# Astro 282

# Lecture Notes: Main Set

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# CMBology

- Universe is currently bathed in $2.725$K blackbody radiation which composes the majority of the radiation density of the universe

  mm-cm wavelength, 100 GHz photons near peak

  400 photon cm$^{-3}$

- Radiation is extremely isotropic: aside from the $10^{-3}$ temperature variations due to the Doppler shift of our own motion, fluctuations in the temperature are at the $10^{-5}$ level.
- Fluctuations are the imprint of the origin of structure
- Fluctuations are polarized at the 10% level reflecting scattering processes by which they last interacted with matter
- Place CMB in cosmological context

# Astro 282

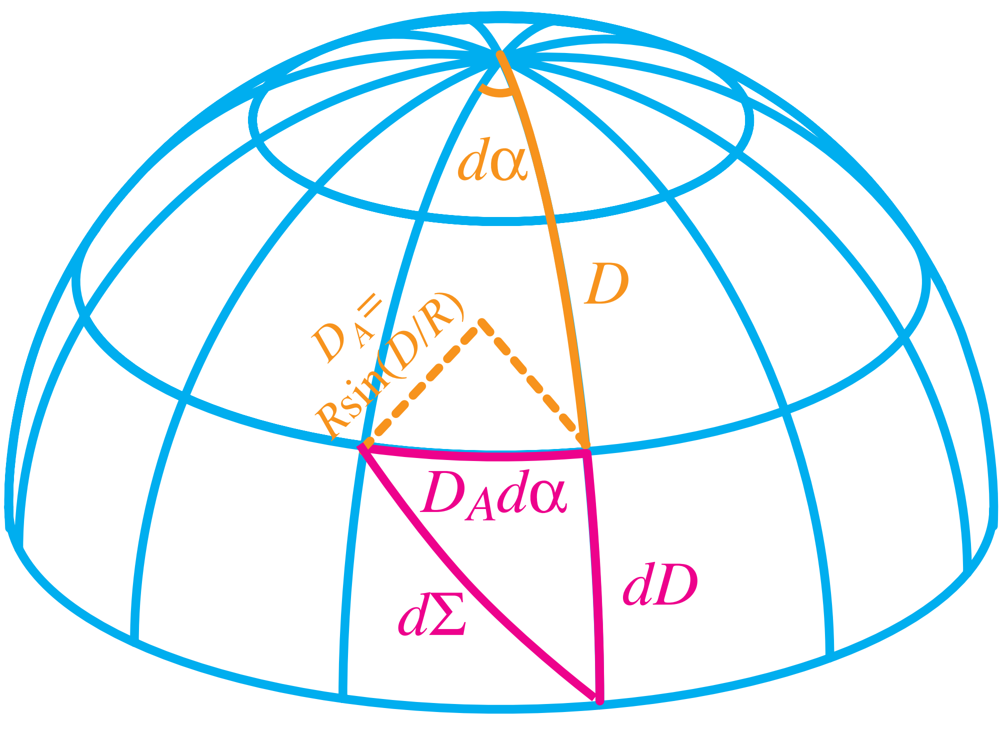
# FRW Cosmology

# FRW Cosmology

- FRW cosmology = homogeneous and isotropic on large scales
- Universe observed to be nearly isotropic (e.g. CMB, radio point sources, galaxy surveys)
- Copernican principle: must be isotropic to all observers (all locations)
- Implies homogeneity; also galaxy redshift surveys (LCRS, 2dF, SDSS) have seen the "end of greatness", large scale homogeneity directly
- FRW cosmology (homogeneity, isotropy & Einstein equations) generically implies the expansion of the universe, except for special unstable cases

# FRW Geometry

- Spatial geometry is that of a constant curvature (positive, negative, zero) surface
- Metric tells us how to measure distances on this surface
- Consider the closed geometry of a sphere of radius $R$ and suppress one dimension



# Angular Diameter Distance

- Spatial distance: restore 3rd dimension with the usual spherical polar angles

$$
\begin{aligned}
d\Sigma^2 &= dD^2 + D_A^2 d\alpha^2 \\
&= dD^2 + D_A^2 (d\theta^2 + \sin^2\theta d\phi^2)
\end{aligned}
$$

- $D_A$ is called the angular diameter distance since $D_A d\alpha$ corresponds to the transverse separation or size as opposed to the Euclidean $Dd\alpha$, i.e. is the apparent distance to an object through the gravitational lens of the background geometry
- In a positively curved geometry $D_A < D$ and objects are further than they appear
- In a negatively curved universe $R$ is imaginary and $R\sin(D/R) = i|R|\sin(D/i|R|) = |R|\sinh(D/|R|)$ – and $D_A > D$ objects are closer than they appear

# Volume Element

- Metric also defines the volume element

$$
\begin{aligned}
dV &= (dD)(D_A d\theta)(D_A \sin\theta d\phi) \\
&= D_A^2 dD d\Omega
\end{aligned}
$$

- Most of classical cosmology boils down to these three quantities, (comoving) distance, (comoving) angular diameter distance, and volume element
- For example, distance to a high redshift supernova, angular size of the horizon at last scattering, number density of clusters...

# Comoving Coordinates

- Remaining degree of freedom (preserving homogeneity and isotropy) is an overall scale factor that relates the geometry (fixed by the radius of curvature $R$) to physical coordinates – a function of time only

$$d\sigma^2 = a^2(t)d\Sigma^2$$

  our conventions are that the scale factor today $a(t_0) \equiv 1$

- Similarly physical distances are given by $d(t) = a(t)D$, $d_A(t) = a(t)D_A$.
- Distances in capital case are *comoving* i.e. they comove with the expansion and do not change with time – simplest coordinates to work out geometrical effects

# Redshift

- Wavelength of light “stretches” with the scale factor, so that it is convenient to define a shift-to-the-red or redshift as the scale factor increases

$$\lambda(a) = a(t)\Lambda$$
$$\frac{\lambda(1)}{\lambda(a)} = \frac{1}{a} \equiv (1+z)$$
$$\frac{\delta\lambda}{\lambda} = -\frac{\delta\nu}{\nu} = z$$

- Given known frequency of emission $\nu(a)$, redshift can be precisely measured (modulo Doppler shifts from peculiar velocities) – interpreting the redshift as a Doppler shift, objects receed in an expanding universe - $v = zc$

# Time and Conformal Time

- As in special relativity, time comes in with the opposite signature in measuring space-time separation
- Proper time

$$
\begin{aligned}
d\tau^2 &= dt^2 - d\sigma^2 \\
&= dt^2 - a^2(t) d\Sigma^2 \\
&\equiv a^2(t) \left( d\eta^2 - d\Sigma^2 \right)
\end{aligned}
$$

- Special relativity: physics invariant under the set of linear coordinate transformations (Lorentz transformation) that preserve lengths ($d\tau^2$)
- General relativity: physics invariant under a general coordinate transformation that preserves lengths

# A GR Aside

- We will generally skirt around General Relativity but knowledge of the language will be useful
- Proper time defines the metric $g_{\mu\nu}$

$$d\tau^2 \equiv g_{\mu\nu} dx^\mu dx^\nu$$

- Usually we will use comoving coordinates and conformal time as the "$x$" 's unless otherwise specified – metric for other choices are related by $a(t)$ – e.g. in spherical coordinates $\mu \in \eta, \theta, \phi, D$

$$g_{\mu\nu} = a^2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -D_A^2 & 0 & 0 \\ 0 & 0 & -D_A^2 \sin^2\theta & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

# Photon Cartography

- Classical cosmology is photon cartography – mapping out the expansion by tracking the distance a photon travels as a function of scale factor or redshift
- Taking out the scale factor in the time coordinate $d\eta = dt/a$ defines conformal time – useful in that photons travelling radially from observer then obey

$$\Delta D = \Delta\eta = \int \frac{dt}{a}$$

so that time and distance may be interchanged

# Horizon

- Distance travelled by a photon in the whole lifetime of the universe defines the horizon
- Since $d\tau = 0$, the horizon is simply the conformal time elapsed

$$D_{\rm horizon}(t) = \int_0^t \frac{dt'}{a} = \eta(t)$$

- Since the horizon always grows with time, there is always a point in time before which two observers separated by a distance $D$ could not have been in causal contact
- Horizon problem: why is the universe homogeneous and isotropic on large scales, near the current horizon – problem deepens for objects seen at early times, e.g. CMB

# Hubble Parameter

- Useful to define the expansion rate or Hubble parameter

$$H(t) \equiv \frac{1}{a}\frac{da}{dt}$$

  since dynamics (Einstein equations) will give this directly as $H(a) \equiv H(t(a))$

- Time becomes

$$t = \int dt = \int \frac{da}{aH(a)}$$

- Conformal time becomes

$$\eta = \int \frac{dt}{a} = \int \frac{da}{a^2 H(a)}$$

# Distance-Redshift Relation

- All distance redshift relations based on comoving distance $D(z)$

$$D(a) = \int dD = \int_a^1 \frac{da'}{a^2 H(a)}$$
$$(da = -(1+z)^{-2} dz = -a^2 dz)$$
$$D(z) = -\int_z^0 \frac{dz'}{H(z')} = \int_0^z \frac{dz'}{H(z')}$$

- Note limiting case is the Hubble law

$$\lim_{z \to 0} D(z) = z/H(z=0) \equiv z/H_0$$

redshift (recession velocity) increases linearly with distance

- Hubble constant usually quoted as $H_0 = 100h$ km s$^{-1}$ Mpc$^{-1}$, observationally $h \sim 0.7$; in natural units $H_0 = (2997.9)^{-1} h$ Mpc$^{-1}$ defines an inverse length scale

# Distance-Redshift Relation

- Example: object of known physical size $\lambda = a(t)\Lambda$ ("standard ruler") subtending an (observed) angle on the sky $\alpha$

$$\alpha = \frac{\Lambda}{D_A(z)} = \frac{\lambda}{aR\sin(D(z)/R)}$$
$$= \frac{\lambda}{R\sin(D(z)/R)}(1+z) \equiv \frac{\lambda}{d_A(z)}$$

- Example: object of known luminosity $L$ ("standard candle") with a measured flux $S$. Comoving surface area $4\pi D_A^2$, frequency/energy $(1+z)$, time-dilation or arrival rate of photons (crests) $(1+z)$:

$$S = \frac{L}{4\pi D_A^2}\frac{1}{(1+z)^2}$$
$$\equiv \frac{L}{4\pi d_L^2} \quad (d_L = (1+z)D_A = (1+z)^2 d_A)$$

# Relative Measures

- If absolute calibration of standards unknown, then absolute distance (or Hubble constant) unknown

$$d_A(z) = \lambda/\alpha(z); d_L(z) = \sqrt{L/4\pi S(z)}$$

- Ratio at two different redshifts drops out the unknown standards $\lambda, L$ and measures evolution of the distance-redshift relation $H_0 D(z)$:

$$\frac{d_{A,L}(z_2)}{d_{A,L}(z_1)} \approx \frac{H_0}{z_1} d_{A,L}(z_2) \quad [z_1 \ll 1]$$

- Alternately, distances & curvature are measured in units of $h^{-1}$ Mpc.

