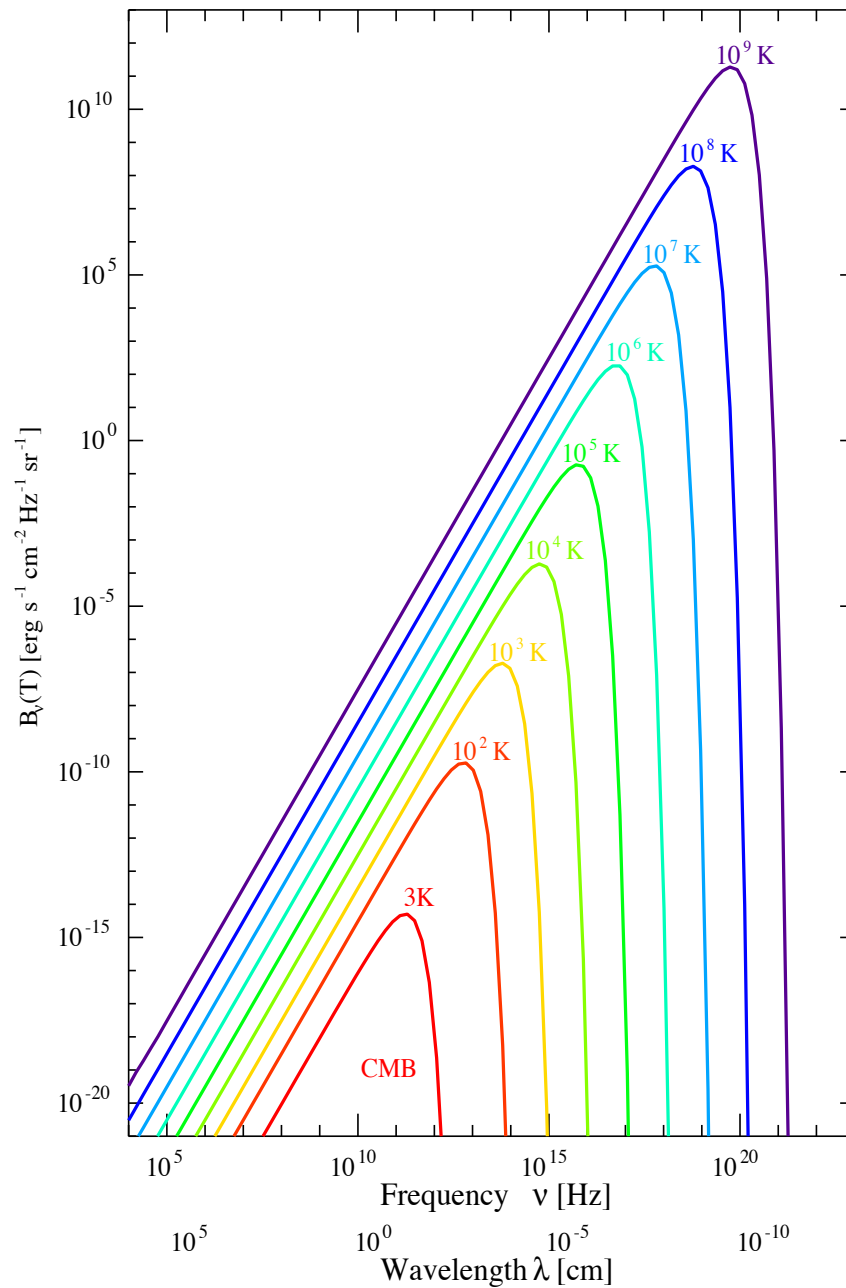
- At low frequencies $h\nu \ll kT$ (Rayleigh Jeans)

$$\exp(h\nu/kT) - 1 \approx 1 + h\nu/kT - 1 = h\nu/kT$$

$$B_\nu = \frac{2h\nu^3}{c^2} \frac{kT}{h\nu} = 2 \frac{\nu^2}{c^2} kT$$

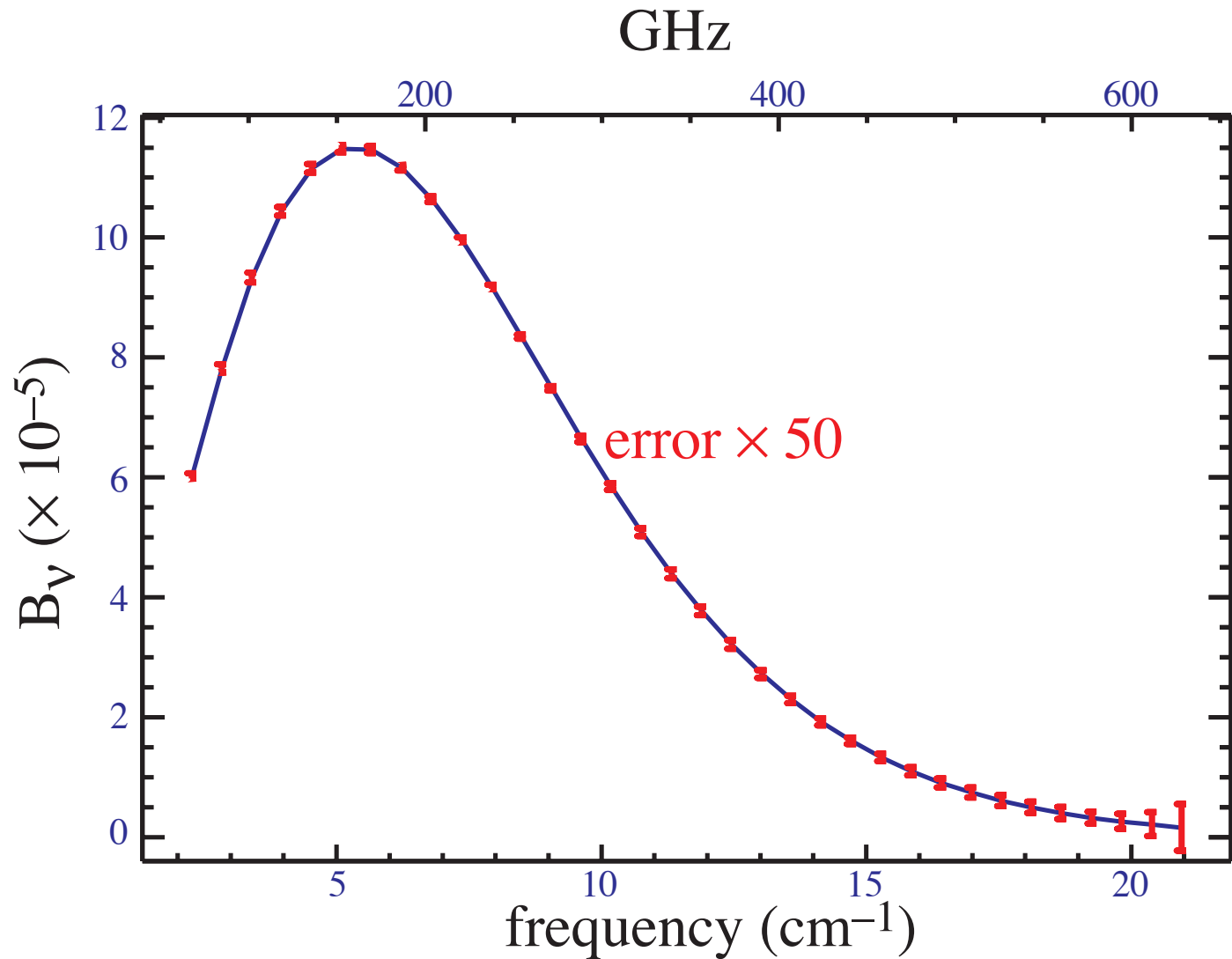
independent of h (classical, many photon limit)

Planck (Black Body) Distribution



Cosmic Microwave Background

- FIRAS observations



Planck (Black Body) Distribution

- $B_\nu \propto \nu^2$ would imply an ultraviolet catastrophe $S = \int B_\nu d\nu$
- At high frequencies $h\nu \gg kT$ (Wien tail)

$$\exp(h\nu/kT) - 1 \approx e^{h\nu/kT}$$

$$B_\nu = \frac{2h\nu^3}{c^2} e^{-h\nu/kT}$$

exhibits the Boltzmann suppression, particle nature of light

- Scaling with T

$$\frac{\partial B_\nu}{\partial T} = \frac{2h\nu^3}{c^2} \frac{\partial f}{\partial T} = \frac{2h\nu^3}{c^2} \left(\frac{-1}{(e^{h\nu/kT} - 1)^2} \right) \frac{-h\nu}{kT^2} > 0$$

so that specific intensity at all ν increases with T

- Setting $\partial B_\nu / \partial \nu = 0$ defines the maximum $h\nu_{\max} = 2.82kT$

Planck (Black Body) Distribution

- Surface Brightness

$$\begin{aligned} S &= \int_0^\infty B_\nu d\nu = \frac{2h}{c^2} \int_0^\infty \frac{\nu^3}{e^{h\nu/kT} - 1} d\nu \\ &= \frac{2h}{c^2} \left(\frac{kT}{h} \right)^4 \int_0^\infty dx \frac{x^3}{e^x - 1} = \frac{2\pi^4 k^4}{15c^2 h^3} T^4 \equiv \frac{\sigma_B T^4}{\pi} \end{aligned}$$

where $\sigma_B = 2\pi^5 k^4 / 15c^2 h^3$ is the Stephan-Boltzmann constant and the π accounts for the emergent flux at the radius R of a uniform sphere where angles up to the $\pi/2$ tangent can be viewed

$$F \equiv \int S \cos \theta d\Omega = S \int_0^{2\pi} d\phi \int_0^{\pi/2} \cos \theta \sin \theta d\theta = \pi S$$

Planck (Black Body) Distribution

- Energy density

$$\rho = \int \frac{B_\nu}{c} d\nu d\Omega = \frac{4\pi}{c} \frac{\sigma_B T^4}{\pi} = \frac{4\sigma_B}{c} T^4$$

- Number density

$$\begin{aligned} n &= 2 \int d\Omega \frac{1}{c^3} \int \nu^2 d\nu \frac{1}{e^{h\nu/kT} - 1} \\ &= \frac{8\pi}{c^3} \left(\frac{kT}{h} \right)^3 \int_0^\infty \frac{x^2 dx}{e^x - 1} = \frac{16\pi\zeta(3)}{c^3} \left(\frac{kT}{h} \right)^3 \\ &= \frac{2\zeta(3)}{\pi^2 c^3} \left(\frac{kT}{\hbar} \right)^3 \end{aligned}$$

where $\zeta(3) \approx 1.202$

Astro 282

Recombination

Saha Equation

- What is the equilibrium ionization state of a gas at a given temperature?
- Hydrogen example: $e + p \leftrightarrow H + \gamma$
- Define $n_{\text{tot}} = n_p + n_H$ and an ionization fraction $x_e \equiv n_p/n_{\text{tot}}$

$$\frac{n_p n_e}{n_H n_{\text{tot}}} = \frac{x_e^2}{1 - x_e}$$

- Number densities defined by distribution function in thermal equilibrium. e and p are non-relativistic at the eV energy scales of recombination
- Maxwell-Boltzmann distribution

$$f = e^{-(mc^2 - \mu)/kT} e^{-q^2/2mkT}$$

Saha Equation

- Number density:

$$\begin{aligned}n &= g \int \frac{d^3q}{(2\pi\hbar)^3} f = \frac{g e^{-(mc^2 - \mu)/kT}}{2\pi^2\hbar^3} \int_0^\infty q^2 dq e^{-q^2/2mkT} \\&= g \frac{e^{-(mc^2 - \mu)/kT}}{2\pi^2\hbar^3} (2mkT)^{3/2} \left[\int_0^\infty x^2 dx e^{-x^2} = \frac{\sqrt{\pi}}{4} \right] \quad (x = p/\sqrt{2mkT}) \\&= g e^{-(mc^2 - \mu)/kT} \left(\frac{mkT}{2\pi\hbar^2} \right)^{3/2}\end{aligned}$$

- Hydrogen recombination ($n_{\text{tot}} = n_p + n_H$)

$$n_p = g_p e^{-(m_p c^2 - \mu_p)/kT} \left(m_p kT / 2\pi\hbar^2 \right)^{3/2}$$

$$n_e = g_e e^{-(m_e c^2 - \mu_e)/kT} \left(m_e kT / 2\pi\hbar^2 \right)^{3/2}$$

$$n_H = g_H e^{-(m_H c^2 - \mu_H)/kT} \left(m_H kT / 2\pi\hbar^2 \right)^{3/2}$$

Saha Equation

- Hydrogen binding energy $B = 13.6\text{eV}$: $m_H = m_p + m_e - B/c^2$

$$\frac{n_p n_e}{n_H n_{\text{tot}}} = \frac{x_e^2}{1 - x_e} \approx \frac{g_p g_e}{g_H n_{\text{tot}}} e^{-B/kT} e^{\mu_p + \mu_e - \mu_H} \left(\frac{m_e kT}{2\pi \hbar^2} \right)^{3/2}$$

- Spin degeneracy: spin 1/2 $g_p = 2$, $g_e = 2$; $g_H = 4$ product
- Equilibrium $\mu_p + \mu_e = \mu_H$

$$\frac{x_e^2}{1 - x_e} \approx \frac{1}{n_{\text{tot}}} e^{-B/kT} \left(\frac{m_e kT}{2\pi \hbar^2} \right)^{3/2}$$

- Quadratic equation involving T and the total density - explicit solution for $x_e(T)$
- Exponential dominant factor: ionization drops quickly as kT drops below B - exactly where the sharp transition occurs depends on the density n_{tot}

Saha Equation

- Photon perspective: compare photon number density at T to n_{tot}

$$n_{\gamma} = \frac{2\zeta(3)}{\pi^2\hbar^3} \left(\frac{kT}{c}\right)^3$$

$$\begin{aligned} \frac{x_e^2}{1-x_e} &= \left(\frac{n_{\gamma}}{n_{\text{tot}}}\right) e^{-B/kT} \frac{\pi^2\hbar^3}{2\zeta(3)} \left(\frac{c}{kT}\right)^3 \left(\frac{m_e kT}{2\pi\hbar^2}\right)^{3/2} \\ &= \left(\frac{n_{\gamma}}{n_{\text{tot}}}\right) e^{-B/kT} \frac{\pi^{1/2}}{2^{5/2}\zeta(3)} \left(\frac{m_e c^2}{kT}\right)^{3/2} \end{aligned}$$

- Photon-baryon ratio controls when recombination occurs:
typically a very large number since baryon number is conserved ($\mu \neq 0$) - a low baryon density medium is easy to keep ionized with the high energy photons in tail of the black body
- Cosmologically, recombination occurs at an energy scale of $kT \sim 0.3\text{eV}$