# Fundamental Observable

- Fundamental dependence (aside from $(1+z)$ factors)

$$
\begin{aligned}
H_0 D_A(z) &= H_0 R \sin(D(z)/R) \\
&= \tilde{R} \sin(H_0 D(z)/\tilde{R}), \quad \tilde{R} = H_0 R
\end{aligned}
$$

$$
H_0 D(z) = \int \frac{da}{a^2} \frac{H_0}{H(a)}
$$

- Maps out the kinematics of the expansion
- Current best standard ruler: acoustic oscillations; current best standard candle supernovae type Ia
- Adding in the dynamics of the expansion, measurements of $D(z)$ indicate a flat universe whose expansion is accelerating

# Evolution of Scale Factor

- FRW cosmology is fully specified if the function $a(t)$ is given
- General relativity relates the scale factor with the matter content of universe.
- Build the Einstein tensor $G^{\mu}{}_{\nu}$ out of the metric and use Einstein equation

$$G^{\mu}{}_{\nu} = -8\pi G T^{\mu}{}_{\nu}$$

$$G^{0}{}_{0} = -\frac{3}{a^2}\left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{1}{R^2}\right]$$

$$G^{i}{}_{j} = -\frac{1}{a^2}\left[2\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 + \frac{1}{R^2}\right]\delta^{i}{}_{j}$$

# Einstein Equations

- Isotropy demands that the stress-energy tensor take the form

$$T^0{}_0 = \rho$$

$$T^i{}_j = -p\delta^i{}_j$$

where $\rho$ is the energy density and $p$ is the pressure

- So Einstein equations become

$$\left(\frac{\dot a}{a}\right)^2 + \frac{1}{R^2} = \frac{8\pi G}{3} a^2 \rho$$

$$2\frac{\ddot a}{a} - \left(\frac{\dot a}{a}\right)^2 + \frac{1}{R^2} = -8\pi G a^2 p$$

$$\text{or} \quad \frac{\ddot a}{a} - \left(\frac{\dot a}{a}\right)^2 = -\frac{4\pi G}{3} a^2 (\rho + 3p)$$

# Friedman Equations

- More usual to see Einstein equations expressed in time not conformal time

$$\frac{\dot a}{a} = \frac{da}{d\eta}\frac{1}{a} = \frac{da}{dt} = aH(a)$$

$$\frac{\ddot a}{a} - \left(\frac{\dot a}{a}\right)^2 = \frac{d}{d\eta}\left(\frac{\dot a}{a}\right) = a\frac{d}{dt}\left(\frac{da}{dt}\right) = a\frac{d^2 a}{dt^2}$$

- Friedmann equations:

$$H^2(a) + \frac{1}{a^2R^2} = \frac{8\pi G}{3}\rho$$

$$\frac{1}{a}\frac{d^2a}{dt^2} = -\frac{4\pi G}{3}(\rho + 3p)$$

- Convenient fiction to describe curvature as an energy density component $\rho_K = -3/(8\pi G a^2 R^2) \propto a^{-2}$ that does not accelerate the expansion, $p_K = -\rho_K/3$

# Critical Density

- Friedmann equation for $H$ then reads

$$H^2(a) = \frac{8\pi G}{3}(\rho + \rho_K) \equiv \frac{8\pi G}{3}\rho_c$$

defining a critical density today $\rho_c$ in terms of the expansion rate

- In particular, its value today is given by the Hubble constant as

$$\rho_{\rm c}(z=0) = 3H_0^2/8\pi G = 1.8788 \times 10^{-29} h^2 {\rm g\,cm}^{-3}$$

- Energy density today is given as a fraction of critical $\Omega \equiv \rho/\rho_c|_{z=0}$. Radius of curvature then given by $R^{-2} = H_0^2(\Omega - 1)$
- If $\Omega \approx 1$, $\rho \approx \rho_c$, then $\rho_K \ll \rho_c$ or $H_0 R \ll 1$, universe is flat across the Hubble distance. $\Omega < 1$ negatively curved; $\Omega > 1$ positively curved

# Newtonian Interpretation

- Consider a test particle of mass $m$ in expanding spherical region of radius $r$ and total mass $M$. Energy conservation

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r} = \text{const}$$

$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2 - \frac{GM}{r} = \text{const}$$

$$\frac{1}{2}\left(\frac{1}{r}\frac{dr}{dt}\right)^2 - \frac{GM}{r^3} = \frac{\text{const}}{r^2}$$

$$H^2 = \frac{8\pi G\rho}{3} - \frac{\text{const}}{a^2}$$

- Constant determines whether the system is bound and in the Friedmann equation is associated with curvature – not general since neglects pressure

# Conservation Law

- Second Friedmann equation, or acceleration equation, simply expresses energy conservation (why: stress energy is automatically conserved in GR via Bianchi identity)

$$d\rho V + p dV = 0$$

$$d\rho a^3 + p da^3 = 0$$

$$\dot{\rho} a^3 + 3\frac{\dot{a}}{a}\rho a^3 + 3\frac{\dot{a}}{a} p a^3 = 0$$

$$\frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a} \qquad w \equiv p/\rho$$

- If $w =$ const. then the energy density depends on the scale factor as $\rho \propto a^{-3(1+w)}$.

# Multicomponent Universe

- The total energy density can be composed of a sum of components with differing equations of state

$$\rho(a) = \sum_i \rho_i(a) = \sum_i \rho_i(a=1) a^{-3(1+w_i)}, \quad \Omega_i \equiv \rho_i/\rho_c|_{a=1}$$

- Important cases: nonrelativistic matter $\rho_m = m n_m \propto a^{-3}$, $w_m = 0$; relativistic radiation $\rho_r = E n_r \propto \nu n_r \propto a^{-4}$, $w_r = 1/3$; "curvature" $\rho_K \propto a^{-2}$, $w_K = -1/3$; constant energy density or cosmological constant $\rho_\Lambda \propto a^0$, $w_\Lambda = -1$
- Or generally with $w_c = p_c/\rho_c = (p + p_K)/(\rho + \rho_K)$

$$\rho_c(a) = \rho_c(a=1) e^{-\int d\ln a \, 3(1+w_c(a))}$$

$$H^2(a) = H_0^2 e^{-\int d\ln a \, 3(1+w_c(a))}$$

# Acceleration Equation

- Time derivative of (first) Friedman equation

$$
\begin{aligned}
2\frac{1}{a}\frac{da}{dt}\left[\frac{1}{a}\frac{d^2a}{dt^2} - H^2(a)\right] &= \frac{8\pi G}{3}\frac{d\rho_c}{dt} \\
\left[\frac{1}{a}\frac{d^2a}{dt^2} - \frac{8\pi G}{3}\rho_c\right] &= \frac{4\pi G}{3}[-3(1+w_c)\rho_c] \\
\frac{1}{a}\frac{d^2a}{dt^2} &= -\frac{4\pi G}{3}[(1+3w_c)\rho_c] \\
&= -\frac{4\pi G}{3}(\rho + \rho_K + 3p + 3p_K) \\
&= -\frac{4\pi G}{3}(1+3w)\rho
\end{aligned}
$$

- Acceleration equation says that universe decelerates if $w > -1/3$

# Expansion Required

- Friedmann equations "predict" the expansion of the universe. Non-expanding conditions $da/dt = 0$ and $d^2a/dt^2 = 0$ require

$$\rho = -\rho_K \qquad \rho = -3p$$

i.e. a positive curvature and a total equation of state $w \equiv p/\rho = -1/3$

- Since matter is known to exist, one can in principle achieve this with

$$\rho = \rho_m + \rho_\Lambda = -\rho_K = -3p = 3\rho_\Lambda$$

$$\rho_\Lambda = -\frac{1}{3}\rho_K \quad \rho_m = -\frac{2}{3}\rho_K$$

Einstein introduced $\rho_\Lambda$ for exactly this reason – "biggest blunder"; but note that this balance is unstable: $\rho_m$ can be perturbed but $\rho_\Lambda$, a true constant cannot

# Dark Energy

- Distance redshift relation depends on energy density components

$$H_0 D(z) = \int \frac{da}{a^2} \frac{H_0}{H(a)}$$
$$= \int \frac{da}{a^2} e^{\int d\ln a \, \frac{3}{2}(1+w_c(a))}$$

- Distant supernova Ia as standard candles imply that $w_c < -1/3$ so that the expansion is accelerating
- Consistent with a cosmological constant that is $\Omega_\Lambda = \rho_\Lambda/\rho_{\rm crit} = 2/3$ of the total energy density
- Coincidence problem: different components of matter scale differently with $a$. Why are (at least) two components comparable today? – Anthropic?

# Dark Matter

- Since Zwicky in the 1930's non-luminous or dark matter has been known to dominate over luminous matter in stars (and hot gas)
- Arguments are basically from a balance of gravitational force against "pressure" from internal motions: rotation velocity in galactic disks, velocity dispersion of galaxies in clusters, gas pressure in clusters, radiation pressure in CMB
- Assuming that the object is somehow typical in its non-luminous to luminous density, these measures are converted to an overall dark matter density through a "mass-to-light ratio"
- From galaxy surveys the luminosity density in solar units is

$$\rho_L = 2 \pm 0.7 \times 10^8 h \, L_\odot \mathrm{Mpc}^{-3}$$

($h$'s: distances in $h^{-1}$ Mpc; luminosity inferred from flux $L \propto Sd^2 \propto h^{-2}$; inverse volume $\propto h^3$)

# Dark Matter

- Critical density in solar units is $\rho_c = 2.7754 \times 10^{11} h^2 \, M_\odot \mathrm{Mpc}^{-3}$ so that the critical mass-to-light ratio in solar units is

$$\left(\frac{M}{L}\right) \approx 1400h$$

- Flat rotation curves: $GM/r^2 \approx v^2/r \rightarrow M \approx v^2 r/G$, so the observed flat rotation curve implies $M \propto r$ out to $30h^{-1}$ kpc, beyond the light. Implies $M/L > 30h$ and perhaps more – closure if flat out to $\sim 1$ Mpc.

- Similar argument holds in clusters of galaxies where velocity dispersion replaces circular velocity and centripetal force is replaced by a "pressure gradient" $T/m = \sigma^2$ and $p = \rho T/m = \rho\sigma^2$– generalization of hydrostatic equilibrium: Zwicky got $M/L \approx$ 300h.

# Hydrostatic Equilibrium

- Evidence for dark matter in $X$-ray clusters also comes from direct hydrostatic equilibrium inference from the gas: balance radial pressure gradient with gravitational potential gradient
- Infinitesimal volume of area $dA$ and thickness $dr$ at radius $r$ and interior mass $M(r)$: pressure difference supports the gas

$$[p_g(r) - p_g(r+dr)]dA = \frac{GmM}{r^2} = \frac{G\rho_g M}{r^2}dV$$

$$\frac{dp_g}{dr} = -\rho_g \frac{d\Phi}{dr}$$

with $p_g = \rho_g T_g/m$ becomes

$$\frac{GM}{r} = -\frac{T_g}{m}\left(\frac{d\ln\rho_g}{d\ln r} + \frac{d\ln T_g}{d\ln r}\right)$$

- $\rho_g$ from X-ray luminosity; $T_g$ sometimes taken as isothermal

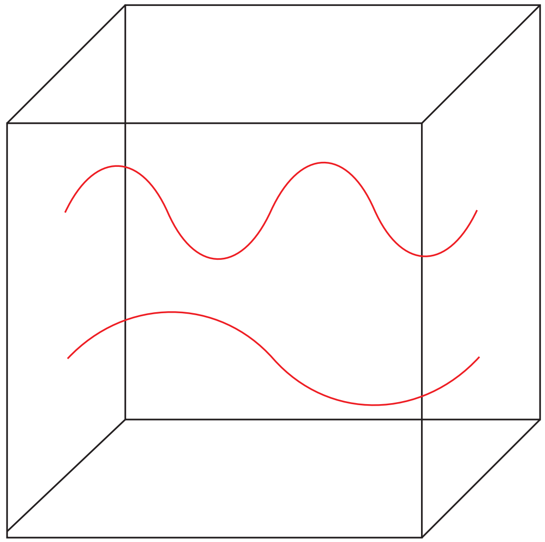
# Gravitational Lensing

- Mass can be directly measured in the gravitational lensing of sources behind the cluster
- Strong lensing (giant arcs) probes central region of clusters
- Weak lensing (1-10% ) elliptical distortion to galaxy image probes outer regions of cluster and total mass
- All techniques agree on the necessity of dark matter and are roughly consistent with a dark matter density $\Omega_m \sim 0.2 - 0.4$
- $\Omega_m + \Omega_\Lambda \approx 1$ from matter density + dark energy
- CMB provides a test of $D_A \neq D$ through the standard rulers of the acoustic peaks and shows that the universe is close to flat $\Omega \approx 1$
- Consistency has lead to the standard model for the cosmological matter budget

# Astro 282 Supplement

## Statistical Mechanics

# How Many Particles Fit in a Box?



- Counting momentum states due to the wave nature of particles with momentum $q$ and de Broglie wavelength (in this supplement we retain $\hbar$ and $c$ to be explicit)

$$\lambda = \frac{h}{q} = \frac{2\pi\hbar}{q}$$

- In a discrete volume $L^3$ there is a discrete set of states that satisfy periodic boundary conditions

# How Many Particles Fit in a Box?

- As in Fourier analysis: $e^{2\pi i x/\lambda} = e^{i(q/\hbar)x} = e^{i(q/\hbar)(x+L)}$ yields a discrete set of allowed states

$$\lambda_i = \frac{L}{m_i} = \frac{2\pi\hbar}{q_i}, \quad m_i = 1, 2, 3\ldots$$

$$q_i = m_i \frac{2\pi\hbar}{L}$$

- In each of 3 directions: $\sum_{m_{xi} m_{yj} m_{zk}} \to \int d^3 m$
- The differential number of allowed momenta in the volume

$$d^3 m = \left(\frac{L}{2\pi\hbar}\right)^3 d^3 q$$

# Density of States

- The total number of states allows for a number of internal degrees of freedom, e.g. spin, quantified by the degeneracy factor $g$: total density of states:

$$\frac{dN_s}{V} = \frac{g}{V} d^3m = \frac{g}{(2\pi\hbar)^3} d^3q$$

- If all states were occupied by a single particle, then the particle density

$$n_s = \frac{N_s}{V} = \frac{1}{V} \int dN_s = \int \frac{g}{(2\pi\hbar)^3} d^3q$$

# Distribution Function

- The distribution function $f$ quantifies the occupation of the allowed momentum states

$$n = \frac{N}{V} = \frac{1}{V}\int f dN_s = \int \frac{g}{(2\pi\hbar)^3} f d^3q$$

- $f$, aka phase space occupation number, also quantifies the density of particles per unit phase space $dN/(\Delta x)^3(\Delta q)^3$
- For photons, the spin degeneracy $g = 2$ accounting for the 2 polarization states
- Energy $E(q) = (q^2c^2 + m^2c^4)^{1/2}$
- Momentum $\rightarrow$ frequency $q = h/\lambda = h\nu/c = E/c$ (where $m = 0$ and $\lambda\nu = c$)

# Number Density

- Momentum state defines the direction of the radiation

$$
\begin{aligned}
n &= g \int \frac{d^3q}{(2\pi\hbar)^3} f \\
&= 2 \int \frac{d\Omega q^2 dq}{(2\pi\hbar)^3} f \\
&= 2 \int d\Omega \left(\frac{h}{c}\right)^3 \frac{1}{h^3} \int \nu^2 d\nu f \\
&= 2 \int d\Omega \frac{1}{c^3} \int \nu^2 d\nu f
\end{aligned}
$$

- Gives number density in a given direction and frequency band

# Energy Density

- In general the energy density is

$$\rho = g \int \frac{d^3q}{(2\pi\hbar)^3} E(q) f$$

- For radiation

$$\rho = g \int \frac{d^3q}{(2\pi\hbar)^3} E(q) f = 2 \int d\Omega \frac{1}{c^3} \int \nu^2 d\nu h\nu f$$

- So specific energy density

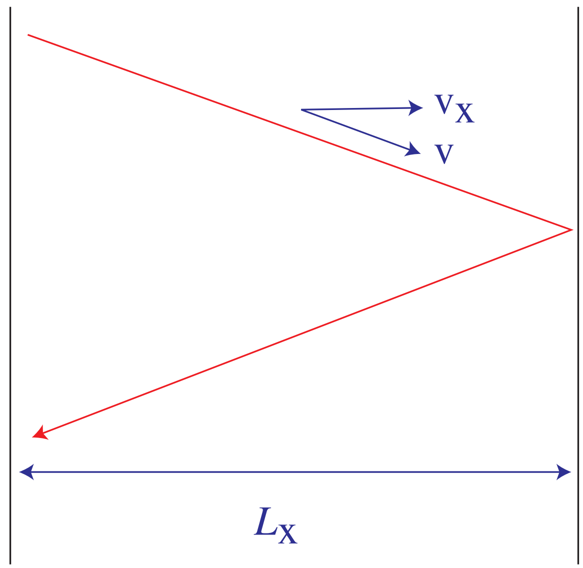
$$\rho_\nu(\Omega) = \frac{d^2\rho}{d\Omega d\nu} = \frac{2\nu^3 h}{c^3} f$$

- And specific intensity

$$I_\nu(\Omega) = \rho_\nu(\Omega) c = \frac{2\nu^3 h}{c^2} f$$

# Pressure

- Pressure: particles bouncing off a surface of area $A$ in a volume spanned by $L_x$: per momentum state




$$p_q = \frac{F}{A} = \frac{N_{\rm part}}{A}\frac{\Delta q_x}{\Delta t}$$

$$(\Delta q_x = 2|q_x|, \quad \Delta t = 2L_x/v_x, \quad q/E = v/c^2)$$

$$= \frac{N_{\rm part}}{V}|q_x||v_x| = f\frac{|q||v|}{3} = f\frac{q^2c^2}{3E}$$

($\cos^2$ term in radiative pressure calc.)

# Moments

- So that summed over states

$$p = g \int \frac{d^3q}{(2\pi\hbar)^3} \frac{|q|^2 c^2}{3E(q)} f$$

- Radiation

$$p = g \int \frac{d^3q}{(2\pi\hbar)^3} \frac{E(q)}{3} f = \frac{1}{3}\rho$$

- Energy and pressure are part of the angular moments of the distribution function – the isotropic ones
- First order anisotropy is the bulk momentum density or dipole of the distribution:

$$(u + p)\mathbf{v}/c = g \int \frac{d^3q}{(2\pi\hbar)^3} \mathbf{q} c f$$

# Fluid Approximation Redux

- Continue with the second moments: radiative viscosity or anisotropic stress

$$\pi_{ij} = g \int \frac{d^3 q}{(2\pi\hbar)^3} \frac{3q_i q_j - q^2 \delta_{ij}}{3E(q)} f$$

- Fluid approximation is that all the higher order moments from the radiative viscosity onward vanishes - due isotropization from a high collision rate
- Since particle kinetics must obey energy and momentum conservation, in the fluid limit there are two equations of motion: continuity and Euler equations
- Three quantities of interest: energy density, pressure, bulk velocity means that a third relation is needed: $p(\rho)$ the equation of state

# Equilibrium

- Thermal physics describes the equilibrium distribution of particles for a medium at temperature $T$
- Expect that the typical energy of a particle by equipartition is $E \sim kT$, so that $f(E/kT, ?)$ in equilibrium
- Must be a second variable of import. Number density

$$n = g \int \frac{d^3q}{(2\pi\hbar)^3} f(E/kT) =? \quad n(T)$$

- If particles are conserved then $n$ cannot simply be a function of temperature.
- The integration constant that concerns particle conservation is called the chemical potential. Relevant for photons when creation and annihilation processes are ineffective