Saha Equation

- Electron perspective: the relevant length scale is the (“thermal”) de Broglie wavelength for a typical particle

$$m_e v^2 \sim kT, \quad q^2 \sim m_e^2 v^2 \sim (m_e kT)$$

$$\lambda_{Te} = \frac{h}{q} = \frac{h}{(2\pi m_e kT)^{1/2}} = \left(\frac{2\pi \hbar^2}{m_e kT} \right)^{1/2}$$

which is the factor in the Saha equation

$$\frac{x_e^2}{1 - x_e} = \frac{1}{n_{\text{tot}} \lambda_{Te}^3} e^{-B/kT}$$

$N_{Te} = n_e \lambda_{Te}^3 = \#$ electrons in a de Broglie volume and is $\ll 1$ for non-degenerate matter

Saha Equation

- Saha equation

$$\frac{x_e}{1 - x_e} = \frac{1}{N_{Te}} e^{-B/kT}$$

- Electron chemical potential

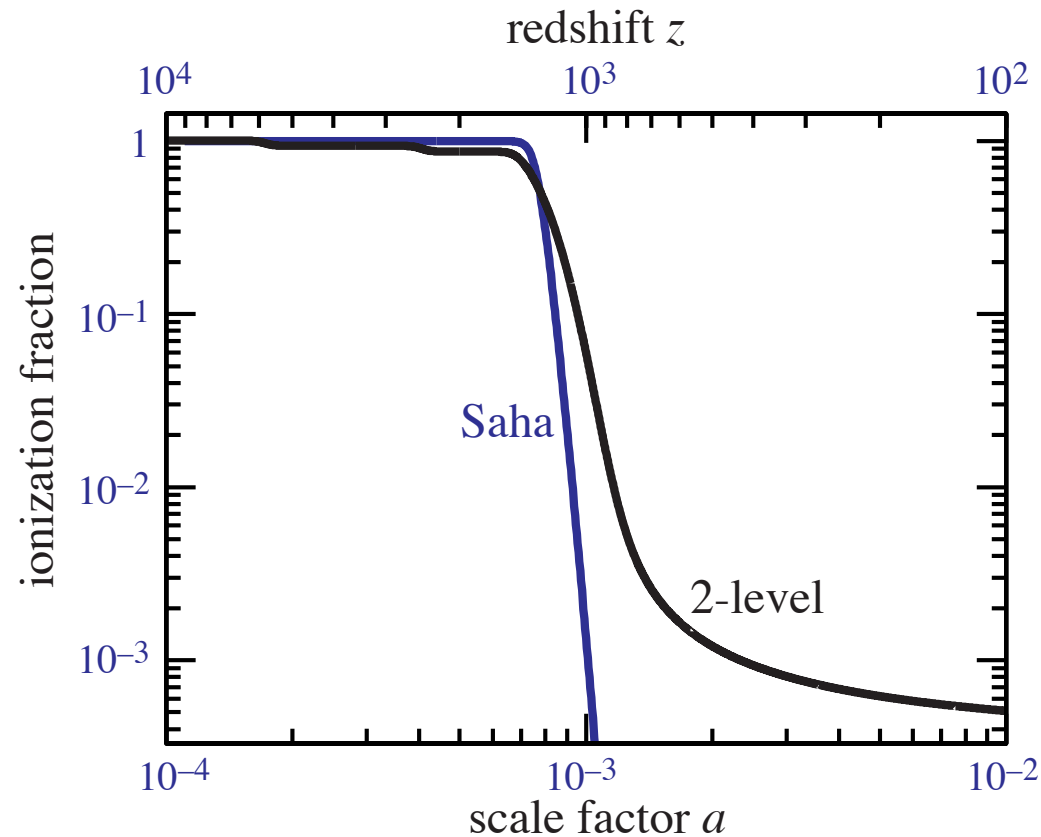
$$N_{Te} = 2e^{-(m_e c^2 - \mu_e)/kT}$$

$$\frac{x_e}{1 - x_e} = \frac{1}{2} e^{-[B - (m_e c^2 - \mu_e)]/kT}$$

- Transition occurs when $B_{\text{eff}} = B - m_e c^2 + \mu_e = kT$ - chemical potential or number density determines correction to $B \sim kT$ rule
- However equilibrium may not be maintained - 2 body interaction may not be rapid enough in low density environment - e.g. freezeout cosmologically

Cosmic Recombination

- Rates insufficient to maintain equilibrium - due to Ly α opacity cosmic recombination relies on forbidden 2 photon decay and redshift



Astro 282

Acoustic Kinematics

Temperature Fluctuations

- Observe blackbody radiation with a temperature that differs at 10^{-5} coming from the surface of last scattering, with distribution function (specific intensity $I_\nu = 4\pi\nu^3 f(\nu)$ each polarization)

$$f(\nu) = [\exp(2\pi\nu/T(\hat{\mathbf{n}})) - 1]^{-1}$$

- Decompose the temperature perturbation in spherical harmonics

$$T(\hat{\mathbf{n}}) = \sum_{\ell m} T_{\ell m} Y_{\ell m}(\hat{\mathbf{n}})$$

- For Gaussian random fluctuations, the statistical properties of the temperature field are determined by the power spectrum

$$\langle T_{\ell m}^* T_{\ell' m'} \rangle = \delta_{\ell\ell'} \delta_{mm'} C_\ell$$

where the δ -function comes from statistical isotropy

Spatial vs Angular Power

- Take the radiation distribution at last scattering to also be described by an isotropic temperature field $T(\mathbf{x})$ and recombination to be instantaneous

$$T(\hat{\mathbf{n}}) = \int dD T(\mathbf{x}) \delta(D - D_*)$$

where D is the comoving distance and D_* denotes recombination.

- Describe the temperature field by its Fourier moments

$$T(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} T(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}$$

with a power spectrum

$$\langle T(\mathbf{k})^* T(\mathbf{k}') \rangle = (2\pi)^3 \delta(\mathbf{k} - \mathbf{k}') P_T(k)$$

Spatial vs Angular Power

- Note that the variance of the field

$$\begin{aligned}\langle T(\mathbf{x})T(\mathbf{x}) \rangle &= \int \frac{d^3k}{(2\pi)^3} P(k) \\ &= \int d \ln k \frac{k^3 P(k)}{2\pi^2} \equiv \int d \ln k \Delta_T^2(k)\end{aligned}$$

so it is more convenient to think in the log power spectrum $\Delta_T^2(k)$

- Temperature field

$$T(\hat{\mathbf{n}}) = \int \frac{d^3k}{(2\pi)^3} T(\mathbf{k}) e^{i\mathbf{k} \cdot D_* \hat{\mathbf{n}}}$$

- Expand out plane wave in spherical coordinates

$$e^{i\mathbf{k} D_* \cdot \hat{\mathbf{n}}} = 4\pi \sum_{\ell m} i^\ell j_\ell(k D_*) Y_{\ell m}^*(\mathbf{k}) Y_{\ell m}(\hat{\mathbf{n}})$$

Spatial vs Angular Power

- Multipole moments

$$T_{\ell m} = \int \frac{d^3 k}{(2\pi)^3} T(\mathbf{k}) 4\pi i^\ell j_\ell(kD_*) Y_{\ell m}(\mathbf{k})$$

- Power spectrum

$$\begin{aligned} \langle T_{\ell m}^* T_{\ell' m'} \rangle &= \int \frac{d^3 k}{(2\pi)^3} (4\pi)^2 (i)^{\ell-\ell'} j_\ell(kD_*) j_{\ell'}(kD_*) Y_{\ell m}^*(\mathbf{k}) Y_{\ell' m'}(\mathbf{k}) P_T(k) \\ &= \delta_{\ell\ell'} \delta_{mm'} 4\pi \int d \ln k j_\ell^2(kD_*) \Delta_T^2(k) \end{aligned}$$

with $\int_0^\infty j_\ell^2(x) d \ln x = 1/(2\ell(\ell + 1))$, slowly varying Δ_T^2

$$C_\ell = \frac{4\pi \Delta_T^2(\ell/D_*)}{2\ell(\ell + 1)} = \frac{2\pi}{\ell(\ell + 1)} \Delta_T^2(\ell/D_*)$$

so $\ell(\ell + 1)C_\ell/2\pi = \Delta_T^2$ is commonly used log power

Scale Invariant Fluctuations

- Scale invariant temperature fluctuations have $\Delta_T^2 = \text{const}$
- Equal contributions to the rms temperature fluctuation per decade in frequency k
- Observed angular fluctuations then have $\ell(\ell + 1)C_\ell/2\pi = \text{const}$
- Weaker assumption of scale free initial temperature fluctuations $\Delta_T^2 \propto k^{n-1}$, where n is called the tilt.
- $n = 1$ is scale invariant for historical reasons.
- However fluctuations evolve from their initial conditions due to gravitational and pressure forces

Thomson Scattering

- Thomson scattering of photons off of free electrons is the most important CMB process with a cross section (averaged over polarization states) of

$$\sigma_T = \frac{8\pi\alpha^2}{3m_e^2} = 6.65 \times 10^{-25} \text{cm}^2$$

- Density of free electrons in a fully ionized $x_e = 1$ universe

$$n_e = (1 - Y_p/2)x_e n_b \approx 10^{-5} \Omega_b h^2 (1+z)^3 \text{cm}^{-3},$$

where $Y_p \approx 0.24$ is the Helium mass fraction, creates a high (comoving) Thomson **opacity**

$$\dot{\tau} \equiv n_e \sigma_T a$$

where dots are conformal time $\eta \equiv \int dt/a$ derivatives and τ is the optical depth.

Tight Coupling Approximation

- Near recombination $z \approx 10^3$ and $\Omega_b h^2 \approx 0.02$, the (comoving) mean free path of a photon

$$\lambda_C \equiv \frac{1}{\dot{\tau}} \sim 2.5 \text{Mpc}$$

small by cosmological standards!

- On scales $\lambda \gg \lambda_C$ photons are **tightly coupled** to the electrons by Thomson scattering which in turn are tightly coupled to the baryons by Coulomb interactions
- Specifically, their bulk velocities are defined by a **single fluid velocity** $v_\gamma = v_b$ and the photons carry **no anisotropy** in the rest frame of the baryons
- \rightarrow No **heat conduction** or **viscosity** (anisotropic stress) in fluid

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Zeroth Order Approximation

- Momentum density of a fluid is $(\rho + p)v$, where p is the pressure
- Neglect the momentum density of the baryons

$$R \equiv \frac{(\rho_b + p_b)v_b}{(\rho_\gamma + p_\gamma)v_\gamma} = \frac{\rho_b + p_b}{\rho_\gamma + p_\gamma} = \frac{3\rho_b}{4\rho_\gamma}$$
$$\approx 0.6 \left(\frac{\Omega_b h^2}{0.02} \right) \left(\frac{a}{10^{-3}} \right)$$

since $\rho_\gamma \propto T^4$ is fixed by the CMB temperature $T = 2.73(1 + z)\text{K}$
– OK substantially before recombination

- Neglect radiation in the expansion

$$\frac{\rho_m}{\rho_r} = 3.6 \left(\frac{\Omega_m h^2}{0.15} \right) \left(\frac{a}{10^{-3}} \right)$$

Number Continuity

- Photons are **not created** or destroyed. Without expansion

$$\dot{n}_\gamma + \nabla \cdot (n_\gamma \mathbf{v}_\gamma) = 0$$

but the **expansion** or Hubble flow causes $n_\gamma \propto a^{-3}$ or

$$\dot{n}_\gamma + 3n_\gamma \frac{\dot{a}}{a} + \nabla \cdot (n_\gamma \mathbf{v}_\gamma) = 0$$

- **Linearize** $\delta n_\gamma = n_\gamma - \bar{n}_\gamma$

$$(\delta n_\gamma)^\cdot = -3\delta n_\gamma \frac{\dot{a}}{a} - n_\gamma \nabla \cdot \mathbf{v}_\gamma$$

$$\left(\frac{\delta n_\gamma}{n_\gamma} \right)^\cdot = -\nabla \cdot \mathbf{v}_\gamma$$

Continuity Equation

- Number density $n_\gamma \propto T^3$ so define temperature fluctuation Θ

$$\frac{\delta n_\gamma}{n_\gamma} = 3 \frac{\delta T}{T} \equiv 3\Theta$$

- Real space continuity equation

$$\dot{\Theta} = -\frac{1}{3} \nabla \cdot \mathbf{v}_\gamma$$

- Fourier space

$$\dot{\Theta} = -\frac{1}{3} i\mathbf{k} \cdot \mathbf{v}_\gamma$$

Momentum Conservation

- No expansion: $\dot{\mathbf{q}} = \mathbf{F}$
- De Broglie wavelength stretches with the expansion

$$\dot{\mathbf{q}} + \frac{\dot{a}}{a}\mathbf{q} = \mathbf{F}$$

for photons this the **redshift**, for non-relativistic particles **expansion drag** on peculiar velocities

- Collection of particles: momentum \rightarrow **momentum density**
 $(\rho_\gamma + p_\gamma)\mathbf{v}_\gamma$ and force \rightarrow **pressure gradient**

$$[(\rho_\gamma + p_\gamma)\mathbf{v}_\gamma]^\cdot = -4\frac{\dot{a}}{a}(\rho_\gamma + p_\gamma)\mathbf{v}_\gamma - \nabla p_\gamma$$

$$\frac{4}{3}\rho_\gamma\dot{\mathbf{v}}_\gamma = \frac{1}{3}\nabla\rho_\gamma$$

$$\dot{\mathbf{v}}_\gamma = -\nabla\Theta$$

Euler Equation

- Fourier space

$$\dot{\mathbf{v}}_\gamma = -ik\Theta$$

- Pressure gradients (any gradient of a scalar field) generates a curl-free flow
- For convenience define velocity amplitude:

$$\mathbf{v}_\gamma \equiv -iv_\gamma \hat{\mathbf{k}}$$

- Euler Equation:

$$\dot{v}_\gamma = k\Theta$$

- Continuity Equation:

$$\dot{\Theta} = -\frac{1}{3}kv_\gamma$$

Oscillator: Take One

- Combine these to form the simple harmonic oscillator equation

$$\ddot{\Theta} + c_s^2 k^2 \Theta = 0$$

where the adiabatic sound speed is defined through

$$c_s^2 \equiv \frac{\dot{p}_\gamma}{\dot{\rho}_\gamma}$$

here $c_s^2 = 1/3$ since we are photon-dominated

- General solution:

$$\Theta(\eta) = \Theta(0) \cos(k s) + \frac{\dot{\Theta}(0)}{k c_s} \sin(k s)$$

where the sound horizon is defined as $s \equiv \int c_s d\eta$

Harmonic Extrema

- All modes are **frozen** in at recombination (denoted with a subscript *) yielding temperature perturbations of **different amplitude** for different modes. For the adiabatic (curvature mode) $\dot{\Theta}(0) = 0$

$$\Theta(\eta_*) = \Theta(0) \cos(k s_*)$$

- Modes caught in the **extrema** of their oscillation will have enhanced fluctuations

$$k_n s_* = n\pi$$

yielding a **fundamental scale** or frequency, related to the inverse **sound horizon**

$$k_A = \pi / s_*$$

and a **harmonic relationship** to the other extrema as 1 : 2 : 3...