# Temperature and Chemical Potential

- Fundamental assumption of statistical mechanics is that all accessible states have an equal probability of being populated. The number of states $G$ defines the entropy $S(U,N,V) = k \ln G$ where $U$ is the energy, $N$ is the number of particles and $V$ is the volume
- When two systems are placed in thermal contact they may exchange energy, leading to a wider range of accessible states

$$G(U,N,V) = \sum_{U_1} G_1(U_1,N_1,V_1) G_2(U-U_1, N-N_1, V-V_1)$$

- The most likely distribution of $U_1$ and $U_2$ is given for the maximum $dG/dU_1 = 0$

$$\left(\frac{\partial G_1}{\partial U_1}\right)_{N_1,V_1} G_2 dU_1 + G_1 \left(\frac{\partial G_2}{\partial U_2}\right)_{N_2,V_2} dU_2 = 0 \qquad dU_1 + dU_2 = 0$$

# Temperature and Chemical Potential

- Or equilibrium requires

$$\left(\frac{\partial \ln G_1}{\partial U_1}\right)_{N_1,V_1} = \left(\frac{\partial \ln G_2}{\partial U_2}\right)_{N_2,V_2} \equiv \frac{1}{kT}$$

which is the definition of the temperature (equal for systems in thermal contact

- Likewise define a chemical potential $\mu$ for a system in diffusive equilibrium

$$\left(\frac{\partial \ln G_1}{\partial N_1}\right)_{U_1,V_1} = \left(\frac{\partial \ln G_2}{\partial N_2}\right)_{U_2,V_2} \equiv -\frac{\mu}{kT}$$

defines the most likely distribution of particle numbers as a system with equal chemical potentials: generalize to multiple types of particles undergoing "chemical" reaction $\rightarrow$ law of mass action $\sum_i \mu_i dN_i = 0$

# Temperature and Chemical Potential

- Equivalent definition: the chemical potential is the free energy cost associated with adding a particle at fixed temperature and volume

$$\mu = \frac{\partial F}{\partial N}\bigg|_{T,V}, \qquad F = U - TS$$

  free energy: balance between minimizing energy and maximizing entropy $S$

- Temperature and chemical potential determine the probability of a state being occupied if the system is in thermal and diffusive contact with a large reservoir at temperature $T$

# Gibbs or Boltzmann Factor

- Suppose the system has two states unoccupied $N_1 = 0$, $U_1 = 0$ and occupied $N_1 = 1$, $U_1 = E$ then the ratio of probabilities in the occupied to unoccupied states is given by

$$P = \frac{\exp[\ln G_{\rm res}(U - E, N - 1, V)]}{\exp[\ln G_{\rm res}(U, N, V)]}$$

- Taylor expand

$$\ln G_{\rm res}(U - E, N - 1, V) \approx \ln G_{\rm res}(U, N, V) - E\frac{1}{kT} + \frac{\mu}{kT}$$

$$P \approx \exp[-(E - \mu)/kT]$$

- This is the Gibbs factor.

# Gibbs or Boltzmann Factor

- More generally the probability of a system being in a state of energy $E_i$ and particle number $N_i$ is given by the Gibbs factor

$$P(E_i, N_i) \propto \exp[-(E_i - \mu N_i)/kT]$$

- Unlikely to be in an energy state $E_i \gg kT$ mitigated by the number of particles
- Dropping the diffusive contact, this is the Boltzmann factor

# Mean Occupation

- Mean occupation in thermal equilibrium

$$f = \langle N \rangle = \frac{\sum_i N_i P(E_i, N_i)}{\sum P(E_i, N_i)}$$

- Take $E_i = N_i E$ where $E$ is the particle energy (zero point drops out)
- For fermions: occupancy $N_i = 0, 1$

$$f = \frac{P(E,1)}{P(0,0) + P(E,1)} = \frac{\exp[-(E-\mu)/kT]}{1 + \exp[-(E-\mu)/kT]}$$

$$= \frac{1}{\exp[(E-\mu)/kT] + 1} \quad \text{Fermi-Dirac Distribution}$$

- $T \to 0$, $f \to [e^{\pm\infty} + 1]^{-1}$ ($E > \mu, f = 0$); ($E < \mu, f = 1$), occupied out to a sharp energy or Fermi surface with $\delta E = kT$

# Bose-Einstein Distribution

- For bosons:

$$\sum_i P[E_i, N_i] = \sum_{N_i=0}^{\infty} \exp[-N_i(E-\mu)/kT] = \sum_{N_i=0}^{\infty} [e^{-(E-\mu)/kT}]^{N_i}$$

$$= \frac{1}{1 - e^{-(E-\mu)/kT}}$$

$$\sum_i N_i P[E_i, N_i] = \sum_{N_i=0}^{\infty} N_i \exp[-N_i(E-\mu)/kT]$$

$$= \frac{\partial}{\partial \mu/kT} \sum_{N_i=0}^{\infty} [e^{-(E-\mu)/kT}]^{N_i}$$

$$= \frac{\partial}{\partial \mu/kT} \left( \frac{1}{1 - e^{-(E-\mu)/kT}} \right) = \frac{e^{-(E-\mu)/kT}}{(1 - e^{-(E-\mu)/kT})^2}$$

# Bose-Einstein Distribution

- Bose Einstein distribution:

$$f = \frac{\sum_i N_i P[E_i, N_i]}{\sum_i P[E_i, N_i]} = \frac{1}{e^{(E-\mu)/kT} - 1}$$

  For $E - \mu \gg kT$, $f \to 0$. For $E - \mu < kT \ln 2$, $f > 1$, high occupation (Bose-Einstein condensate).

- General equilibrium distribution

$$f = \frac{1}{e^{(E-\mu)/kT} \pm 1}$$

  $+ =$ fermions, $- =$ bosons

- $\mu$ alters the number of particles at temperature $T$

# Maxwell Boltzmann Distribution

- In both cases, if $(E - \mu) \gg kT$ (including rest mass energy), then

$$f = e^{-(E-\mu)/kT}$$

- For non relativistic particles

$$E = (q^2c^2 + m^2c^4)^{1/2} = mc^2(1 + q^2/m^2c^2)^{1/2}$$
$$\approx mc^2(1 + q^2/2m^2c^2) = mc^2 + \frac{1}{2}mv^2$$

$$f = e^{-(mc^2-\mu)/kT}e^{-mv^2/2kT}$$

# Planck (Black Body) Distribution

- When particles can be freely created and destroyed $\mu \to 0$ and for bosons this is the black body distribution

$$f = \frac{1}{e^{E/kT} - 1}$$

- Specific intensity

$$I_\nu = B_\nu = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1}$$

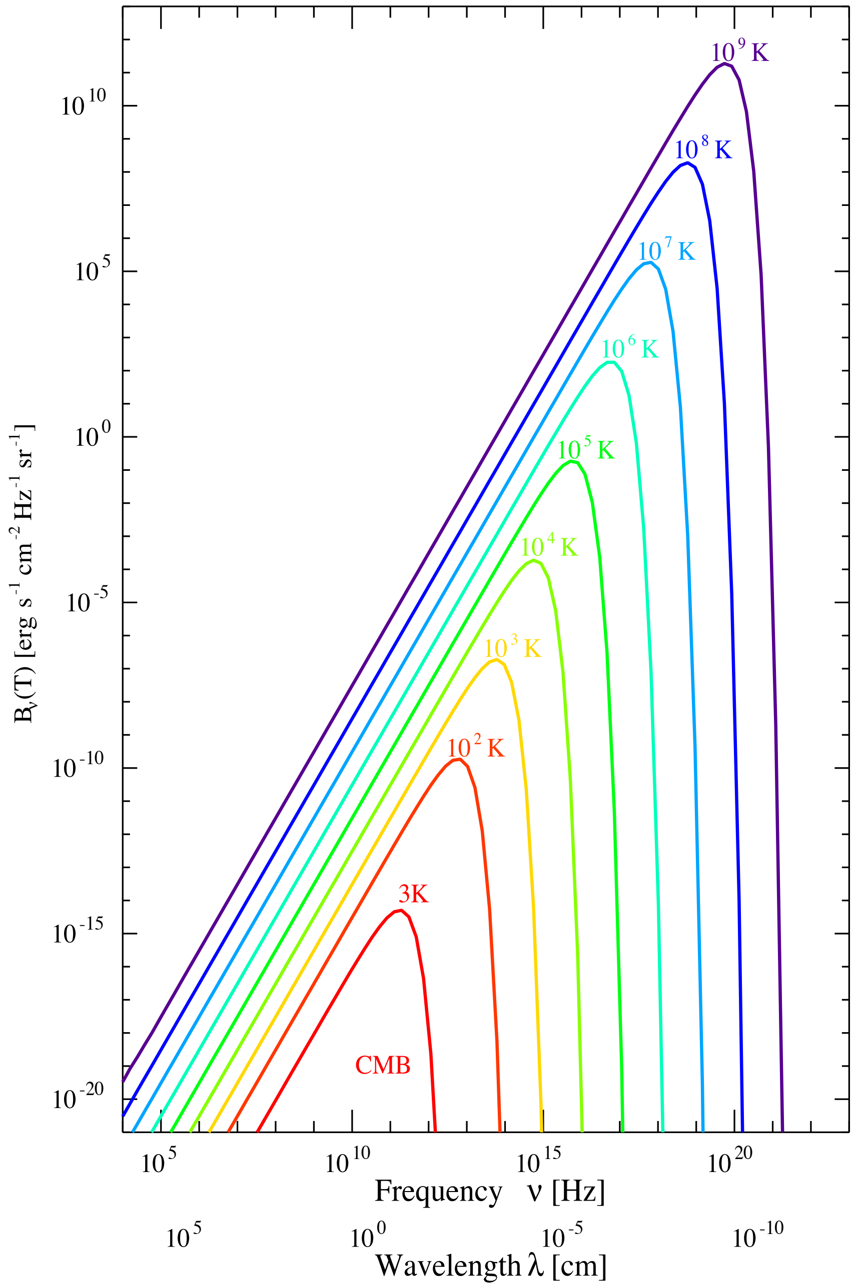
- At low frequencies $h\nu \ll kT$ (Rayleigh Jeans)

$$\exp(h\nu/kT) - 1 \approx 1 + h\nu/kT - 1 = h\nu/kT$$

$$B_\nu = \frac{2h\nu^3}{c^2} \frac{kT}{h\nu} = 2\frac{\nu^2}{c^2} kT$$

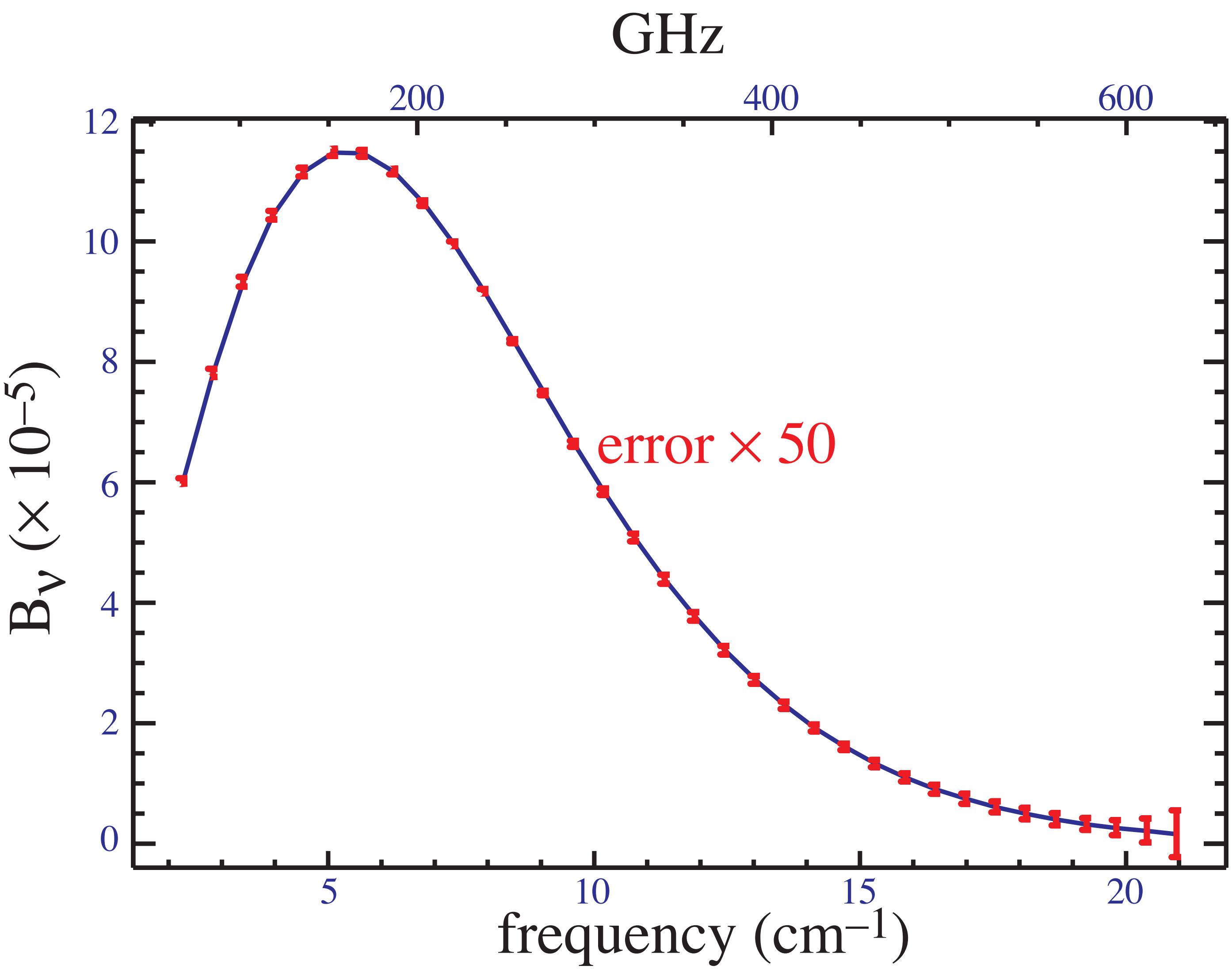
independent of $h$ (classical, many photon limit)

# Planck (Black Body) Distribution



# Cosmic Microwave Background

- FIRAS observations



# Planck (Black Body) Distribution

- $B_\nu \propto \nu^2$ would imply an ultraviolet catastrophy $S = \int B_\nu d\nu$
- At high frequencies $h\nu \gg kT$ (Wien tail)

$$\exp(h\nu/kT) - 1 \approx e^{h\nu/kT}$$

$$B_\nu = \frac{2h\nu^3}{c^2} e^{-h\nu/kT}$$

exhibits the Boltzmann suppression, particle nature of light

- Scaling with $T$

$$\frac{\partial B_\nu}{\partial T} = \frac{2h\nu^3}{c^2}\frac{\partial f}{\partial T} = \frac{2h\nu^3}{c^2}\left(\frac{-1}{(e^{h\nu/kT}-1)^2}\right)\frac{-h\nu}{kT^2} > 0$$

so that specific intensity at all $\nu$ increases with $T$

- Setting $\partial B_\nu/\partial\nu = 0$ defines the maximum $h\nu_{\rm max} = 2.82kT$

# Planck (Black Body) Distribution

- Surface Brightness

$$S = \int_0^\infty B_\nu d\nu = \frac{2h}{c^2} \int_0^\infty \frac{\nu^3}{e^{h\nu/kT} - 1} d\nu$$
$$= \frac{2h}{c^2} \left(\frac{kT}{h}\right)^4 \int_0^\infty dx \frac{x^3}{e^x - 1} = \frac{2\pi^4 k^4}{15 c^2 h^3} T^4 \equiv \frac{\sigma_B T^4}{\pi}$$

where $\sigma_B = 2\pi^5 k^4 / 15 c^2 h^3$ is the Stephan-Boltzmann constant and the $\pi$ accounts for the emergent flux at the radius $R$ of a uniform sphere where angles up to the $\pi/2$ tangent can be viewed

$$F \equiv \int S \cos\theta d\Omega = S \int_0^{2\pi} d\phi \int_0^{\pi/2} \cos\theta \sin\theta d\theta = \pi S$$

# Planck (Black Body) Distribution

- Energy density

$$\rho = \int \frac{B_\nu}{c} d\nu d\Omega = \frac{4\pi}{c}\frac{\sigma_B T^4}{\pi} = \frac{4\sigma_B}{c}T^4$$

- Number density

$$\begin{aligned} n &= 2\int d\Omega \frac{1}{c^3}\int \nu^2 d\nu \frac{1}{e^{h\nu/kT}-1} \\ &= \frac{8\pi}{c^3}\left(\frac{kT}{h}\right)^3 \int_0^\infty \frac{x^2 dx}{e^x - 1} = \frac{16\pi\zeta(3)}{c^3}\left(\frac{kT}{h}\right)^3 \\ &= \frac{2\zeta(3)}{\pi^2 c^3}\left(\frac{kT}{\hbar}\right)^3 \end{aligned}$$

where $\zeta(3) \approx 1.202$

# Astro 282

# Recombination

# Saha Equation

- What is the equilibrium ionization state of a gas at a given temperature?
- Hydrogen example: $e + p \leftrightarrow H + \gamma$
- Define $n_{\rm tot} = n_p + n_H$ and an ionization fraction $x_e \equiv n_p / n_{\rm tot}$

$$\frac{n_p n_e}{n_H n_{\rm tot}} = \frac{x_e^2}{1 - x_e}$$

- Number densities defined by distribution function in thermal equilibrium. $e$ and $p$ are non-relativistic at the eV energy scales of recombination
- Maxwell-Boltzmann distribution

$$f = e^{-(mc^2 - \mu)/kT} e^{-q^2/2mkT}$$

# Saha Equation

- Number density:

$$n = g\int \frac{d^3q}{(2\pi\hbar)^3} f = \frac{g e^{-(mc^2-\mu)/kT}}{2\pi^2\hbar^3}\int_0^\infty q^2 dq e^{-q^2/2mkT}$$

$$= g\frac{e^{-(mc^2-\mu)/kT}}{2\pi^2\hbar^3}(2mkT)^{3/2}\left[\int_0^\infty x^2 dx e^{-x^2} = \frac{\sqrt{\pi}}{4}\right] \quad (x = p/\sqrt{2mkT})$$

$$= g e^{-(mc^2-\mu)/kT}\left(\frac{mkT}{2\pi\hbar^2}\right)^{3/2}$$

- Hydrogen recombination ($n_{\rm tot} = n_p + n_H$)

$$n_p = g_p e^{-(m_p c^2-\mu_p)/kT}\left(m_p kT/2\pi\hbar^2\right)^{3/2}$$

$$n_e = g_e e^{-(m_e c^2-\mu_e)/kT}\left(m_e kT/2\pi\hbar^2\right)^{3/2}$$

$$n_H = g_H e^{-(m_H c^2-\mu_H)/kT}\left(m_H kT/2\pi\hbar^2\right)^{3/2}$$

# Saha Equation

- Hydrogen binding energy $B = 13.6$eV: $m_H = m_p + m_e - B/c^2$

$$\frac{n_p n_e}{n_H n_{\rm tot}} = \frac{x_e^2}{1-x_e} \approx \frac{g_p g_e}{g_H n_{\rm tot}} e^{-B/kT} e^{\mu_p + \mu_e - \mu_H} \left(\frac{m_e kT}{2\pi\hbar^2}\right)^{3/2}$$

- Spin degeneracy: spin 1/2 $g_p = 2$ , $g_e = 2$; $g_H = 4$ product
- Equilibrium $\mu_p + \mu_e = \mu_H$

$$\frac{x_e^2}{1-x_e} \approx \frac{1}{n_{\rm tot}} e^{-B/kT} \left(\frac{m_e kT}{2\pi\hbar^2}\right)^{3/2}$$

- Quadratic equation involving $T$ and the total density - explicit solution for $x_e(T)$
- Exponential dominant factor: ionization drops quickly as $kT$ drops below $B$ - exactly where the sharp transition occurs depends on the density $n_{\rm tot}$

# Saha Equation

- Photon perspective: compare photon number density at $T$ to $n_{\rm tot}$

$$n_\gamma = \frac{2\zeta(3)}{\pi^2\hbar^3}\left(\frac{kT}{c}\right)^3$$

$$\frac{x_e^2}{1-x_e} = \left(\frac{n_\gamma}{n_{\rm tot}}\right) e^{-B/kT}\frac{\pi^2\hbar^3}{2\zeta(3)}\left(\frac{c}{kT}\right)^3\left(\frac{m_e kT}{2\pi\hbar^2}\right)^{3/2}$$
$$= \left(\frac{n_\gamma}{n_{\rm tot}}\right) e^{-B/kT}\frac{\pi^{1/2}}{2^{5/2}\zeta(3)}\left(\frac{m_e c^2}{kT}\right)^{3/2}$$

- Photon-baryon ratio controls when recombination occurs: typically a very large number since baryon number is conserved $(\mu \ne 0)$ -a low baryon density medium is easy to keep ionized with the high energy photons in tail of the black body
- Cosmologically, recombination occurs at an energy scale of $kT \sim 0.3$eV