Peak Location

- The fundamental **physical scale** is translated into a fundamental **angular scale** by simple projection according to the angular diameter distance D_A

$$\theta_A = \lambda_A / D_A$$

$$\ell_A = k_A D_A$$

- In a flat universe, the distance is simply $D_A = D \equiv \eta_0 - \eta_* \approx \eta_0$, the horizon distance, and $k_A = \pi / s_* = \sqrt{3}\pi / \eta_*$ so

$$\theta_A \approx \frac{\eta_*}{\eta_0}$$

- In a **matter-dominated** universe $\eta \propto a^{1/2}$ so $\theta_A \approx 1/30 \approx 2^\circ$ or

$$\ell_A \approx 200$$

Curvature

- In a **curved universe**, the apparent or **angular diameter distance** is no longer the conformal distance $D_A = R \sin(D/R) \neq D$
- Objects in a **closed universe** are **further** than they appear! gravitational **lensing** of the background...
- Curvature scale of the universe must be substantially **larger than current horizon**
- **Flat universe** indicates critical density and implies missing energy given local measures of the matter density “**dark energy**”
- D also depends on **dark energy density** Ω_{DE} and **equation of state** $w = p_{\text{DE}}/\rho_{\text{DE}}$.
- Expansion rate at recombination or **matter-radiation ratio** enters into calculation of k_A .

Doppler Effect

- Bulk motion of fluid changes the observed temperature via Doppler shifts

$$\left(\frac{\Delta T}{T}\right)_{\text{dop}} = \hat{\mathbf{n}} \cdot \mathbf{v}_\gamma$$

- Averaged over directions

$$\left(\frac{\Delta T}{T}\right)_{\text{rms}} = \frac{v_\gamma}{\sqrt{3}}$$

- Acoustic solution

$$\begin{aligned} \frac{v_\gamma}{\sqrt{3}} &= -\frac{\sqrt{3}}{k} \dot{\Theta} = \frac{\sqrt{3}}{k} k c_s \Theta(0) \sin(ks) \\ &= \Theta(0) \sin(ks) \end{aligned}$$

Doppler Peaks?

- Doppler effect for the photon dominated system is of equal amplitude and $\pi/2$ out of phase: extrema of temperature are turning points of velocity
- Effects add in quadrature:

$$\left(\frac{\Delta T}{T}\right)^2 = \Theta^2(0)[\cos^2(ks) + \sin^2(ks)] = \Theta^2(0)$$

- No peaks in k spectrum! However the Doppler effect carries an angular dependence that changes its projection on the sky

$$\hat{\mathbf{n}} \cdot \mathbf{v}_\gamma \propto \hat{\mathbf{n}} \cdot \hat{\mathbf{k}}$$

- Coordinates where $\hat{\mathbf{z}} \parallel \hat{\mathbf{k}}$

$$Y_{10}Y_{\ell 0} \rightarrow Y_{\ell \pm 1 0}$$

recoupling $j'_\ell Y_{\ell 0}$: no peaks in Doppler effect

Astro 282

Acoustic Dynamics

Restoring Gravity: Continuity

- Take a simple photon dominated system with gravity
- Continuity altered since a gravitational potential represents a stretching of the spatial fabric that dilutes number densities – formally a spatial curvature perturbation
- Think of this as a perturbation to the scale factor $a \rightarrow a(1 + \Phi)$ so that the cosmological redshift is generalized to

$$\frac{\dot{a}}{a} \rightarrow \frac{\dot{a}}{a} + \dot{\Phi}$$

$$(\delta n_\gamma)' = -3\delta n_\gamma \frac{\dot{a}}{a} - 3n_\gamma \dot{\Phi} - n_\gamma \nabla \cdot \mathbf{v}_\gamma$$

so that the continuity equation becomes

$$\dot{\Theta} = -\frac{1}{3}k v_\gamma - \dot{\Phi}$$

Restoring Gravity: Euler

- Gravitational force in momentum conservation $\mathbf{F} = -m\nabla\Psi$ generalized to momentum density modifies the Euler equation to

$$\dot{v}_\gamma = k(\Theta + \Psi)$$

- General relativity says that Φ and Ψ are the relativistic analogues of the Newtonian potential and that $\Phi \approx -\Psi$.
- In our matter-dominated approximation, Φ represents matter density fluctuations through the cosmological Poisson equation

$$k^2\Phi = 4\pi G a^2 \rho_m \Delta_m$$

where the difference comes from the use of comoving coordinates for k (a^2 factor), the removal of the background density into the background expansion ($\rho_m \Delta_m$) and finally a coordinate subtlety that enters into the definition of Δ_m

Constant Potentials

- In the matter dominated epoch potentials are constant because infall generates velocities as $v_m \sim k\eta\Psi$
- Velocity divergence generates density perturbations as $\Delta_m \sim -k\eta v_m \sim -(k\eta)^2\Psi$
- And density perturbations generate potential fluctuations as $\Phi \sim \Delta_m/(k\eta)^2 \sim -\Psi$, keeping them constant. Note that because of the expansion, density perturbations must grow to keep potentials constant.
- Here we have used the Friedman equation $H^2 = 8\pi G\rho_m/3$ and $\eta = \int d\ln a/(aH) \sim 1/(aH)$
- More generally, if stress perturbations are negligible compared with density perturbations ($\delta p \ll \delta\rho$) then potential will remain roughly constant – more specifically a variant called the Bardeen or comoving curvature ζ is constant

Oscillator: Take Two

- Combine these to form the **simple harmonic oscillator** equation

$$\ddot{\Theta} + c_s^2 k^2 \Theta = -\frac{k^2}{3} \Psi - \ddot{\Phi}$$

- In a **CDM dominated** expansion $\dot{\Phi} = \dot{\Psi} = 0$. Also for **photon domination** $c_s^2 = 1/3$ so the oscillator equation becomes

$$\ddot{\Theta} + \ddot{\Psi} + c_s^2 k^2 (\Theta + \Psi) = 0$$

- Solution is just an **offset version** of the original

$$[\Theta + \Psi](\eta) = [\Theta + \Psi](0) \cos(ks)$$

- $\Theta + \Psi$ is also the **observed temperature fluctuation** since photons lose energy climbing out of **gravitational potentials** at recombination

Effective Temperature

- Photons climb out of potential wells at last scattering
- Lose energy to gravitational redshifts
- Observed or effective temperature

$$\Theta + \Psi$$

- Effective temperature oscillates around zero with amplitude given by the initial conditions
- Note: initial conditions are set when the perturbation is outside of horizon, need inflation or other modification to matter-radiation FRW universe.
- GR says that initial temperature is given by initial potential

Sachs-Wolfe Effect and the Magic 1/3

- A gravitational potential is a perturbation to the temporal coordinate [formally a gauge transformation]

$$\frac{\delta t}{t} = \Psi$$

- Convert this to a perturbation in the scale factor,

$$t = \int \frac{da}{aH} \propto \int \frac{da}{a\rho^{1/2}} \propto a^{3(1+w)/2}$$

where $w \equiv p/\rho$ so that during matter domination

$$\frac{\delta a}{a} = \frac{2}{3} \frac{\delta t}{t}$$

- CMB temperature is cooling as $T \propto a^{-1}$ so

$$\Theta + \Psi \equiv \frac{\delta T}{T} + \Psi = -\frac{\delta a}{a} + \Psi = \frac{1}{3}\Psi$$

Baryon Loading

- Baryons add extra **mass** to the photon-baryon fluid
- Controlling parameter is the **momentum density ratio**:

$$R \equiv \frac{p_b + \rho_b}{p_\gamma + \rho_\gamma} \approx 30\Omega_b h^2 \left(\frac{a}{10^{-3}} \right)$$

of order **unity** at recombination

- Momentum density of the **joint system** is conserved

$$\begin{aligned} (\rho_\gamma + p_\gamma)v_\gamma + (\rho_b + p_b)v_b &\approx (p_\gamma + p_\gamma + \rho_b + \rho_\gamma)v_\gamma \\ &= (1 + R)(\rho_\gamma + p_\gamma)v_{\gamma b} \end{aligned}$$

where the controlling parameter is the **momentum density ratio**:

$$R \equiv \frac{p_b + \rho_b}{p_\gamma + \rho_\gamma} \approx 30\Omega_b h^2 \left(\frac{a}{10^{-3}} \right)$$

of order **unity** at recombination

New Euler Equation

- Momentum density ratio enters as

$$\begin{aligned} [(1 + R)(\rho_\gamma + p_\gamma)\mathbf{v}_{\gamma b}]^\cdot &= -4\frac{\dot{a}}{a}(1 + R)(\rho_\gamma + p_\gamma)\mathbf{v}_{\gamma b} \\ &\quad - \nabla p_\gamma - (1 + R)(\rho_\gamma + p_\gamma)\nabla\Psi \end{aligned}$$

same as before except for $(1 + R)$ terms so

$$[(1 + R)v_{\gamma b}]^\cdot = k\Theta + (1 + R)k\Psi$$

- Photon continuity remains the same

$$\dot{\Theta} = -\frac{k}{3}v_{\gamma b} - \dot{\Phi}$$

- Modification of oscillator equation

$$[(1 + R)\dot{\Theta}]^\cdot + \frac{1}{3}k^2\Theta = -\frac{1}{3}k^2(1 + R)\Psi - [(1 + R)\dot{\Phi}]^\cdot$$

Oscillator: Take Three

- Combine these to form the not-quite-so simple harmonic oscillator equation

$$c_s^2 \frac{d}{d\eta} (c_s^{-2} \dot{\Theta}) + c_s^2 k^2 \Theta = -\frac{k^2}{3} \Psi - c_s^2 \frac{d}{d\eta} (c_s^{-2} \dot{\Phi})$$

where $c_s^2 \equiv \dot{p}_{\gamma b} / \dot{\rho}_{\gamma b}$

$$c_s^2 = \frac{1}{3} \frac{1}{1 + R}$$

- In a CDM dominated expansion $\dot{\Phi} = \dot{\Psi} = 0$ and the adiabatic approximation $\dot{R}/R \ll \omega = kc_s$

$$[\Theta + (1 + R)\Psi](\eta) = [\Theta + (1 + R)\Psi](0) \cos(k s)$$

Baryon Peak Phenomenology

- Photon-baryon ratio enters in **three ways**
- Overall larger **amplitude**:

$$[\Theta + (1 + R)\Psi](0) = \frac{1}{3}(1 + 3R)\Psi(0)$$

- Even-odd peak **modulation** of effective temperature

$$[\Theta + \Psi]_{\text{peaks}} = [\pm(1 + 3R) - 3R] \frac{1}{3} \Psi(0)$$

$$[\Theta + \Psi]_1 - [\Theta + \Psi]_2 = [-6R] \frac{1}{3} \Psi(0)$$

- Shifting of the **sound horizon** down or ℓ_A up

$$\ell_A \propto \sqrt{1 + R}$$

- Actual effects **smaller** since R evolves

Photon Baryon Ratio Evolution

- Oscillator equation has time evolving mass

$$c_s^2 \frac{d}{d\eta} (c_s^{-2} \dot{\Theta}) + c_s^2 k^2 \Theta = 0$$

- Effective mass is $m_{\text{eff}} = 3c_s^{-2} = (1 + R)$
- Adiabatic invariant

$$\frac{E}{\omega} = \frac{1}{2} m_{\text{eff}} \omega A^2 = \frac{1}{2} 3c_s^{-2} k c_s A^2 \propto A^2 (1 + R)^{1/2} = \text{const.}$$

- Amplitude of oscillation $A \propto (1 + R)^{-1/4}$ decays adiabatically as the photon-baryon ratio changes

Oscillator: Take Three and a Half

- The not-quite-so simple harmonic oscillator equation is a forced harmonic oscillator

$$c_s^2 \frac{d}{d\eta} (c_s^{-2} \dot{\Theta}) + c_s^2 k^2 \Theta = -\frac{k^2}{3} \Psi - c_s^2 \frac{d}{d\eta} (c_s^{-2} \Phi)$$

changes in the gravitational potentials alter the form of the acoustic oscillations

- If the forcing term has a temporal structure that is related to the frequency of the oscillation, this becomes a driven harmonic oscillator
- Term involving Ψ is the ordinary gravitational force
- Term involving Φ involves the $\dot{\Phi}$ term in the continuity equation as a (curvature) perturbation to the scale factor

Potential Decay

- Matter-to-radiation ratio

$$\frac{\rho_m}{\rho_r} \approx 24\Omega_m h^2 \left(\frac{a}{10^{-3}} \right)$$

of order **unity** at recombination in a low Ω_m universe

- Radiation is not stress free and so **impedes** the growth of structure

$$k^2\Phi = 4\pi G a^2 \rho_r \Delta_r$$

$\Delta_r \sim 4\Theta$ **oscillates** around a constant value, $\rho_r \propto a^{-4}$ so the Newtonian **curvature decays**.