# Saha Equation

- Electron perspective: the relevant length scale is the ("thermal") de Broglie wavelength for a typical particle

$$m_e v^2 \sim kT, \quad q^2 \sim m_e^2 v^2 \sim (m_e kT)$$

$$\lambda_{Te} = \frac{h}{q} = \frac{h}{(2\pi m_e kT)^{1/2}} = \left(\frac{2\pi\hbar^2}{m_e kT}\right)^{1/2}$$

which is the factor in the Saha equation

$$\frac{x_e^2}{1-x_e} = \frac{1}{n_{\rm tot}\lambda_{Te}^3} e^{-B/kT}$$

$N_{Te} = n_e \lambda_{Te}^3 =$ # electrons in a de Broglie volume and is $\ll 1$ for non-degenerate matter

# Saha Equation

- Saha equation

$$\frac{x_e}{1-x_e} = \frac{1}{N_{Te}} e^{-B/kT}$$

- Electron chemical potential

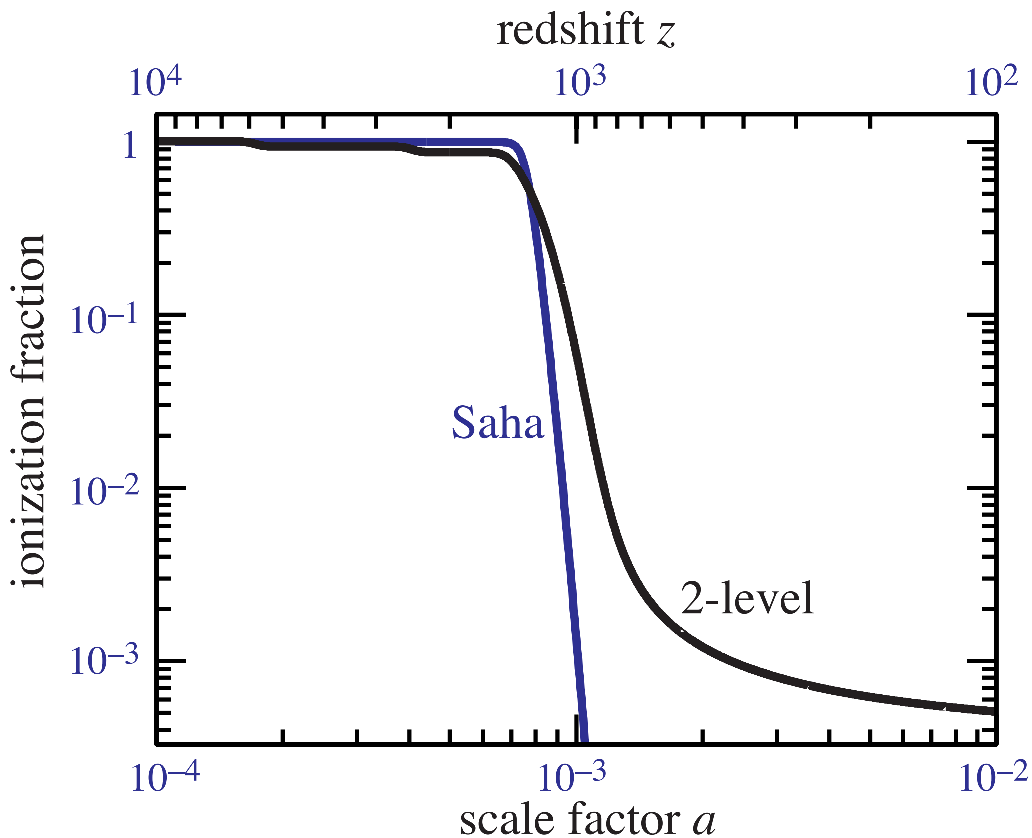
$$N_{Te} = 2e^{-(m_e c^2 - \mu_e)/kT}$$

$$\frac{x_e}{1-x_e} = \frac{1}{2} e^{-[B-(m_e c^2 - \mu_e)]/kT}$$

- Transition occurs when $B_{\rm eff} = B - m_e c^2 + \mu_e = kT$ - chemical potential or number density determines correction to $B \sim kT$ rule
- However equilibrium may not be maintained - 2 body interaction may not be rapid enough in low density environment - e.g. freezeout cosmologically

# Cosmic Recombination

- Rates insufficient to maintain equilibrium - due to Ly$\alpha$ opacity cosmic recombination relies on forbidden 2 photon decay and redshift



# Astro 282

# Acoustic Kinematics

# Temperature Fluctuations

- Observe blackbody radiation with a temperature that differs at $10^{-5}$ coming from the surface of last scattering, with distribution function (specific intensity $I_\nu = 4\pi\nu^3 f(\nu)$ each polarization)

$$f(\nu) = [\exp(2\pi\nu/T(\hat{\mathbf{n}})) - 1]^{-1}$$

- Decompose the temperature perturbation in spherical harmonics

$$T(\hat{\mathbf{n}}) = \sum_{\ell m} T_{\ell m} Y_{\ell m}(\hat{\mathbf{n}})$$

- For Gaussian random fluctuations, the statistical properties of the temperature field are determined by the power spectrum

$$\langle T^*_{\ell m} T_{\ell' m'} \rangle = \delta_{\ell\ell'} \delta_{mm'} C_\ell$$

where the $\delta$-function comes from statistical isotropy

# Spatial vs Angular Power

- Take the radiation distribution at last scattering to also be described by an isotropic temperature field $T(\mathbf{x})$ and recombination to be instantaneous

$$T(\hat{\mathbf{n}}) = \int dD \, T(\mathbf{x}) \delta(D - D_*)$$

where $D$ is the comoving distance and $D_*$ denotes recombination.

- Describe the temperature field by its Fourier moments

$$T(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} T(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}$$

with a power spectrum

$$\langle T(\mathbf{k})^* T(\mathbf{k}') \rangle = (2\pi)^3 \delta(\mathbf{k} - \mathbf{k}') P_T(k)$$

# Spatial vs Angular Power

- Note that the variance of the field

$$\langle T(\mathbf{x})T(\mathbf{x})\rangle = \int \frac{d^3k}{(2\pi)^3} P(k)$$
$$= \int d\ln k \, \frac{k^3 P(k)}{2\pi^2} \equiv \int d\ln k \, \Delta_T^2(k)$$

so it is more convenient to think in the log power spectrum $\Delta_T^2(k)$

- Temperature field

$$T(\hat{\mathbf{n}}) = \int \frac{d^3k}{(2\pi)^3} T(\mathbf{k}) e^{i\mathbf{k}\cdot D_* \hat{\mathbf{n}}}$$

- Expand out plane wave in spherical coordinates

$$e^{i\mathbf{k}D_*\cdot\hat{\mathbf{n}}} = 4\pi \sum_{\ell m} i^\ell j_\ell(kD_*) Y_{\ell m}^*(\mathbf{k}) Y_{\ell m}(\hat{\mathbf{n}})$$

# Spatial vs Angular Power

- Multipole moments

$$T_{\ell m} = \int \frac{d^3k}{(2\pi)^3} T(\mathbf{k}) 4\pi i^\ell j_\ell(kD_*) Y_{\ell m}(\mathbf{k})$$

- Power spectrum

$$\langle T^*_{\ell m} T_{\ell' m'} \rangle = \int \frac{d^3k}{(2\pi)^3} (4\pi)^2 (i)^{\ell-\ell'} j_\ell(kD_*) j_{\ell'}(kD_*) Y^*_{\ell m}(\mathbf{k}) Y_{\ell' m'}(\mathbf{k}) P_T(k)$$

$$= \delta_{\ell\ell'} \delta_{mm'} 4\pi \int d\ln k \, j_\ell^2(kD_*) \Delta_T^2(k)$$

with $\int_0^\infty j_\ell^2(x) d\ln x = 1/(2\ell(\ell+1))$, slowly varying $\Delta_T^2$

$$C_\ell = \frac{4\pi \Delta_T^2(\ell/D_*)}{2\ell(\ell+1)} = \frac{2\pi}{\ell(\ell+1)} \Delta_T^2(\ell/D_*)$$

so $\ell(\ell+1)C_\ell/2\pi = \Delta_T^2$ is commonly used log power

# Scale Invariant Fluctuations

- Scale invariant temperature fluctuations have $\Delta_T^2$ =const
- Equal contributions to the rms temperature fluctuation per decade in frequency $k$
- Observed angular fluctuations then have $\ell(\ell+1)C_\ell/2\pi$ = const
- Weaker assumption of scale free initial temperature fluctuations $\Delta_T^2 \propto k^{n-1}$, where $n$ is called the tilt.
- $n = 1$ is scale invariant for historical reasons.
- However fluctuations evolve from their initial conditions due to gravitational and pressure forces

# Thomson Scattering

- Thomson scattering of photons off of free electrons is the most important CMB process with a cross section (averaged over polarization states) of

$$\sigma_T = \frac{8\pi\alpha^2}{3m_e^2} = 6.65 \times 10^{-25}\mathrm{cm}^2$$

- Density of free electrons in a fully ionized $x_e = 1$ universe

$$n_e = (1 - Y_p/2)x_e n_b \approx 10^{-5}\Omega_b h^2 (1+z)^3 \mathrm{cm}^{-3}\,,$$

where $Y_p \approx 0.24$ is the Helium mass fraction, creates a high (comoving) Thomson opacity

$$\dot\tau \equiv n_e \sigma_T a$$

where dots are conformal time $\eta \equiv \int dt/a$ derivatives and $\tau$ is the optical depth.

# Tight Coupling Approximation

- Near recombination $z \approx 10^3$ and $\Omega_b h^2 \approx 0.02$, the (comoving) mean free path of a photon

$$\lambda_C \equiv \frac{1}{\dot{\tau}} \sim 2.5\mathrm{Mpc}$$

  small by cosmological standards!

- On scales $\lambda \gg \lambda_C$ photons are tightly coupled to the electrons by Thomson scattering which in turn are tightly coupled to the baryons by Coulomb interactions

- Specifically, their bulk velocities are defined by a single fluid velocity $v_\gamma = v_b$ and the photons carry no anisotropy in the rest frame of the baryons

- $\rightarrow$ No heat conduction or viscosity (anisotropic stress) in fluid

# Tight Coupling Approximation

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- $\rightarrow$ No heat conduction or viscosity (anisotropic stress) in fluid

# Zeroth Order Approximation

- Momentum density of a fluid is $(\rho + p)v$, where $p$ is the pressure
- Neglect the momentum density of the baryons

$$R \equiv \frac{(\rho_b + p_b)v_b}{(\rho_\gamma + p_\gamma)v_\gamma} = \frac{\rho_b + p_b}{\rho_\gamma + p_\gamma} = \frac{3\rho_b}{4\rho_\gamma}$$
$$\approx 0.6 \left(\frac{\Omega_b h^2}{0.02}\right)\left(\frac{a}{10^{-3}}\right)$$

since $\rho_\gamma \propto T^4$ is fixed by the CMB temperature $T = 2.73(1+z)$K – OK substantially before recombination

- Neglect radiation in the expansion

$$\frac{\rho_m}{\rho_r} = 3.6 \left(\frac{\Omega_m h^2}{0.15}\right)\left(\frac{a}{10^{-3}}\right)$$

# Number Continuity

- Photons are not created or destroyed. Without expansion

$$\dot{n}_\gamma + \nabla \cdot (n_\gamma \mathbf{v}_\gamma) = 0$$

but the expansion or Hubble flow causes $n_\gamma \propto a^{-3}$ or

$$\dot{n}_\gamma + 3n_\gamma \frac{\dot{a}}{a} + \nabla \cdot (n_\gamma \mathbf{v}_\gamma) = 0$$

- Linearize $\delta n_\gamma = n_\gamma - \bar{n}_\gamma$

$$(\delta n_\gamma)^\cdot = -3\delta n_\gamma \frac{\dot{a}}{a} - n_\gamma \nabla \cdot \mathbf{v}_\gamma$$

$$\left(\frac{\delta n_\gamma}{n_\gamma}\right)^\cdot = -\nabla \cdot \mathbf{v}_\gamma$$

# Continuity Equation

- Number density $n_\gamma \propto T^3$ so define temperature fluctuation $\Theta$

$$\frac{\delta n_\gamma}{n_\gamma} = 3\frac{\delta T}{T} \equiv 3\Theta$$

- Real space continuity equation

$$\dot{\Theta} = -\frac{1}{3}\nabla \cdot \mathbf{v}_\gamma$$

- Fourier space

$$\dot{\Theta} = -\frac{1}{3} i\mathbf{k} \cdot \mathbf{v}_\gamma$$

# Momentum Conservation

- No expansion: $\dot{\mathbf{q}} = \mathbf{F}$
- De Broglie wavelength stretches with the expansion

$$\dot{\mathbf{q}} + \frac{\dot{a}}{a}\mathbf{q} = \mathbf{F}$$

  for photons this the redshift, for non-relativistic particles expansion drag on peculiar velocities

- Collection of particles: momentum $\rightarrow$ momentum density $(\rho_\gamma + p_\gamma)\mathbf{v}_\gamma$ and force $\rightarrow$ pressure gradient

$$\begin{aligned}
[(\rho_\gamma + p_\gamma)\mathbf{v}_\gamma]^{\cdot} &= -4\frac{\dot{a}}{a}(\rho_\gamma + p_\gamma)\mathbf{v}_\gamma - \nabla p_\gamma \\
\frac{4}{3}\rho_\gamma \dot{\mathbf{v}}_\gamma &= \frac{1}{3}\nabla \rho_\gamma \\
\dot{\mathbf{v}}_\gamma &= -\nabla \Theta
\end{aligned}$$

# Euler Equation

- Fourier space

$$\dot{\mathbf{v}}_\gamma = -ik\Theta$$

- Pressure gradients (any gradient of a scalar field) generates a curl-free flow
- For convenience define velocity amplitude:

$$\mathbf{v}_\gamma \equiv -iv_\gamma \hat{\mathbf{k}}$$

- Euler Equation:

$$\dot{v}_\gamma = k\Theta$$

- Continuity Equation:

$$\dot{\Theta} = -\frac{1}{3}kv_\gamma$$

# Oscillator: Take One

- Combine these to form the simple harmonic oscillator equation

$$\ddot{\Theta} + c_s^2 k^2 \Theta = 0$$

where the adiabatic sound speed is defined through

$$c_s^2 \equiv \frac{\dot{p}_\gamma}{\dot{\rho}_\gamma}$$

here $c_s^2 = 1/3$ since we are photon-dominated

- General solution:

$$\Theta(\eta) = \Theta(0)\cos(ks) + \frac{\dot{\Theta}(0)}{kc_s}\sin(ks)$$

where the sound horizon is defined as $s \equiv \int c_s d\eta$

# Harmonic Extrema

- All modes are frozen in at recombination (denoted with a subscript $*$) yielding temperature perturbations of different amplitude for different modes. For the adiabatic (curvature mode) $\dot{\Theta}(0) = 0$

$$\Theta(\eta_*) = \Theta(0)\cos(ks_*)$$

- Modes caught in the extrema of their oscillation will have enhanced fluctuations

$$k_n s_* = n\pi$$

yielding a fundamental scale or frequency, related to the inverse sound horizon

$$k_A = \pi/s_*$$

and a harmonic relationship to the other extrema as $1 : 2 : 3 \ldots$

# Peak Location

- The fundmental physical scale is translated into a fundamental angular scale by simple projection according to the angular diameter distance $D_A$

$$\theta_A = \lambda_A / D_A$$
$$\ell_A = k_A D_A$$

- In a flat universe, the distance is simply $D_A = D \equiv \eta_0 - \eta_* \approx \eta_0$, the horizon distance, and $k_A = \pi / s_* = \sqrt{3}\pi / \eta_*$ so

$$\theta_A \approx \frac{\eta_*}{\eta_0}$$

- In a matter-dominated universe $\eta \propto a^{1/2}$ so $\theta_A \approx 1/30 \approx 2^\circ$ or

$$\ell_A \approx 200$$

# Curvature

- In a curved universe, the apparent or angular diameter distance is no longer the conformal distance $D_A = R\sin(D/R) \neq D$
- Objects in a closed universe are further than they appear! gravitational lensing of the background...
- Curvature scale of the universe must be substantially larger than current horizon
- Flat universe indicates critical density and implies missing energy given local measures of the matter density "dark energy"
- $D$ also depends on dark energy density $\Omega_{\rm DE}$ and equation of state $w = p_{\rm DE}/\rho_{\rm DE}$.
- Expansion rate at recombination or matter-radiation ratio enters into calculation of $k_A$.

# Doppler Effect

- Bulk motion of fluid changes the observed temperature via Doppler shifts

$$\left(\frac{\Delta T}{T}\right)_{\rm dop} = \hat{\mathbf{n}} \cdot \mathbf{v}_\gamma$$

- Averaged over directions

$$\left(\frac{\Delta T}{T}\right)_{\rm rms} = \frac{v_\gamma}{\sqrt{3}}$$

- Acoustic solution

$$\begin{aligned}\frac{v_\gamma}{\sqrt{3}} &= -\frac{\sqrt{3}}{k}\dot{\Theta} = \frac{\sqrt{3}}{k} k c_s \, \Theta(0) \sin(ks) \\ &= \Theta(0)\sin(ks)\end{aligned}$$

# Doppler Peaks?

- Doppler effect for the photon dominated system is of equal amplitude and $\pi/2$ out of phase: extrema of temperature are turning points of velocity
- Effects add in quadrature:

$$\left(\frac{\Delta T}{T}\right)^2 = \Theta^2(0)[\cos^2(ks) + \sin^2(ks)] = \Theta^2(0)$$

- No peaks in $k$ spectrum! However the Doppler effect carries an angular dependence that changes its projection on the sky $\hat{\mathbf{n}} \cdot \mathbf{v}_\gamma \propto \hat{\mathbf{n}} \cdot \hat{\mathbf{k}}$
- Coordinates where $\hat{\mathbf{z}} \parallel \hat{\mathbf{k}}$

$$Y_{10} Y_{\ell 0} \rightarrow Y_{\ell \pm 1\, 0}$$

recoupling $j_\ell' Y_{\ell 0}$: no peaks in Doppler effect

# Astro 282

# Acoustic Dynamics

# Restoring Gravity: Continuity

- Take a simple photon dominated system with gravity
- Continuity altered since a gravitational potential represents a stretching of the spatial fabric that dilutes number densities – formally a spatial curvature perturbation
- Think of this as a perturbation to the scale factor $a \to a(1+\Phi)$ so that the cosmogical redshift is generalized to

$$\frac{\dot a}{a} \to \frac{\dot a}{a} + \dot\Phi$$

$$(\delta n_\gamma)^{\cdot} = -3\delta n_\gamma \frac{\dot a}{a} - 3 n_\gamma \dot\Phi - n_\gamma \nabla \cdot \mathbf{v}_\gamma$$

so that the continuity equation becomes

$$\dot\Theta = -\frac{1}{3} k v_\gamma - \dot\Phi$$

# Restoring Gravity: Euler

- Gravitational force in momentum conservation $\mathbf{F} = -m\nabla\Psi$ generalized to momentum density modifies the Euler equation to

$$\dot{v}_\gamma = k(\Theta + \Psi)$$

- General relativity says that $\Phi$ and $\Psi$ are the relativistic analogues of the Newtonian potential and that $\Phi \approx -\Psi$.
- In our matter-dominated approximation, $\Phi$ represents matter density fluctuations through the cosmological Poisson equation

$$k^2\Phi = 4\pi G a^2 \rho_m \Delta_m$$

where the difference comes from the use of comoving coordinates for $k$ ($a^2$ factor), the removal of the background density into the background expansion $(\rho_m \Delta_m)$ and finally a coordinate subtlety that enters into the definition of $\Delta_m$

# Constant Potentials

- In the matter dominated epoch potentials are constant because infall generates velocities as $v_m \sim k\eta\Psi$
- Velocity divergence generates density perturbations as $\Delta_m \sim -k\eta v_m \sim -(k\eta)^2\Psi$
- And density perturbations generate potential fluctuations as $\Phi \sim \Delta_m/(k\eta)^2 \sim -\Psi$, keeping them constant. Note that because of the expansion, density perturbations must grow to keep potentials constant.
- Here we have used the Friedman equation $H^2 = 8\pi G\rho_m/3$ and $\eta = \int d\ln a/(aH) \sim 1/(aH)$
- More generally, if stress perturbations are negligible compared with density perturbations ( $\delta p \ll \delta\rho$ ) then potential will remain roughly constant – more specifically a variant called the Bardeen or comoving curvature $\zeta$ is constant