- General rule: potential decays if the dominant energy component has substantial stress fluctuations, i.e. below the generalized sound horizon or Jeans scale

Radiation Driving

- Decay is timed precisely to **drive** the oscillator - close to fully coherent

$$\begin{aligned} [\Theta + \Psi](\eta) &= [\Theta + \Psi](0) + \Delta\Psi - \Delta\Phi \\ &= \frac{1}{3}\Psi(0) - 2\Psi(0) = \frac{5}{3}\Psi(0) \end{aligned}$$

- $5\times$ the amplitude of the Sachs-Wolfe effect!
- Coherent approximation is **exact** for a photon-baryon fluid but reality is reduced to $\sim 4\times$ because of **neutrino contribution** to radiation
- Actual **initial conditions** are $\Theta + \Psi = \Psi/2$ for radiation domination but comparison to matter dominated SW correct

External Potential Approach

- Solution to homogeneous equation

$$(1 + R)^{-1/4} \cos(ks), \quad (1 + R)^{-1/4} \sin(ks)$$

- Give the general solution for an external potential by propagating impulsive forces

$$(1 + R)^{1/4} \Theta(\eta) = \Theta(0) \cos(ks) + \frac{\sqrt{3}}{k} \left[\dot{\Theta}(0) + \frac{1}{4} \dot{R}(0) \Theta(0) \right] \sin ks \\ + \frac{\sqrt{3}}{k} \int_0^\eta d\eta' (1 + R')^{3/4} \sin[ks - ks'] F(\eta')$$

where

$$F = -\ddot{\Phi} - \frac{\dot{R}}{1 + R} \dot{\Phi} - \frac{k^2}{3} \Psi$$

- Useful if general form of potential evolution is known

Damping

- Tight coupling equations assume a perfect fluid: no viscosity, no heat conduction
- Fluid imperfections are related to the mean free path of the photons in the baryons

$$\lambda_C = \dot{\tau}^{-1} \quad \text{where} \quad \dot{\tau} = n_e \sigma_T a$$

is the conformal opacity to Thompson scattering

- Dissipation is related to the diffusion length: random walk approximation

$$\lambda_D = \sqrt{N} \lambda_C = \sqrt{\eta / \lambda_C} \lambda_C = \sqrt{\eta \lambda_C}$$

the geometric mean between the horizon and mean free path

- $\lambda_D / \eta_* \sim \text{few } \%$, so expect the peaks > 3 to be affected by dissipation

Equations of Motion

- Continuity

$$\dot{\Theta} = -\frac{k}{3}v_\gamma - \dot{\Phi}, \quad \dot{\delta}_b = -kv_b - 3\dot{\Phi}$$

where the photon equation remains unchanged and the baryons follow number conservation with $\rho_b = m_b n_b$

- Euler

$$\begin{aligned}\dot{v}_\gamma &= k(\Theta + \Psi) - \frac{k}{6}\pi_\gamma - \dot{\tau}(v_\gamma - v_b) \\ \dot{v}_b &= -\frac{\dot{a}}{a}v_b + k\Psi + \dot{\tau}(v_\gamma - v_b)/R\end{aligned}$$

where the photons gain an anisotropic stress term π_γ from radiation viscosity and a momentum exchange term with the baryons and are compensated by the opposite term in the baryon Euler equation

Viscosity

- Viscosity is generated from radiation streaming from hot to cold regions
- Expect

$$\pi_\gamma \sim v_\gamma \frac{k}{\dot{\tau}}$$

generated by streaming, suppressed by scattering in a wavelength of the fluctuation. Radiative transfer says

$$\pi_\gamma \approx 2A_v v_\gamma \frac{k}{\dot{\tau}}$$

where $A_v = 16/15$

$$\dot{v}_\gamma = k(\Theta + \Psi) - \frac{k}{3} A_v \frac{k}{\dot{\tau}} v_\gamma$$

Oscillator: Penultimate Take

- Adiabatic approximation ($\omega \gg \dot{a}/a$)

$$\dot{\Theta} \approx -\frac{k}{3}v_\gamma$$

- Oscillator equation contains a $\dot{\Theta}$ damping term

$$c_s^2 \frac{d}{d\eta} (c_s^{-2} \dot{\Theta}) + \frac{k^2 c_s^2}{\dot{\tau}} A_v \dot{\Theta} + k^2 c_s^2 \Theta = -\frac{k^2}{3} \Psi - c_s^2 \frac{d}{d\eta} (c_s^{-2} \dot{\Phi})$$

- Heat conduction term similar in that it is proportional to v_γ and is suppressed by scattering $k/\dot{\tau}$. Expansion of Euler equations to leading order in $k/\dot{\tau}$ gives

$$A_h = \frac{R^2}{1 + R}$$

since the effects are only significant if the baryons are dynamically important

Oscillator: Final Take

- Final oscillator equation

$$c_s^2 \frac{d}{d\eta} (c_s^{-2} \dot{\Theta}) + \frac{k^2 c_s^2}{\dot{\tau}} [A_v + A_h] \dot{\Theta} + k^2 c_s^2 \Theta = -\frac{k^2}{3} \Psi - c_s^2 \frac{d}{d\eta} (c_s^{-2} \dot{\Phi})$$

- Solve in the adiabatic approximation

$$\Theta \propto \exp(i \int \omega d\eta)$$

$$-\omega^2 + \frac{k^2 c_s^2}{\dot{\tau}} (A_v + A_h) i\omega + k^2 c_s^2 = 0 \quad (1)$$

Dispersion Relation

- Solve

$$\begin{aligned}\omega^2 &= k^2 c_s^2 \left[1 + i \frac{\omega}{\dot{\tau}} (A_v + A_h) \right] \\ \omega &= \pm k c_s \left[1 + \frac{i}{2} \frac{\omega}{\dot{\tau}} (A_v + A_h) \right] \\ &= \pm k c_s \left[1 \pm \frac{i}{2} \frac{k c_s}{\dot{\tau}} (A_v + A_h) \right]\end{aligned}$$

- Exponentiate

$$\begin{aligned}\exp(i \int \omega d\eta) &= e^{\pm i k s} \exp\left[-k^2 \int d\eta \frac{1}{2} \frac{c_s^2}{\dot{\tau}} (A_v + A_h)\right] \\ &= e^{\pm i k s} \exp\left[-(k/k_D)^2\right]\end{aligned}\tag{2}$$

- Damping is **exponential** under the scale k_D

Diffusion Scale

- Diffusion wavenumber

$$k_D^{-2} = \int d\eta \frac{1}{\dot{\tau}} \frac{1}{6(1+R)} \left(\frac{16}{15} + \frac{R^2}{(1+R)} \right)$$

- Limiting forms

$$\lim_{R \rightarrow 0} k_D^{-2} = \frac{1}{6} \frac{16}{15} \int d\eta \frac{1}{\dot{\tau}}$$

$$\lim_{R \rightarrow \infty} k_D^{-2} = \frac{1}{6} \int d\eta \frac{1}{\dot{\tau}}$$

- Geometric mean between horizon and mean free path as expected from a **random walk**

$$\lambda_D = \frac{2\pi}{k_D} \sim \frac{2\pi}{\sqrt{6}} (\eta \dot{\tau}^{-1})^{1/2}$$

Astro 282

Idealized Data Analysis

Gaussian Statistics

- Statistical isotropy says two-point correlation depends only on the power spectrum

$$\Theta(\hat{\mathbf{n}}) = \sum_{\ell m} \Theta_{\ell m} Y_{\ell m}(\hat{\mathbf{n}})$$

$$\langle \Theta_{\ell m}^* \Theta_{\ell' m'} \rangle = \delta_{\ell \ell'} \delta_{m m'} C_{\ell}^{\Theta\Theta}$$

- Reality of field says $\Theta_{\ell m} = (-1)^m \Theta_{\ell(-m)}$
- For a Gaussian random field, power spectrum defines all higher order statistics, e.g.

$$\langle \Theta_{\ell_1 m_1} \Theta_{\ell_2 m_2} \Theta_{\ell_3 m_3} \Theta_{\ell_4 m_4} \rangle$$

$$= (-1)^{m_1+m_2} \delta_{\ell_1 \ell_3} \delta_{m_1(-m_3)} \delta_{\ell_2 \ell_4} \delta_{m_2(-m_4)} C_{\ell_1}^{\Theta\Theta} C_{\ell_2}^{\Theta\Theta} + \text{all pairs}$$

Idealized Statistical Errors

- Take a noisy estimator of the multipoles in the map

$$\hat{\Theta}_{\ell m} = \Theta_{\ell m} + N_{\ell m}$$

and take the noise to be statistically isotropic

$$\langle N_{\ell m}^* N_{\ell' m'} \rangle = \delta_{\ell\ell'} \delta_{mm'} C_{\ell}^{NN}$$

- Construct an unbiased estimator of the power spectrum

$$\langle \hat{C}_{\ell}^{\Theta\Theta} \rangle = C_{\ell}^{\Theta\Theta}$$

$$\hat{C}_{\ell}^{\Theta\Theta} = \frac{1}{2\ell + 1} \sum_{m=-\ell}^{\ell} \hat{\Theta}_{\ell m}^* \hat{\Theta}_{\ell m} - C_{\ell}^{NN}$$

- Variance in estimator

$$\langle \hat{C}_{\ell}^{\Theta\Theta} \hat{C}_{\ell}^{\Theta\Theta} \rangle - \langle \hat{C}_{\ell}^{\Theta\Theta} \rangle^2 = \frac{2}{2\ell + 1} (C_{\ell}^{\Theta\Theta} + C_{\ell}^{NN})^2$$

Cosmic and Noise Variance

- RMS in estimator is simply the total power spectrum reduced by $\sqrt{2/N_{\text{modes}}}$ where N_{modes} is the number of m -mode measurements
- Even a perfect experiment where $C_\ell^{NN} = 0$ has statistical variance due to the Gaussian random realizations of the field. This cosmic variance is the result of having only one realization to measure.
- Noise variance is often approximated as white detector noise.
Removing the beam to place the measurement on the sky

$$N_\ell^{\Theta\Theta} = \left(\frac{T}{d_T}\right)^2 e^{\ell(\ell+1)\sigma^2} = \left(\frac{T}{d_T}\right)^2 e^{\ell(\ell+1)\text{FWHM}^2/8\ln 2}$$

where d_T can be thought of as a noise level per steradian of the temperature measurement, σ is the Gaussian beam width, FWHM is the full width at half maximum of the beam

Idealized Parameter Forecasts

- A crude propagation of errors is often useful for estimation purposes.
- Suppose $C_{\alpha\beta}$ describes the covariance matrix of the estimators for a given parameter set π_α .
- Define $\mathbf{F} = \mathbf{C}^{-1}$ [formalized as the Fisher matrix later]. Making an infinitesimal transformation to a new set of parameters p_μ

$$F_{\mu\nu} = \sum_{\alpha\beta} \frac{\partial\pi_\alpha}{\partial p_\mu} F_{\alpha\beta} \frac{\partial\pi_\beta}{\partial p_\nu}$$

- In our case π_α are the C_ℓ the covariance is diagonal and p_μ are cosmological parameters

$$F_{\mu\nu} = \sum_{\ell} \frac{2\ell + 1}{2(C_\ell^{\Theta\Theta} + C_\ell^{NN})^2} \frac{\partial C_\ell^{\Theta\Theta}}{\partial p_\mu} \frac{\partial C_\ell^{\Theta\Theta}}{\partial p_\nu}$$

Idealized Parameter Forecasts

- Polarization handled in same way (requires covariance)
- Fisher matrix represents a local approximation to the transformation of the covariance and hence is only accurate for well constrained directions in parameter space
- Derivatives evaluated by finite difference
- Fisher matrix identifies parameter degeneracies but only the local direction – i.e. all errors are ellipses not bananas

Beyond Idealizations: Time Ordered Data

- For the data analyst the starting point is a string of “time ordered” data coming out of the instrument (post removal of systematic errors!)
- Begin with a model of the time ordered data as

$$d_t = P_{ti}\Theta_i + n_t$$

where i denotes pixelized positions indexed by i , d_t is the data in a time ordered stream indexed by t . Number of time ordered data will be of the order 10^{10} for a satellite! number of pixels $10^6 - 10^7$.

- The noise n_t is drawn from a distribution with a known power spectrum

$$\langle n_t n_{t'} \rangle = C_{d,tt'}$$

Pointing Matrix

- The pointing matrix \mathbf{P} is the mapping between pixel space and the time ordered data
- Simplest incarnation: row with all zeros except one column which just says what point in the sky the telescope is pointing at that time

$$\mathbf{P} = \begin{pmatrix} 0 & 0 & 1 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 1 & \dots & 0 \end{pmatrix}$$

- More generally incorporates differencing, beam, rotation (for polarization)

Maximum Likelihood Mapmaking

- What is the best estimator of the underlying map Θ_i
- Likelihood function: the probability of getting the data given the theory $\mathcal{L} \equiv P[\text{data}|\text{theory}]$. In this case, the *theory* is the set of parameters Θ_i .

$$\mathcal{L}_{\Theta}(d_t) = \frac{1}{(2\pi)^{N_t/2} \sqrt{\det \mathbf{C}_d}} \exp \left[-\frac{1}{2} (d_t - P_{ti}\Theta_i) C_{d,tt'}^{-1} (d_{t'} - P_{t'j}\Theta_j) \right].$$

- Bayes theorem says that $P[\Theta_i|d_t]$, the probability that the temperatures are equal to Θ_i given the data, is proportional to the likelihood function times a *prior* $P(\Theta_i)$, taken to be uniform

$$P[\Theta_i|d_t] \propto P[d_t|\Theta_i] \equiv \mathcal{L}_{\Theta}(d_t)$$

Maximum Likelihood Mapping

- Maximizing the likelihood of Θ_i is simple since the log-likelihood is quadratic.
- Differentiating the argument of the exponential with respect to Θ_i and setting to zero leads immediately to the estimator

$$\hat{\Theta}_i = C_{N,ij} P_{jt} C_{d,tt'}^{-1} d_{t'} ,$$

where $C_N \equiv (\mathbf{P}^{\text{tr}} \mathbf{C}_d^{-1} \mathbf{P})^{-1}$ is the covariance of the estimator

- Given the large dimension of the time ordered data, direct matrix manipulation is unfeasible. A key simplifying assumption is the stationarity of the noise, that $C_{d,tt'}$ depends only on $t - t'$ (temporal statistical homogeneity)

Foregrounds

- Maximum likelihood mapmaking can be applied to the time streams of multiple observations frequencies N_ν and hence obtain multiple maps
- A cleaned CMB map can be obtained by modeling the maps as

$$\hat{\Theta}_i^\nu = A_i^\nu \Theta_i + n_i^\nu + f_i^\nu$$

where $A_i^\nu = 1$ if all the maps are at the same resolution (otherwise, embed the beam as in the pointing matrix; f_i^ν is the noise contributed by the foregrounds

- Again, a map making problem. Given a covariance matrix for foregrounds noise (a prior from other data), same solution. Alternately, can derive weights from stats of the recovered maps
- 5 foregrounds: synchrotron, free-free, radio pt sources, at low frequencies and dust and IR pt sources at high frequencies.

Power Spectrum

- The next step in the chain of inference is the power spectrum extraction. Here the correlation between pixels is modelled through the power spectrum

$$C_{S,ij} \equiv \langle \Theta_i \Theta_j \rangle = \sum_{\ell} \Delta_{T,\ell}^2 W_{\ell,ij}$$

- Here W_{ℓ} , the window function, is derived by writing down the expansion of $\Theta(\hat{\mathbf{n}})$ in harmonic space, including smoothing by the beam and pixelization
- For example in the simple case of a gaussian beam of width σ it is proportional to the Legendre polynomial $P_{\ell}(\hat{\mathbf{n}}_i \cdot \hat{\mathbf{n}}_j)$ for the pixel separation multiplied by $b_{\ell}^2 \propto e^{-\ell(\ell+1)\sigma^2}$

Band Powers

- In principle the underlying theory to extract from maximum likelihood is the power spectrum at every ℓ
- However with a finite patch of sky, it is not possible to extract multipoles separated by $\Delta\ell < 2\pi/L$ where L is the dimension of the survey
- So consider instead a theory parameterization of $\Delta_{T,\ell}^2$ constant in bands of $\Delta\ell$ chosen to match the survey forming a set of band powers B_a
- The likelihood of the bandpowers given the pixelized data is

$$\mathcal{L}_B(\Theta_i) = \frac{1}{(2\pi)^{N_p/2} \sqrt{\det \mathbf{C}_\Theta}} \exp\left(-\frac{1}{2} \Theta_i C_{\Theta,ij}^{-1} \Theta_j\right)$$

where $\mathbf{C}_\Theta = \mathbf{C}_S + \mathbf{C}_N$ and N_p is the number of pixels in the map.

Band Power Estimation

- As before, \mathcal{L}_B is Gaussian in the anisotropies Θ_i , but in this case Θ_i are *not* the parameters to be determined; the theoretical parameters are the B_a , upon which the covariance matrix depends.
- The likelihood function is not Gaussian in the parameters, and there is no simple, analytic way to find the maximum likelihood bandpowers
- Iterative approach to maximizing the likelihood: take a trial point $B_a^{(0)}$ and improve estimate based a Newton-Rhapson approach to finding zeros

$$\begin{aligned}\hat{B}_a &= \hat{B}_a^{(0)} + \hat{F}_{B,ab}^{-1} \frac{\partial \ln \mathcal{L}_B}{\partial B_b} \\ &= \hat{B}_a^{(0)} + \frac{1}{2} \hat{F}_{B,ab}^{-1} \left(\Theta_i C_{\Theta,ij}^{-1} \frac{\partial C_{\Theta,jk}}{\partial B_b} C_{\Theta,kl}^{-1} \Theta_l - C_{\Theta,ij}^{-1} \frac{\partial C_{\Theta,ji}}{\partial B_b} \right),\end{aligned}$$

Fisher Matrix

- The expectation value of the local curvature is the Fisher matrix

$$\begin{aligned} F_{B,ab} &\equiv \left\langle -\frac{\partial^2 \ln \mathcal{L}_B}{\partial B_a \partial B_b} \right\rangle \\ &= \frac{1}{2} C_{\Theta,ij}^{-1} \frac{\partial C_{\Theta,jk}}{\partial B_a} C_{\Theta,kl}^{-1} \frac{\partial C_{\Theta,li}}{\partial B_b}. \end{aligned}$$

- This is a general statement: for a gaussian distribution the Fisher matrix

$$F_{ab} = \frac{1}{2} \text{Tr}[\mathbf{C}^{-1} \mathbf{C}_{,a} \mathbf{C}^{-1} \mathbf{C}_{,b}]$$

- Kramer-Rao identity says that the best possible covariance matrix on a set of parameters is $\mathbf{C} = \mathbf{F}^{-1}$
- Thus, the iteration returns an estimate of the covariance matrix of the estimators \mathbf{C}_B

Cosmological Parameters

- The probability distribution of the bandpowers given the cosmological parameters c_i is not Gaussian but it is often an adequate approximation

$$\mathcal{L}_c(\hat{B}_a) \approx \frac{1}{(2\pi)^{N_c/2} \sqrt{\det \mathbf{C}_B}} \exp \left[-\frac{1}{2} (\hat{B}_a - B_a) C_{B,ab}^{-1} (\hat{B}_b - B_b) \right]$$

- Grid based approaches evaluate the likelihood in cosmological parameter space and maximize
- Faster approaches monte carlo the exploration of the likelihood space intelligently (“Monte Carlo Markov Chains”)
- Since the number of cosmological parameters in the working model is $N_c \sim 10$ this represents a final radical compression of information in the original timestream which recall has up to $N_t \sim 10^{10}$ data points.

MCMC

- Monte Carlo Markov Chain: a random walk in parameter space
- Start with a set of cosmological parameters \mathbf{c}^m , compute likelihood
- Take a random step in parameter space to \mathbf{c}^{m+1} of size drawn from a multivariate Gaussian (a guess at the parameter covariance matrix) \mathbf{C}_c (e.g. from the crude Fisher approximation. Compute likelihood.
- Draw a random number between 0,1 and if the likelihood ratio exceeds this value take the step (add to Markov chain); if not then do not take the step (add the original point to the Markov chain). Goto 3.

MCMC

- With a complete chain of N_M elements, compute the mean of the chain and its variance

$$\bar{c}_i = \frac{1}{N_M} \sum_{m=1}^{N_M} c_i^m$$

$$\sigma^2(c_i) = \frac{1}{N_M - 1} \sum_{m=1}^{N_M} (c_i^m - \bar{c}_i)^2$$

- Trick is in assuring burn in (not sensitive to initial point), step size, and convergence
- Usually requires running multiple chains. Typically tens of thousands of elements per chain.

Radical Compression

- Started with time ordered data $\sim 10^{10}$ numbers for a satellite experiment
- Compressed to a map assuming a CMB spectrum (and time independent fluctuations) $\sim 10^7$ numbers
- Compressed to a power spectrum (Gaussian statistics) independent of m (statistical isotropy) $\sim 10^3$ numbers
- Compressed to cosmological parameters (a cosmological model) $\sim 10^3$
- A factor of 10^9 reduction in the representation. Nature is very efficient.

Parameter Forecasts

- The Fisher matrix of the cosmological parameters becomes

$$F_{c,ij} = \frac{\partial B_a}{\partial c_i} C_{B,ab}^{-1} \frac{\partial B_b}{\partial c_j} .$$

which is the error propagation formula discussed above

- The Fisher matrix can be more accurately defined for an experiment by taking the pixel covariance and using the general formula for the Fisher matrix of gaussian data
- Corrects for edge effects with the approximate effect of

$$F_{ij} = \sum_{\ell} \frac{(2\ell + 1) f_{\text{sky}}}{2(C_{\ell}^{\Theta\Theta} + C_{\ell}^{NN})^2} \frac{\partial C_{\ell}^{\Theta\Theta}}{\partial c_i} \frac{\partial C_{\ell}^{\Theta\Theta}}{\partial c_j}$$

where the sky fraction f_{sky} quantifies the loss of independent modes due to the sky cut

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Polarization

Stokes Parameters

- Polarization state of radiation in direction $\hat{\mathbf{n}}$ described by the intensity matrix $\langle E_i(\hat{\mathbf{n}})E_j^*(\hat{\mathbf{n}}) \rangle$, where \mathbf{E} is the electric field vector and the brackets denote time averaging.
- As a hermitian matrix, it can be decomposed into the Pauli basis

$$\begin{aligned}\mathbf{P} &= C \langle \mathbf{E}(\hat{\mathbf{n}}) \mathbf{E}^\dagger(\hat{\mathbf{n}}) \rangle \\ &= \Theta(\hat{\mathbf{n}})\boldsymbol{\sigma}_0 + Q(\hat{\mathbf{n}})\boldsymbol{\sigma}_3 + U(\hat{\mathbf{n}})\boldsymbol{\sigma}_1 + V(\hat{\mathbf{n}})\boldsymbol{\sigma}_2,\end{aligned}$$

where

$$\boldsymbol{\sigma}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \boldsymbol{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \boldsymbol{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \boldsymbol{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- Stokes parameters recovered as $\text{Tr}(\boldsymbol{\sigma}_i \mathbf{P})/2$

Monochromatic Wave

- A pure monochromatic wave is fully polarized

$$\mathbf{E} = E_1 \mathbf{e}_1 + E_2 \mathbf{e}_2$$

where

$$E_{1,2} = \text{Re}[A_{1,2} e^{i\phi_{1,2}} e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)}]$$

- Implies $\Theta^2 = Q^2 + U^2 + V^2$
- However a finite bandwidth leads to a sum of components

$$\mathbf{E} = \sum_{\alpha} \mathbf{E}^{\alpha}$$

Partial Polarization

- A signal of finite bandwidth is only partially polarized since the time averaging will destroy the correlation between the frequency components

$$\langle \mathbf{E} \mathbf{E}^\dagger \rangle = \sum_{\alpha} \langle \mathbf{E}^{\alpha} \mathbf{E}^{\alpha\dagger} \rangle$$

- Stokes parameters then add

$$\Theta = \sum_{\alpha} \Theta^{\alpha}, \quad Q = \sum_{\alpha} Q^{\alpha}, \quad U = \sum_{\alpha} U^{\alpha}, \quad V = \sum_{\alpha} V^{\alpha}$$

- Result is $\Theta^2 > Q^2 + U^2 + V^2$ (since Q, U, V have either sign) or partially polarized radiation - like a mixed state in quantum mechanics)

Linear Polarization

- $Q \propto \langle E_1 E_1^* \rangle - \langle E_2 E_2^* \rangle$, $U \propto \langle E_1 E_2^* \rangle + \langle E_2 E_1^* \rangle$.
- Counterclockwise rotation of axes by $\theta = 45^\circ$

$$E_1 = (E'_1 - E'_2)/\sqrt{2}, \quad E_2 = (E'_1 + E'_2)/\sqrt{2}$$

- $U \propto \langle E'_1 E'_1^* \rangle - \langle E'_2 E'_2^* \rangle$, difference of intensities at 45° or Q'
- More generally, \mathbf{P} transforms as a tensor under rotations and

$$Q' = \cos(2\theta)Q + \sin(2\theta)U$$

$$U' = -\sin(2\theta)Q + \cos(2\theta)U$$

- or

$$Q' \pm iU' = e^{\mp 2i\theta}[Q \pm iU]$$

acquires a phase under rotation and is a spin ± 2 object

Coordinate Independent Representation

- Two directions: orientation of polarization and change in amplitude, i.e. Q and U in the basis of the Fourier wavevector (pointing with angle ϕ_l) for small sections of sky are called E and B components

$$\begin{aligned} E(\mathbf{l}) \pm iB(\mathbf{l}) &= - \int d\hat{\mathbf{n}} [Q'(\hat{\mathbf{n}}) \pm iU'(\hat{\mathbf{n}})] e^{-i\mathbf{l}\cdot\hat{\mathbf{n}}} \\ &= -e^{\mp 2i\phi_l} \int d\hat{\mathbf{n}} [Q(\hat{\mathbf{n}}) \pm iU(\hat{\mathbf{n}})] e^{-i\mathbf{l}\cdot\hat{\mathbf{n}}} \end{aligned}$$

- For the B -mode to not vanish, the polarization must point in a direction not related to the wavevector - not possible for density fluctuations in linear theory
- Generalize to all-sky: plane waves are eigenmodes of the Laplace operator on the tensor \mathbf{P} .

Spin Harmonics

- Laplace Eigenfunctions

$$\nabla^2_{\pm 2} Y_{\ell m}[\boldsymbol{\sigma}_3 \mp i\boldsymbol{\sigma}_1] = -[l(l+1) - 4]_{\pm 2} Y_{\ell m}[\boldsymbol{\sigma}_3 \mp i\boldsymbol{\sigma}_1]$$

- Spin s spherical harmonics: orthogonal and complete

$$\int d\hat{\mathbf{n}} {}_s Y_{\ell m}^*(\hat{\mathbf{n}}) {}_s Y_{\ell' m'}(\hat{\mathbf{n}}) = \delta_{\ell\ell'} \delta_{mm'}$$
$$\sum_{\ell m} {}_s Y_{\ell m}^*(\hat{\mathbf{n}}) {}_s Y_{\ell m}(\hat{\mathbf{n}}') = \delta(\phi - \phi') \delta(\cos\theta - \cos\theta')$$

where the ordinary spherical harmonics are $Y_{\ell m} = {}_0 Y_{\ell m}$

- Given in terms of the rotation matrix

$${}_s Y_{\ell m}(\beta\alpha) = (-1)^m \sqrt{\frac{2\ell+1}{4\pi}} D_{-ms}^{\ell}(\alpha\beta 0)$$

Statistical Representation

- All-sky decomposition

$$[Q(\hat{\mathbf{n}}) \pm iU(\hat{\mathbf{n}})] = \sum_{\ell m} [E_{\ell m} \pm iB_{\ell m}]_{\pm 2} Y_{\ell m}(\hat{\mathbf{n}})$$

- Power spectra

$$\langle E_{\ell m}^* E_{\ell m} \rangle = \delta_{\ell\ell'} \delta_{mm'} C_{\ell}^{EE}$$

$$\langle B_{\ell m}^* B_{\ell m} \rangle = \delta_{\ell\ell'} \delta_{mm'} C_{\ell}^{BB}$$

- Cross correlation

$$\langle E_{\ell m}^* E_{\ell m} \rangle = \delta_{\ell\ell'} \delta_{mm'} C_{\ell}^{\ominus E}$$

others vanish if parity is conserved

Thomson Scattering

- Differential cross section

$$\frac{d\sigma}{d\Omega} = \frac{3}{8\pi} |\hat{\mathbf{E}}' \cdot \hat{\mathbf{E}}|^2 \sigma_T ,$$

where $\sigma_T = 8\pi\alpha^2/3m_e$ is the Thomson cross section, $\hat{\mathbf{E}}'$ and $\hat{\mathbf{E}}$ denote the incoming and outgoing directions of the electric field or polarization vector.

- Summed over angle and incoming polarization

$$\sum_{i=1,2} \int d\hat{\mathbf{n}}' \frac{d\sigma}{d\Omega} = \sigma_T$$

Polarization Generation

- Heuristic: incoming radiation shakes an electron in direction of electric field vector $\hat{\mathbf{E}}'$
- Radiates photon with polarization also in direction $\hat{\mathbf{E}}'$
- But photon cannot be longitudinally polarized so that scattering into 90° can only pass one polarization
- Linearly polarized radiation like polarization by reflection
- Unlike reflection of sunlight, incoming radiation is nearly isotropic
- Missing linear polarization supplied by scattering from direction orthogonal to original incoming direction
- Only quadrupole anisotropy generates polarization by Thomson scattering

Acoustic Polarization

- Break down of tight-coupling leads to quadrupole anisotropy of

$$\pi_\gamma \approx \frac{k}{\dot{\tau}} v_\gamma$$

- Scaling $k_D = (\dot{\tau}/\eta_*)^{1/2} \rightarrow \dot{\tau} = k_D^2 \eta_*$
- Know: $k_D s_* \approx k_D \eta_* \approx 10$
- So:

$$\pi_\gamma \approx \frac{k}{k_D} \frac{1}{10} v_\gamma$$

$$\Delta_P \approx \frac{\ell}{\ell_D} \frac{1}{10} \Delta_T$$

Acoustic Polarization

- Gradient of velocity is along direction of wavevector, so polarization is pure E -mode
- Velocity is 90° out of phase with temperature – turning points of oscillator are zero points of velocity:

$$\Theta + \Psi \propto \cos(ks); \quad v_\gamma \propto \sin(ks)$$

- Polarization peaks are at troughs of temperature power

Cross Correlation

- Cross correlation of temperature and polarization

$$(\Theta + \Psi)(v_\gamma) \propto \cos(ks) \sin(ks) \propto \sin(2ks)$$

- Oscillation at twice the frequency
- Correlation: radial or tangential around hot spots
- Partial correlation: easier to measure if polarization data is noisy, harder to measure if polarization data is high S/N or if bands do not resolve oscillations
- Good check for systematics and foregrounds
- Comparison of temperature and polarization is proof against features in initial conditions mimicking acoustic features

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Formal CMB Theory & CMBFAST

Boltzmann Equation

- CMB radiation is generally described by the phase space distribution function for each polarization state $f_a(\mathbf{x}, \mathbf{q}, \eta)$, where \mathbf{x} is the comoving position and \mathbf{q} is the photon momentum
- Boltzmann equation describes the evolution of the distribution function under gravity and collisions
- Low order moments of the Boltzmann equation are simply the covariant conservation equations
- Higher moments provide the closure condition to the conservation law (specification of stress tensor) and the CMB observable – fine scale anisotropy
- Higher moments mainly describe the simple geometry of source projection

Liouville Equation

- In absence of scattering, the phase space distribution of photons is conserved along the propagation path
- Rewrite variables in terms of the photon propagation direction $\mathbf{q} = q\hat{\mathbf{n}}$, so $f_a(\mathbf{x}, \hat{\mathbf{n}}, q, \eta)$ and

$$\begin{aligned} \frac{d}{d\eta} f_a(\mathbf{x}, \hat{\mathbf{n}}, q, \eta) &= 0 \\ &= \left(\frac{\partial}{\partial \eta} + \frac{d\mathbf{x}}{d\eta} \cdot \frac{\partial}{\partial \mathbf{x}} + \frac{d\hat{\mathbf{n}}}{d\eta} \cdot \frac{\partial}{\partial \hat{\mathbf{n}}} + \frac{dq}{d\eta} \cdot \frac{\partial}{\partial q} \right) f_a \end{aligned}$$

- For simplicity, assume spatially flat universe $K = 0$ then $d\hat{\mathbf{n}}/d\eta = 0$ and $d\mathbf{x} = \hat{\mathbf{n}}d\eta$

$$\dot{f}_a + \hat{\mathbf{n}} \cdot \nabla f_a + \dot{q} \frac{\partial}{\partial q} f_a = 0$$

Correspondence to Einstein Eqn.

- Geodesic equation gives the redshifting term

$$\frac{\dot{q}}{q} = -\frac{\dot{a}}{a} - \frac{1}{2}n^i n^j \dot{H}_{Tij} - \dot{H}_L + n^i \dot{B}_i - \hat{\mathbf{n}} \cdot \nabla A$$

- which is incorporated in the conservation and gauge transformation equations
- Stress energy tensor involves integrals over the distribution function the two polarization states

$$T^{\mu\nu} = \int \frac{d^3q}{(2\pi)^3} \frac{q^\mu q^\nu}{E} (f_a + f_b)$$

- Components are simply the low order angular moments of the distribution function

Angular Moments

- Define the angularly dependent temperature perturbation

$$\Theta(\mathbf{x}, \hat{\mathbf{n}}, \eta) = \frac{1}{4\rho_\gamma} \int \frac{q^3 dq}{2\pi^2} (f_a + f_b) - 1$$

and likewise for the linear polarization states Q and U

- Decompose into normal modes: plane waves for spatial part and spherical harmonics for angular part

$$G_\ell^m(\mathbf{k}, \mathbf{x}, \hat{\mathbf{n}}) \equiv (-i)^\ell \sqrt{\frac{4\pi}{2\ell + 1}} Y_\ell^m(\hat{\mathbf{n}}) \exp(i\mathbf{k} \cdot \mathbf{x})$$

$$\pm_2 G_\ell^m(\mathbf{k}, \mathbf{x}, \hat{\mathbf{n}}) \equiv (-i)^\ell \sqrt{\frac{4\pi}{2\ell + 1}} \pm_2 Y_\ell^m(\hat{\mathbf{n}}) \exp(i\mathbf{k} \cdot \mathbf{x})$$

- In a spatially curved universe generalize the plane wave part

Normal Modes

- Temperature and polarization fields

$$\Theta(\mathbf{x}, \hat{\mathbf{n}}, \eta) = \int \frac{d^3k}{(2\pi)^3} \sum_{\ell m} \Theta_{\ell}^{(m)} G_{\ell}^m$$

$$[Q \pm iU](\mathbf{x}, \hat{\mathbf{n}}, \eta) = \int \frac{d^3k}{(2\pi)^3} \sum_{\ell m} [E_{\ell}^{(m)} \pm iB_{\ell}^{(m)}]_{\pm 2} G_{\ell}^m$$

- For each \mathbf{k} mode, work in coordinates where $\mathbf{k} \parallel \mathbf{z}$ and so $m = 0$ represents scalar modes, $m = \pm 1$ vector modes, $m = \pm 2$ tensor modes, $|m| > 2$ vanishes. Since modes add incoherently and $Q \pm iU$ is invariant up to a phase, rotation back to a fixed coordinate system is trivial.

Scalar, Vector, Tensor

- Normalization of modes is chosen so that the lowest angular mode for scalars, vectors and tensors are normalized in the same way as the mode function

$$\begin{aligned}G_0^0 &= Q^{(0)} & G_1^0 &= n^i Q_i^{(0)} & G_2^0 &\propto n^i n^j Q_{ij}^{(0)} \\G_1^{\pm 1} &= n^i Q_i^{(\pm 1)} & G_2^{\pm 1} &\propto n^i n^j Q_{ij}^{(\pm 1)} \\G_2^{\pm 2} &= n^i n^j Q_{ij}^{(\pm 2)}\end{aligned}$$

where recall

$$\begin{aligned}Q^{(0)} &= \exp(i\mathbf{k} \cdot \mathbf{x}) \\Q_i^{(\pm 1)} &= \frac{-i}{\sqrt{2}}(\hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2)_i \exp(i\mathbf{k} \cdot \mathbf{x}) \\Q_{ij}^{(\pm 2)} &= -\sqrt{\frac{3}{8}}(\hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2)_i (\hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2)_j \exp(i\mathbf{k} \cdot \mathbf{x})\end{aligned}$$

Geometrical Projection

- Main content of Liouville equation is purely geometrical and describes the projection of inhomogeneities into anisotropies
- Spatial gradient term hits plane wave:

$$\hat{\mathbf{n}} \cdot \nabla e^{i\mathbf{k} \cdot \mathbf{x}} = i\hat{\mathbf{n}} \cdot \mathbf{k} e^{i\mathbf{k} \cdot \mathbf{x}} = i\sqrt{\frac{4\pi}{3}} k Y_1^0(\hat{\mathbf{n}}) e^{i\mathbf{k} \cdot \mathbf{x}}$$

- Dipole term adds to angular dependence through the addition of angular momentum

$$\sqrt{\frac{4\pi}{3}} Y_1^0 Y_\ell^m = \frac{\kappa_\ell^m}{\sqrt{(2\ell+1)(2\ell-1)}} Y_{\ell-1}^m + \frac{\kappa_{\ell+1}^m}{\sqrt{(2\ell+1)(2\ell+3)}} Y_{\ell+1}^m$$

where $\kappa_\ell^m = \sqrt{\ell^2 - m^2}$ is given by Clebsch-Gordon coefficients.

Temperature Hierarchy

- Absorb recoupling of angular momentum into evolution equation for normal modes

$$\dot{\Theta}_\ell^{(m)} = k \left[\frac{\kappa_\ell^m}{2\ell + 1} \Theta_{\ell-1}^{(m)} - \frac{\kappa_{\ell+1}^m}{2\ell + 3} \Theta_{\ell+1}^{(m)} \right] - \dot{\tau} \Theta_\ell^{(m)} + S_\ell^{(m)}$$

where $S_\ell^{(m)}$ are the gravitational (and later scattering sources; added scattering suppression of anisotropy)

- An originally isotropic $\ell = 0$ temperature perturbation will eventually become a high order anisotropy by “free streaming” or simple projection
- Original CMB codes solved the full hierarchy equations out to the ℓ of interest.

Integral Solution

- Hierarchy equation simply represents geometric projection, exactly as we have seen before in the projection of temperature perturbations on the last scattering surface
- In general, the solution describes the decomposition of the source $S_\ell^{(m)}$ with its local angular dependence as seen at a distance $\mathbf{x} = D\hat{\mathbf{n}}$.
- Proceed by decomposing the angular dependence of the plane wave

$$e^{i\mathbf{k}\cdot\mathbf{x}} = \sum_{\ell} (-i)^{\ell} \sqrt{4\pi(2\ell + 1)} j_{\ell}(kD) Y_{\ell}^0(\hat{\mathbf{n}})$$

- Recouple to the local angular dependence of G_{ℓ}^m

$$G_{\ell_s}^m = \sum_{\ell} (-i)^{\ell} \sqrt{4\pi(2\ell + 1)} \alpha_{\ell_s \ell}^{(m)}(kD) Y_{\ell}^m(\hat{\mathbf{n}})$$

Integral Solution

- Projection kernels:

$$\ell_s = 0, \quad m = 0 \qquad \alpha_{0\ell}^{(0)} \equiv j_\ell$$

$$\ell_s = 1, \quad m = 0 \qquad \alpha_{1\ell}^{(0)} \equiv j'_\ell$$

- Integral solution:

$$\frac{\Theta_\ell^{(m)}(k, \eta_0)}{2\ell + 1} = \int_0^{\eta_0} d\eta e^{-\tau} \sum_{\ell_s} S_{\ell_s}^{(m)} \alpha_{\ell_s \ell}^{(m)}(k(\eta_0 - \eta))$$

- Power spectrum:

$$C_\ell = \frac{2}{\pi} \int \frac{dk}{k} \sum_m \frac{k^3 \langle \Theta_\ell^{(m)*} \Theta_\ell^{(m)} \rangle}{(2\ell + 1)^2}$$

- Solving for C_ℓ reduces to solving for the behavior of a handful of sources

Polarization Hierarchy

- In the same way, the coupling of a gradient or dipole angular momentum to the spin harmonics leads to the polarization hierarchy:

$$\begin{aligned}\dot{E}_\ell^{(m)} &= k \left[\frac{{}_2\kappa_\ell^m}{2\ell - 1} E_{\ell-1}^{(m)} - \frac{2m}{\ell(\ell + 1)} B_\ell^{(m)} - \frac{{}_2\kappa_{\ell+1}^m}{2\ell + 3} \right] - \dot{\tau} E_\ell^{(m)} + \mathcal{E}_\ell^{(m)} \\ \dot{B}_\ell^{(m)} &= k \left[\frac{{}_2\kappa_\ell^m}{2\ell - 1} B_{\ell-1}^{(m)} + \frac{2m}{\ell(\ell + 1)} B_\ell^{(m)} - \frac{{}_2\kappa_{\ell+1}^m}{2\ell + 3} \right] - \dot{\tau} E_\ell^{(m)} + \mathcal{B}_\ell^{(m)}\end{aligned}$$

where ${}_2\kappa_\ell^m = \sqrt{(\ell^2 - m^2)(\ell^2 - 4)}/\ell^2$ is given by the Clebsch-Gordon coefficients and \mathcal{E} , \mathcal{B} are the sources (scattering only).

- Note that for vectors and tensors $|m| > 0$ and B modes may be generated from E modes by projection. Cosmologically $\mathcal{B}_\ell^{(m)} = 0$

Polarization Integral Solution

- Again, we can recouple the plane wave angular momentum of the source inhomogeneity to its local angular dependence directly

$$\frac{E_\ell^{(m)}(k, \eta_0)}{2\ell + 1} = \int_0^{\eta_0} d\eta e^{-\tau} \mathcal{E}_{\ell_s}^{(m)} \epsilon_{\ell_s \ell}^{(m)}(k(\eta_0 - \eta))$$

$$\frac{B_\ell^{(m)}(k, \eta_0)}{2\ell + 1} = \int_0^{\eta_0} d\eta e^{-\tau} \mathcal{E}_{\ell_s}^{(m)} \beta_{\ell_s \ell}^{(m)}(k(\eta_0 - \eta))$$

- The only source to the polarization is from the quadrupole anisotropy so we only need $\ell_s = 2$, e.g. for scalars

$$\epsilon_{2\ell}^{(0)}(x) = \sqrt{\frac{3(\ell + 2)!}{8(\ell - 2)!}} \frac{j_\ell(x)}{x^2} \quad \beta_{2\ell}^{(0)} = 0$$

Truncated Hierarchy

- CMBFast uses the integral solution and relies on a fast j_ℓ generator
- However sources are not external to system and are defined through the Boltzmann hierarchy itself
- Solution: recall that we used this technique in the tight coupling regime by applying a closure condition from tight coupling
- CMBFast extends this idea by solving a truncated hierarchy of equations, e.g. out to $\ell = 25$ with non-reflecting boundary conditions

Thomson Collision Term

- Full Boltzmann equation

$$\frac{d}{d\eta} f_{a,b} = C[f_a, f_b]$$

- Collision term describes the scattering out of and into a phase space element
- Thomson collision based on differential cross section

$$\frac{d\sigma}{d\Omega} = \frac{3}{8\pi} |\hat{\mathbf{E}}' \cdot \hat{\mathbf{E}}|^2 \sigma_T ,$$

where $\hat{\mathbf{E}}'$ and $\hat{\mathbf{E}}$ denote the incoming and outgoing directions of the electric field or polarization vector.

Scattering Calculation

- Start in the electron rest frame and in a coordinate system fixed by the scattering plane, spanned by incoming and outgoing directional vectors $-\hat{\mathbf{n}}' \cdot \hat{\mathbf{n}} = \cos \beta$, where β is the scattering angle
- Θ_{\parallel} : in-plane polarization state; Θ_{\perp} : \perp -plane polarization state
- Transfer probability (constant set by $\dot{\tau}$)

$$\Theta_{\parallel} \propto \cos^2 \beta \Theta'_{\parallel}, \quad \Theta_{\perp} \propto \Theta'_{\perp}$$

- and with the 45° axes as

$$\hat{\mathbf{E}}_1 = \frac{1}{\sqrt{2}}(\hat{\mathbf{E}}_{\parallel} + \hat{\mathbf{E}}_{\perp}), \quad \hat{\mathbf{E}}_2 = \frac{1}{\sqrt{2}}(\hat{\mathbf{E}}_{\parallel} - \hat{\mathbf{E}}_{\perp})$$

Stokes Parameters

- Define the temperature in this basis

$$\begin{aligned}\Theta_1 &\propto |\hat{\mathbf{E}}_1 \cdot \hat{\mathbf{E}}_1|^2 \Theta'_1 + |\hat{\mathbf{E}}_1 \cdot \hat{\mathbf{E}}_2|^2 \Theta'_2 \\ &\propto \frac{1}{4}(\cos \beta + 1)^2 \Theta'_1 + \frac{1}{4}(\cos \beta - 1)^2 \Theta'_2\end{aligned}$$

$$\begin{aligned}\Theta_2 &\propto |\hat{\mathbf{E}}_2 \cdot \hat{\mathbf{E}}_2|^2 \Theta'_2 + |\hat{\mathbf{E}}_2 \cdot \hat{\mathbf{E}}_1|^2 \Theta'_1 \\ &\propto \frac{1}{4}(\cos \beta + 1)^2 \Theta'_2 + \frac{1}{4}(\cos \beta - 1)^2 \Theta'_1\end{aligned}$$

or $\Theta_1 - \Theta_2 \propto \cos \beta (\Theta'_1 - \Theta'_2)$

- Define Θ , Q , U in the scattering coordinates

$$\Theta \equiv \frac{1}{2}(\Theta_{\parallel} + \Theta_{\perp}), \quad Q \equiv \frac{1}{2}(\Theta_{\parallel} - \Theta_{\perp}), \quad U \equiv \frac{1}{2}(\Theta_1 - \Theta_2)$$

Scattering Matrix

- Transfer of Stokes states, e.g.

$$\Theta = \frac{1}{2}(\Theta_{\parallel} + \Theta_{\perp}) \propto \frac{1}{4}(\cos^2 \beta + 1)\Theta' + \frac{1}{4}(\cos^2 \beta - 1)Q'$$

- Transfer matrix of Stokes state $\mathbf{T} \equiv (\Theta, Q + iU, Q - iU)$

$$\mathbf{T} \propto \mathbf{S}(\beta)\mathbf{T}'$$

$$\mathbf{S}(\beta) = \frac{3}{4} \begin{pmatrix} \cos^2 \beta + 1 & -\frac{1}{2} \sin^2 \beta & -\frac{1}{2} \sin^2 \beta \\ -\frac{1}{2} \sin^2 \beta & \frac{1}{2}(\cos \beta + 1)^2 & \frac{1}{2}(\cos \beta - 1)^2 \\ -\frac{1}{2} \sin^2 \beta & \frac{1}{2}(\cos \beta - 1)^2 & \frac{1}{2}(\cos \beta + 1)^2 \end{pmatrix}$$

normalization factor of 3 is set by photon conservation in scattering

Scattering Matrix

- Transform to a fixed basis, by a rotation of the incoming and outgoing states $\mathbf{T} = \mathbf{R}(\psi)\mathbf{T}$ where

$$\mathbf{R}(\psi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-2i\psi} & 0 \\ 0 & 0 & e^{2i\psi} \end{pmatrix}$$

giving the scattering matrix

$$\mathbf{R}(-\gamma)\mathbf{S}(\beta)\mathbf{R}(\alpha) =$$

$$\frac{1}{2}\sqrt{\frac{4\pi}{5}} \begin{pmatrix} Y_2^0(\beta, \alpha) + 2\sqrt{5}Y_0^0(\beta, \alpha) & -\sqrt{\frac{3}{2}}Y_2^{-2}(\beta, \alpha) & -\sqrt{\frac{3}{2}}Y_2^2(\beta, \alpha) \\ -\sqrt{6}{}_2Y_2^0(\beta, \alpha)e^{2i\gamma} & 3{}_2Y_2^{-2}(\beta, \alpha)e^{2i\gamma} & 3{}_2Y_2^2(\beta, \alpha)e^{2i\gamma} \\ -\sqrt{6}{}_{-2}Y_2^0(\beta, \alpha)e^{-2i\gamma} & 3{}_{-2}Y_2^{-2}(\beta, \alpha)e^{-2i\gamma} & 3{}_{-2}Y_2^2(\beta, \alpha)e^{-2i\gamma} \end{pmatrix}$$

Addition Theorem for Spin Harmonics

- Spin harmonics are related to rotation matrices as

$${}_s Y_\ell^m(\theta, \phi) = \sqrt{\frac{2\ell + 1}{4\pi}} \mathcal{D}_{-ms}^\ell(\phi, \theta, 0)$$

Note: for explicit evaluation sign convention differs from usual (e.g. Jackson) by $(-1)^m$

- Multiplication of rotations

$$\sum_{m''} \mathcal{D}_{mm''}^\ell(\alpha_2, \beta_2, \gamma_2) \mathcal{D}_{m''m}^\ell(\alpha_1, \beta_1, \gamma_1) = \mathcal{D}_{mm'}^\ell(\alpha, \beta, \gamma)$$

- Implies

$$\sum_m {}_{s_1} Y_\ell^{m*}(\theta', \phi') {}_{s_2} Y_\ell^m(\theta, \phi) = (-1)^{s_1 - s_2} \sqrt{\frac{2\ell + 1}{4\pi}} {}_{s_2} Y_\ell^{-s_1}(\beta, \alpha) e^{is_2\gamma}$$

Sky Basis

- Scattering into the state (rest frame)

$$\begin{aligned}
 C_{\text{in}}[\mathbf{T}] &= \dot{\tau} \int \frac{d\hat{\mathbf{n}}'}{4\pi} \mathbf{R}(-\gamma) \mathbf{S}(\beta) \mathbf{R}(\alpha) \mathbf{T}(\hat{\mathbf{n}}'), \\
 &= \dot{\tau} \int \frac{d\hat{\mathbf{n}}'}{4\pi} (\Theta', 0, 0) + \frac{1}{10} \dot{\tau} \int d\hat{\mathbf{n}}' \sum_{m=-2}^2 \mathbf{P}^{(m)}(\hat{\mathbf{n}}, \hat{\mathbf{n}}') \mathbf{T}(\hat{\mathbf{n}}').
 \end{aligned}$$

where the quadrupole coupling term is $\mathbf{P}^{(m)}(\hat{\mathbf{n}}, \hat{\mathbf{n}}') =$

$$\begin{pmatrix}
 Y_2^{m*}(\hat{\mathbf{n}}') Y_2^m(\hat{\mathbf{n}}) & -\sqrt{\frac{3}{2}} {}_2Y_2^{m*}(\hat{\mathbf{n}}') Y_2^m(\hat{\mathbf{n}}) & -\sqrt{\frac{3}{2}} {}_{-2}Y_2^{m*}(\hat{\mathbf{n}}') Y_2^m(\hat{\mathbf{n}}) \\
 -\sqrt{6} Y_2^{m*}(\hat{\mathbf{n}}') {}_2Y_2^m(\hat{\mathbf{n}}) & 3 {}_2Y_2^{m*}(\hat{\mathbf{n}}') {}_2Y_2^m(\hat{\mathbf{n}}) & 3 {}_{-2}Y_2^{m*}(\hat{\mathbf{n}}') {}_2Y_2^m(\hat{\mathbf{n}}) \\
 -\sqrt{6} Y_2^{m*}(\hat{\mathbf{n}}') {}_{-2}Y_2^m(\hat{\mathbf{n}}) & 3 {}_2Y_2^{m*}(\hat{\mathbf{n}}') {}_{-2}Y_2^m(\hat{\mathbf{n}}) & 3 {}_{-2}Y_2^{m*}(\hat{\mathbf{n}}') {}_{-2}Y_2^m(\hat{\mathbf{n}})
 \end{pmatrix},$$

expression uses angle addition relation above. We call this term C_Q .

Scattering Matrix

- Full scattering matrix involves difference of scattering into and out of state

$$C[\mathbf{T}] = C_{\text{in}}[\mathbf{T}] - C_{\text{out}}[\mathbf{T}]$$

- In the electron rest frame

$$C[\mathbf{T}] = \dot{\tau} \int \frac{d\hat{\mathbf{n}}'}{4\pi} (\Theta', 0, 0) - \dot{\tau} \mathbf{T} + C_Q[\mathbf{T}]$$

which describes isotropization in the rest frame. All moments have $e^{-\tau}$ suppression except for isotropic temperature Θ_0 .

Transformation into the background frame simply induces a dipole term

$$C[\mathbf{T}] = \dot{\tau} \left(\hat{\mathbf{n}} \cdot \mathbf{v}_b + \int \frac{d\hat{\mathbf{n}}'}{4\pi} \Theta', 0, 0 \right) - \dot{\tau} \mathbf{T} + C_Q[\mathbf{T}]$$

Source Terms

- Temperature source terms $S_l^{(m)}$ (rows $\pm|m|$; flat assumption)

$$\begin{pmatrix} \dot{\tau}\Theta_0^{(0)} - \dot{H}_L^{(0)} & \dot{\tau}v_b^{(0)} + \dot{B}^{(0)} & \dot{\tau}P^{(0)} - \frac{2}{3}\dot{H}_T^{(0)} \\ 0 & \dot{\tau}v_b^{(\pm 1)} + \dot{B}^{(\pm 1)} & \dot{\tau}P^{(\pm 1)} - \frac{\sqrt{3}}{3}\dot{H}_T^{(\pm 1)} \\ 0 & 0 & \dot{\tau}P^{(\pm 2)} - \dot{H}_T^{(\pm 2)} \end{pmatrix}$$

where

$$P^{(m)} \equiv \frac{1}{10}(\Theta_2^{(m)} - \sqrt{6}E_2^{(m)})$$

- Polarization source term

$$\mathcal{E}_\ell^{(m)} = -\dot{\tau}\sqrt{6}P^{(m)}\delta_{\ell,2}$$

$$\mathcal{B}_\ell^{(m)} = 0$$

Astro 282
Secondaries

Reionization

- Ionization depth during reionization

$$\begin{aligned}\tau(z) &= \int d\eta n_e \sigma_T a = \int d \ln a \frac{n_e \sigma_T}{H(a)} \propto (\Omega_b h^2) (\Omega_m h^2)^{-1/2} (1+z)^{3/2} \\ &= \left(\frac{\Omega_b h^2}{0.02} \right) \left(\frac{\Omega_m h^2}{0.15} \right)^{-1/2} \left(\frac{1+z}{61} \right)^{3/2}\end{aligned}$$

- Quasars say $z_{ri} \geq 7$ so $\tau > 0.04$
- During reionization, cosmic quadrupole of $\sim 30 \mu\text{K}$ from the Sachs-Wolfe effect scatters into E -polarization
- Few percent optical depth leads to fraction of a μK signal
- Peaks at horizon scale at recombination: quadrupole source $j_2(kD_*)$ maximal at $kD_* \approx k\eta \approx 2$

Breaking degeneracies

- First objects, breaking degeneracy of initial amplitude vs optical depth in the peak heights

$$C_\ell \propto e^{-2\tau}$$

only below horizon scale at reionization

- Breaks degeneracies in angular diameter distance by removing an ambiguity for ISW-dark energy measure, helps in $\Omega_{DE} - w_{DE}$ plane

Gravitational Wave

- Gravitational waves produce a quadrupolar distortion in the temperature of the CMB like effect on a ring of test particles
- Like ISW effect, source is a metric perturbation with time dependent amplitude
- After recombination, is a source of observable temperature anisotropy – but is therefore confined to low order multipoles
- Generated during inflation by quantum fluctuations

Gravitational Wave Polarization

- In the tight coupling regime, quadrupole anisotropy suppressed by scattering

$$\pi_\gamma \approx \frac{\dot{h}}{\dot{\tau}}$$

- Since gravitational waves oscillate and decay at horizon crossing, the polarization peaks at the horizon scale at recombination not the damping scale
- More distinct signature in the B -mode polarization since symmetry of plane wave is broken by the transverse nature of gravity wave polarization

Secondary Anisotropy

- CMB photons traverse the large-scale structure of the universe from $z = 1000$ to the present.
- With the nearly scale-invariant adiabatic fluctuations observed in the CMB, structures form from the bottom up, i.e. small scales first, a.k.a. hierarchical structure formation.
- First objects reionize the universe between $z \sim 7 - 30$
- Main sources of secondary anisotropy
- Gravitational: Integrated Sachs-Wolfe effect (gravitational redshift) and gravitational lensing
- Scattering: peak suppression, large-angle polarization, Doppler effect(s), inverse Compton scattering

Transfer Function

- **Transfer function** transfers the initial Newtonian curvature to its value today (linear response theory)

$$T(k) = \frac{\Phi(k, a = 1)}{\Phi(k, a_{\text{init}})} \frac{\Phi(k_{\text{norm}}, a_{\text{init}})}{\Phi(k_{\text{norm}}, a = 1)}$$

- Conservation of Bardeen curvature: Newtonian curvature is a **constant** when **stress perturbations are negligible**: above the horizon during radiation and dark energy domination, on all scales during matter domination
- When stress fluctuations dominate, perturbations are stabilized by the **Jeans mechanism**
- Hybrid **Poisson equation**: Newtonian curvature, comoving density perturbation $\Delta \equiv (\delta\rho/\rho)_{\text{com}}$ implies Φ decays

$$(k^2 - 3K)\Phi = 4\pi G\rho\Delta \sim \eta^{-2}\Delta$$

Transfer Function

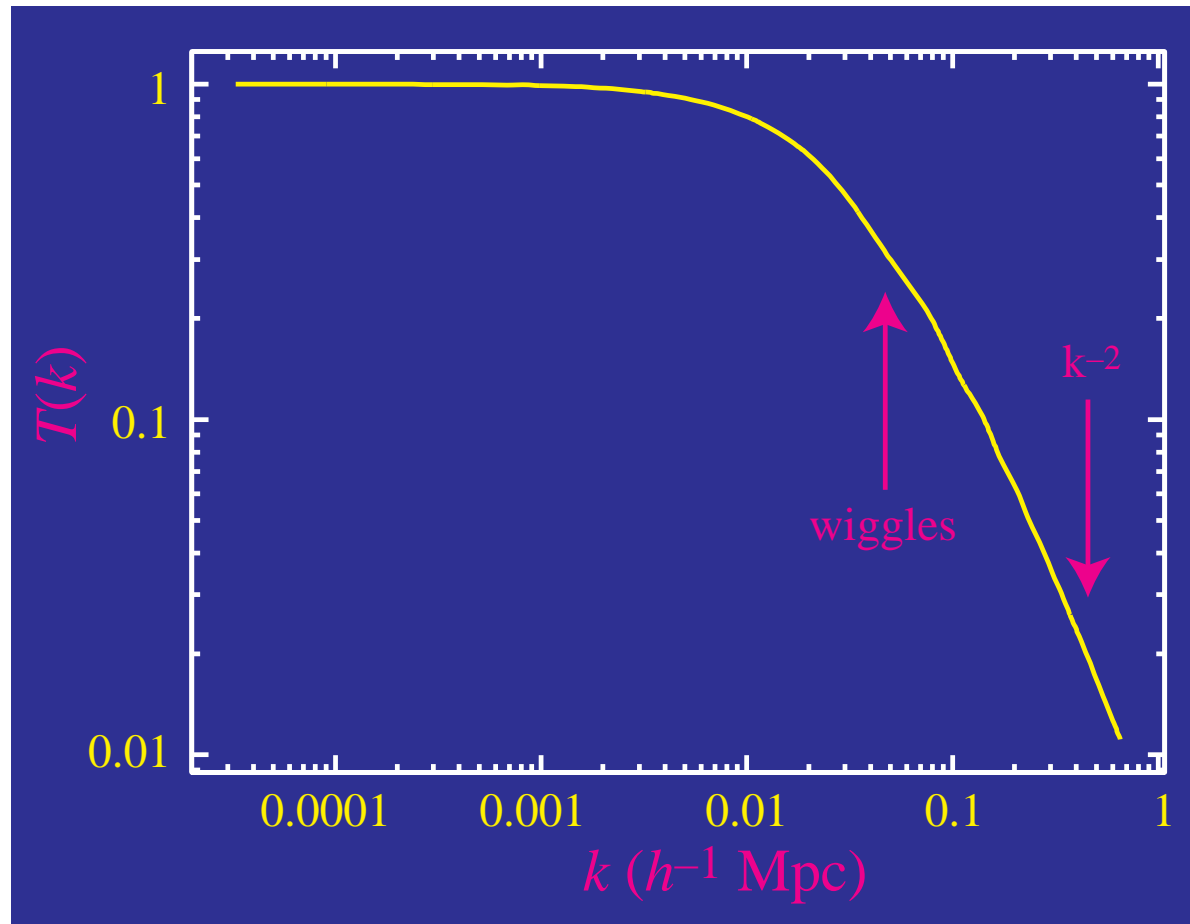
- Matter-radiation example: Jeans scale is horizon scale and Δ freezes into its value at horizon crossing $\Delta_H \approx \Phi_{\text{init}}$
- Freezing of Δ stops at η_{eq}

$$\Phi \sim (k\eta_{\text{eq}})^{-2} \Delta_H \sim (k\eta_{\text{eq}})^{-2} \Phi_{\text{init}}$$

- Conventionally k_{norm} is chosen as a scale between the horizon at matter radiation equality and dark energy domination.
- Small correction since growth with a smooth radiation component is logarithmic not frozen
- Run CMBfast to get transfer function or use fits

Transfer Function

- Transfer function has a k^{-2} fall-off beyond $k_{\text{eq}} \sim \eta_{\text{eq}}^{-1}$



- Additional baryon wiggles are due to acoustic oscillations at recombination – an interesting means of measuring distances

Growth Function

- Same physics applies to the dark energy dominated universe
- Under the dark energy sound horizon, dark energy density frozen. Potential decays at the same rate for all scales

$$g(a) = \frac{\Phi(k_{\text{norm}}, a)}{\Phi(k_{\text{norm}}, a_{\text{init}})}$$

- Pressure growth suppression: $\delta \equiv \delta\rho_m/\rho_m \propto ag$

$$\frac{d^2g}{d\ln a^2} + \left[\frac{5}{2} - \frac{3}{2}w(z)\Omega_{DE}(z) \right] \frac{dg}{d\ln a} + \frac{3}{2}[1 - w(z)]\Omega_{DE}(z)g = 0,$$

where $w \equiv p_{DE}/\rho_{DE}$ and $\Omega_{DE} \equiv \rho_{DE}/(\rho_m + \rho_{DE})$ with initial conditions $g = 1, dg/d\ln a = 0$

- As $\Omega_{DE} \rightarrow 0$ $g = \text{const.}$ is a solution. The other solution is the decaying mode, eliminated by initial conditions

ISW effect

- Potential decay leads to gravitational redshifts through the integrated Sachs-Wolfe effect
- Intrinsically a large effect since $2\Delta\Phi = 6\Psi_{\text{init}}/3$
- But net redshift is integral along line of sight

$$\begin{aligned}\frac{\Theta_\ell(k, \eta_0)}{2\ell + 1} &= \int_0^{\eta_0} d\eta e^{-\tau} [2\dot{\Phi}(k, \eta)] j_\ell(k(\eta_0 - \eta)) \\ &= 2\Phi(k, \eta_{MD}) \int_0^{\eta_0} d\eta e^{-\tau} \dot{g}(D) j_\ell(kD)\end{aligned}$$

- On small scales where $k \gg \dot{g}/g$, can pull source out of the integral

$$\int_0^{\eta_0} d\eta \dot{g}(D) j_\ell(kD) \approx \dot{g}(D = \ell/k) \frac{1}{k} \sqrt{\frac{\pi}{2\ell}}$$

evaluated at peak, where we have used $\int dx j_\ell(x) = \sqrt{\pi/2\ell}$

ISW effect

- Power spectrum

$$\begin{aligned} C_\ell &= \frac{2}{\pi} \int \frac{dk}{k} \frac{k^3 \langle \Theta_\ell^*(k, \eta_0) \Theta_\ell(k, \eta_0) \rangle}{(2\ell + 1)^2} \\ &= \frac{2\pi^2}{l^3} \int d\eta D \dot{g}^2(\eta) \Delta_\Phi^2(\ell/D, \eta_{MD}) \end{aligned}$$

- Or $l^2 C_l / 2\pi \propto 1/\ell$ for scale invariant potential. This is the Limber equation in spherical coordinates. Projection of 3D power retains only the transverse piece. For a general dark energy model, add in the scale dependence of growth rate on large scales.
- Cancellation of redshifts and blueshifts as the photon traverses many crests and troughs of a small scale fluctuation during decay. Enhancement of the $\ell < 10$ multipoles. Difficult to extract from cosmic variance and galaxy. Current ideas: cross correlation with other tracers of structure

Gravitational Lensing

- Lensing is a surface brightness conserving remapping of source to image planes by the gradient of the projected potential

$$\phi(\hat{\mathbf{n}}) = 2 \int_{\eta_*}^{\eta_0} d\eta \frac{(D_* - D)}{D D_*} \Phi(D\hat{\mathbf{n}}, \eta) .$$

such that the fields are remapped as

$$x(\hat{\mathbf{n}}) \rightarrow x(\hat{\mathbf{n}} + \nabla\phi) ,$$

where $x \in \{\Theta, Q, U\}$ temperature and polarization.

- Taylor expansion leads to product of fields and Fourier mode-coupling

Flat-sky Treatment

- Taylor expand

$$\begin{aligned}\Theta(\hat{\mathbf{n}}) &= \tilde{\Theta}(\hat{\mathbf{n}} + \nabla\phi) \\ &= \tilde{\Theta}(\hat{\mathbf{n}}) + \nabla_i\phi(\hat{\mathbf{n}})\nabla^i\tilde{\Theta}(\hat{\mathbf{n}}) + \frac{1}{2}\nabla_i\phi(\hat{\mathbf{n}})\nabla_j\phi(\hat{\mathbf{n}})\nabla^i\nabla^j\tilde{\Theta}(\hat{\mathbf{n}}) + \dots\end{aligned}$$

- Fourier decomposition

$$\begin{aligned}\phi(\hat{\mathbf{n}}) &= \int \frac{d^2l}{(2\pi)^2} \phi(\mathbf{l}) e^{i\mathbf{l}\cdot\hat{\mathbf{n}}} \\ \tilde{\Theta}(\hat{\mathbf{n}}) &= \int \frac{d^2l}{(2\pi)^2} \tilde{\Theta}(\mathbf{l}) e^{i\mathbf{l}\cdot\hat{\mathbf{n}}}\end{aligned}$$

Flat-sky Treatment

- Mode coupling of harmonics

$$\begin{aligned}\Theta(\mathbf{l}) &= \int d\hat{\mathbf{n}} \Theta(\hat{\mathbf{n}}) e^{-i\mathbf{l}\cdot\hat{\mathbf{n}}} \\ &= \tilde{\Theta}(\mathbf{l}) - \int \frac{d^2\mathbf{l}_1}{(2\pi)^2} \tilde{\Theta}(\mathbf{l}_1) L(\mathbf{l}, \mathbf{l}_1),\end{aligned}$$

where

$$\begin{aligned}L(\mathbf{l}, \mathbf{l}_1) &= \phi(\mathbf{l} - \mathbf{l}_1) (\mathbf{l} - \mathbf{l}_1) \cdot \mathbf{l}_1 \\ &+ \frac{1}{2} \int \frac{d^2\mathbf{l}_2}{(2\pi)^2} \phi(\mathbf{l}_2) \phi^*(\mathbf{l}_2 + \mathbf{l}_1 - \mathbf{l}) (\mathbf{l}_2 \cdot \mathbf{l}_1) (\mathbf{l}_2 + \mathbf{l}_1 - \mathbf{l}) \cdot \mathbf{l}_1.\end{aligned}$$

- Represents a coupling of harmonics separated by $L \approx 60$ peak of deflection power

Power Spectrum

- Power spectra

$$\langle \Theta^*(\mathbf{l}) \Theta(\mathbf{l}') \rangle = (2\pi)^2 \delta(\mathbf{l} - \mathbf{l}') C_l^{\Theta\Theta},$$

$$\langle \phi^*(\mathbf{l}) \phi(\mathbf{l}') \rangle = (2\pi)^2 \delta(\mathbf{l} - \mathbf{l}') C_l^{\phi\phi},$$

becomes

$$C_l^{\Theta\Theta} = (1 - l^2 R) \tilde{C}_l^{\Theta\Theta} + \int \frac{d^2 \mathbf{l}_1}{(2\pi)^2} \tilde{C}_{|\mathbf{l} - \mathbf{l}_1|}^{\Theta\Theta} C_{l_1}^{\phi\phi} [(\mathbf{l} - \mathbf{l}_1) \cdot \mathbf{l}_1]^2,$$

where

$$R = \frac{1}{4\pi} \int \frac{dl}{l} l^4 C_l^{\phi\phi}. \quad (3)$$

Smoothing Power Spectrum

- If $\tilde{C}_l^{\Theta\Theta}$ slowly varying then two term cancel

$$\tilde{C}_l^{\Theta\Theta} \int \frac{d^2\mathbf{l}_1}{(2\pi)^2} C_l^{\phi\phi} (\mathbf{l} \cdot \mathbf{l}_1)^2 \approx l^2 R \tilde{C}_l^{\Theta\Theta}.$$

- So lensing acts to smooth features in the power spectrum.
Smoothing kernel is $L \sim 60$ the peak of deflection power spectrum
- Because acoustic feature appear on a scale $l_A \sim 300$, smoothing is a subtle effect in the power spectrum.
- Lensing generates power below the damping scale which directly reflect power in deflections on the same scale

Polarization Lensing

- Polarization field harmonics lensed similarly

$$[Q \pm iU](\hat{\mathbf{n}}) = - \int \frac{d^2l}{(2\pi)^2} [E \pm iB](\mathbf{l}) e^{\pm 2i\phi_{\mathbf{l}}} e^{\mathbf{l} \cdot \hat{\mathbf{n}}}$$

so that

$$\begin{aligned} [Q \pm iU](\hat{\mathbf{n}}) &= [\tilde{Q} \pm i\tilde{U}](\hat{\mathbf{n}} + \nabla\phi) \\ &\approx [\tilde{Q} \pm i\tilde{U}](\hat{\mathbf{n}}) + \nabla_i\phi(\hat{\mathbf{n}})\nabla^i[\tilde{Q} \pm i\tilde{U}](\hat{\mathbf{n}}) \\ &\quad + \frac{1}{2}\nabla_i\phi(\hat{\mathbf{n}})\nabla_j\phi(\hat{\mathbf{n}})\nabla^i\nabla^j[\tilde{Q} \pm i\tilde{U}](\hat{\mathbf{n}}) \end{aligned}$$

Polarization Power Spectra

- Carrying through the algebra

$$C_l^{EE} = (1 - l^2 R) \tilde{C}_l^{EE} + \frac{1}{2} \int \frac{d^2 \mathbf{l}_1}{(2\pi)^2} [(\mathbf{1} - \mathbf{l}_1) \cdot \mathbf{l}_1]^2 C_{|\mathbf{1}-\mathbf{l}_1|}^{\phi\phi} \\ \times [(\tilde{C}_{l_1}^{EE} + \tilde{C}_{l_1}^{BB}) + \cos(4\varphi_{l_1})(\tilde{C}_{l_1}^{EE} - \tilde{C}_{l_1}^{BB})],$$

$$C_l^{BB} = (1 - l^2 R) \tilde{C}_l^{BB} + \frac{1}{2} \int \frac{d^2 \mathbf{l}_1}{(2\pi)^2} [(\mathbf{1} - \mathbf{l}_1) \cdot \mathbf{l}_1]^2 C_{|\mathbf{1}-\mathbf{l}_1|}^{\phi\phi} \\ \times [(\tilde{C}_{l_1}^{EE} + \tilde{C}_{l_1}^{BB}) - \cos(4\varphi_{l_1})(\tilde{C}_{l_1}^{EE} - \tilde{C}_{l_1}^{BB})],$$

$$C_l^{\Theta E} = (1 - l^2 R) \tilde{C}_l^{\Theta E} + \int \frac{d^2 \mathbf{l}_1}{(2\pi)^2} [(\mathbf{1} - \mathbf{l}_1) \cdot \mathbf{l}_1]^2 C_{|\mathbf{1}-\mathbf{l}_1|}^{\phi\phi} \\ \times \tilde{C}_{l_1}^{\Theta E} \cos(2\varphi_{l_1}),$$

- Lensing generates B -modes out of the acoustic polarization E -modes contaminates gravitational wave signature if $E_i < 10^{16} \text{ GeV}$.

Reconstruction from the CMB

- Correlation between Fourier moments reflect lensing potential

$$\langle x(\mathbf{l})x'(\mathbf{l}') \rangle_{\text{CMB}} = f_\alpha(\mathbf{l}, \mathbf{l}')\phi(\mathbf{l} + \mathbf{l}'),$$

where $x \in$ temperature, polarization fields and f_α is a fixed weight that reflects geometry

- Each pair forms a **noisy estimate** of the potential or projected mass
- just like a pair of galaxy shears
- **Minimum variance weight** all pairs to form an estimator of the lensing mass

Scattering Secondaries

- Optical depth during reionization

$$\tau \approx 0.066 \left(\frac{\Omega_b h^2}{0.02} \right) \left(\frac{\Omega_m h^2}{0.15} \right)^{-1/2} \left(\frac{1+z}{10} \right)^{3/2}$$

- Anisotropy suppressed as $e^{-\tau}$. Integral solution

$$\frac{\Theta_\ell(k, \eta_0)}{2\ell + 1} = \int_0^{\eta_0} d\eta e^{-\tau} S_0^{(0)} j_\ell(k(\eta_0 - \eta)) + \dots$$

- Isotropic (large scale) fluctuations not suppressed since suppression represents isotropization by scattering
- Quadrupole from the Sachs-Wolfe effect scatters into a large angle polarization bump

Doppler Effects

- Velocity fields of 10^{-3} and optical depths of 10^{-2} would imply large Doppler effect due to reionization
- Limber approximation says only fluctuations transverse to line of sight survive
- In linear theory, transverse fluctuations have no line of sight velocity and so Doppler effect is highly suppressed.
- Beyond linear theory: modulate the optical depth in the transverse direction using density fluctuations or ionization fraction fluctuations. Generate a modulated Doppler effect
- Linear fluctuations: Vishniac effect; Clusters: kinetic SZ effect; ionization patches: inhomogeneous reionization effect

Thermal SZ Effect

- Thermal velocities also lead to Doppler effect but first order contribution cancels because of random directions
- Residual effect is of order $v^2\tau \approx T_e/m_e\tau$ and can reach a sizeable level for clusters with $T_e \approx 10\text{keV}$.
- Raleigh-Jeans decrement and Wien enhancement described by second order collision term in Boltzmann equation: Kompaneets equation
- Clusters are rare objects so contribution to power spectrum suppressed, but may have been detected by CBI/BIMA: extremely sensitive to power spectrum normalization σ_8
- White noise on large-scales ($l < 2000$), turnover as cluster profile is resolved