# Oscillator: Take Two

- Combine these to form the simple harmonic oscillator equation

$$\ddot{\Theta} + c_s^2 k^2 \Theta = -\frac{k^2}{3}\Psi - \ddot{\Phi}$$

- In a CDM dominated expansion $\dot{\Phi} = \dot{\Psi} = 0$. Also for photon domination $c_s^2 = 1/3$ so the oscillator equation becomes

$$\ddot{\Theta} + \ddot{\Psi} + c_s^2 k^2 (\Theta + \Psi) = 0$$

- Solution is just an offset version of the original

$$[\Theta + \Psi](\eta) = [\Theta + \Psi](0) \cos(ks)$$

- $\Theta + \Psi$ is also the observed temperature fluctuation since photons lose energy climbing out of gravitational potentials at recombination

# Effective Temperature

- Photons climb out of potential wells at last scattering
- Lose energy to gravitational redshifts
- Observed or effective temperature

$$\Theta + \Psi$$

- Effective temperature oscillates around zero with amplitude given by the initial conditions
- Note: initial conditions are set when the perturbation is outside of horizon, need inflation or other modification to matter-radiation FRW universe.
- GR says that initial temperature is given by initial potential

# Sachs-Wolfe Effect and the Magic 1/3

- A gravitational potential is a perturbation to the temporal coordinate [formally a gauge transformation]

$$\frac{\delta t}{t} = \Psi$$

- Convert this to a perturbation in the scale factor,

$$t = \int \frac{da}{aH} \propto \int \frac{da}{a\rho^{1/2}} \propto a^{3(1+w)/2}$$

where $w \equiv p/\rho$ so that during matter domination

$$\frac{\delta a}{a} = \frac{2}{3}\frac{\delta t}{t}$$

- CMB temperature is cooling as $T \propto a^{-1}$ so

$$\Theta + \Psi \equiv \frac{\delta T}{T} + \Psi = -\frac{\delta a}{a} + \Psi = \frac{1}{3}\Psi$$

# Baryon Loading

- Baryons add extra mass to the photon-baryon fluid
- Controlling parameter is the momentum density ratio:

$$R \equiv \frac{p_b + \rho_b}{p_\gamma + \rho_\gamma} \approx 30\Omega_b h^2 \left(\frac{a}{10^{-3}}\right)$$

of order unity at recombination

- Momentum density of the joint system is conserved

$$\begin{aligned}(\rho_\gamma + p_\gamma)v_\gamma + (\rho_b + p_b)v_b &\approx (p_\gamma + p_\gamma + \rho_b + \rho_\gamma)v_\gamma \\ &= (1 + R)(\rho_\gamma + p_\gamma)v_{\gamma b}\end{aligned}$$

where the controlling parameter is the momentum density ratio:

$$R \equiv \frac{p_b + \rho_b}{p_\gamma + \rho_\gamma} \approx 30\Omega_b h^2 \left(\frac{a}{10^{-3}}\right)$$

of order unity at recombination

# New Euler Equation

- Momentum density ratio enters as

$$[(1+R)(\rho_\gamma + p_\gamma)\mathbf{v}_{\gamma b}]^{\cdot} = -4\frac{\dot{a}}{a}(1+R)(\rho_\gamma + p_\gamma)\mathbf{v}_{\gamma b} - \nabla p_\gamma - (1+R)(\rho_\gamma + p_\gamma)\nabla\Psi$$

same as before except for $(1+R)$ terms so

$$[(1+R)v_{\gamma b}]^{\cdot} = k\Theta + (1+R)k\Psi$$

- Photon continuity remains the same

$$\dot{\Theta} = -\frac{k}{3}v_{\gamma b} - \dot{\Phi}$$

- Modification of oscillator equation

$$[(1+R)\dot{\Theta}]^{\cdot} + \frac{1}{3}k^2\Theta = -\frac{1}{3}k^2(1+R)\Psi - [(1+R)\dot{\Phi}]^{\cdot}$$

# Oscillator: Take Three

- Combine these to form the not-quite-so simple harmonic oscillator equation

$$c_s^2 \frac{d}{d\eta}(c_s^{-2}\dot{\Theta}) + c_s^2 k^2 \Theta = -\frac{k^2}{3}\Psi - c_s^2 \frac{d}{d\eta}(c_s^{-2}\dot{\Phi})$$

where $c_s^2 \equiv \dot{p}_{\gamma b}/\dot{\rho}_{\gamma b}$

$$c_s^2 = \frac{1}{3}\frac{1}{1+R}$$

- In a CDM dominated expansion $\dot{\Phi} = \dot{\Psi} = 0$ and the adiabatic approximation $\dot{R}/R \ll \omega = kc_s$

$$[\Theta + (1+R)\Psi](\eta) = [\Theta + (1+R)\Psi](0)\cos(ks)$$

# Baryon Peak Phenomenology

- Photon-baryon ratio enters in three ways
- Overall larger amplitude:

$$[\Theta + (1 + R)\Psi](0) = \frac{1}{3}(1 + 3R)\Psi(0)$$

- Even-odd peak modulation of effective temperature

$$[\Theta + \Psi]_{\mathrm{peaks}} = [\pm(1 + 3R) - 3R]\,\frac{1}{3}\Psi(0)$$

$$[\Theta + \Psi]_1 - [\Theta + \Psi]_2 = [-6R]\frac{1}{3}\Psi(0)$$

- Shifting of the sound horizon down or $\ell_A$ up

$$\ell_A \propto \sqrt{1 + R}$$

- Actual effects smaller since $R$ evolves

# Photon Baryon Ratio Evolution

- Oscillator equation has time evolving mass

$$c_s^2 \frac{d}{d\eta}(c_s^{-2}\dot{\Theta}) + c_s^2 k^2 \Theta = 0$$

- Effective mass is is $m_{\rm eff} = 3c_s^{-2} = (1 + R)$
- Adiabatic invariant

$$\frac{E}{\omega} = \frac{1}{2} m_{\rm eff} \omega A^2 = \frac{1}{2} 3 c_s^{-2} k c_s A^2 \propto A^2 (1 + R)^{1/2} = const.$$

- Amplitude of oscillation $A \propto (1 + R)^{-1/4}$ decays adiabatically as the photon-baryon ratio changes

# Oscillator: Take Three and a Half

- The not-quite-so simple harmonic oscillator equation is a forced harmonic oscillator

$$c_s^2 \frac{d}{d\eta}(c_s^{-2}\dot{\Theta}) + c_s^2 k^2 \Theta = -\frac{k^2}{3}\Psi - c_s^2 \frac{d}{d\eta}(c_s^{-2}\Phi)$$

  changes in the gravitational potentials alter the form of the acoustic oscillations

- If the forcing term has a temporal structure that is related to the frequency of the oscillation, this becomes a driven harmonic oscillator

- Term involving $\Psi$ is the ordinary gravitational force

- Term involving $\Phi$ involves the $\dot{\Phi}$ term in the continuity equation as a (curvature) perturbation to the scale factor

# Potential Decay

- Matter-to-radiation ratio

$$\frac{\rho_m}{\rho_r} \approx 24\Omega_m h^2 \left(\frac{a}{10^{-3}}\right)$$

  of order unity at recombination in a low $\Omega_m$ universe

- Radiation is not stress free and so impedes the growth of structure

$$k^2\Phi = 4\pi G a^2 \rho_r \Delta_r$$

  $\Delta_r \sim 4\Theta$ oscillates around a constant value, $\rho_r \propto a^{-4}$ so the Netwonian curvature decays.

- General rule: potential decays if the dominant energy component has substantial stress fluctuations, i.e. below the generalized sound horizon or Jeans scale

# Radiation Driving

- Decay is timed precisely to drive the oscillator - close to fully coherent

$$
\begin{aligned}
[\Theta + \Psi](\eta) &= [\Theta + \Psi](0) + \Delta\Psi - \Delta\Phi \\
&= \frac{1}{3}\Psi(0) - 2\Psi(0) = \frac{5}{3}\Psi(0)
\end{aligned}
$$

- $5\times$ the amplitude of the Sachs-Wolfe effect!
- Coherent approximation is exact for a photon-baryon fluid but reality is reduced to $\sim 4\times$ because of neutrino contribution to radiation
- Actual initial conditions are $\Theta + \Psi = \Psi/2$ for radiation domination but comparison to matter dominated SW correct

# External Potential Approach

- Solution to homogeneous equation

$$(1+R)^{-1/4}\cos(ks)\,, \qquad (1+R)^{-1/4}\sin(ks)$$

- Give the general solution for an external potential by propagating impulsive forces

$$(1+R)^{1/4}\Theta(\eta) = \Theta(0)\cos(ks) + \frac{\sqrt{3}}{k}\left[\dot\Theta(0) + \frac{1}{4}\dot R(0)\Theta(0)\right]\sin ks$$
$$+ \frac{\sqrt{3}}{k}\int_0^\eta d\eta'(1+R')^{3/4}\sin[ks - ks']F(\eta')$$

where

$$F = -\ddot\Phi - \frac{\dot R}{1+R}\dot\Phi - \frac{k^2}{3}\Psi$$

- Useful if general form of potential evolution is known

# Damping

- Tight coupling equations assume a perfect fluid: no viscosity, no heat conduction
- Fluid imperfections are related to the mean free path of the photons in the baryons

$$\lambda_C = \dot{\tau}^{-1} \quad \text{where} \quad \dot{\tau} = n_e \sigma_T a$$

is the conformal opacity to Thompson scattering

- Dissipation is related to the diffusion length: random walk approximation

$$\lambda_D = \sqrt{N} \lambda_C = \sqrt{\eta / \lambda_C}\, \lambda_C = \sqrt{\eta \lambda_C}$$

the geometric mean between the horizon and mean free path

- $\lambda_D / \eta_* \sim$ few %, so expect the peaks :> 3 to be affected by dissipation

# Equations of Motion

- Continuity

$$\dot{\Theta} = -\frac{k}{3}v_\gamma - \dot{\Phi}\,, \quad \dot{\delta}_b = -kv_b - 3\dot{\Phi}$$

where the photon equation remains unchanged and the baryons follow number conservation with $\rho_b = m_b n_b$

- Euler

$$\begin{aligned} \dot{v}_\gamma &= k(\Theta + \Psi) - \frac{k}{6}\pi_\gamma - \dot{\tau}(v_\gamma - v_b) \\ \dot{v}_b &= -\frac{\dot{a}}{a}v_b + k\Psi + \dot{\tau}(v_\gamma - v_b)/R \end{aligned}$$

where the photons gain an anisotropic stress term $\pi_\gamma$ from radiation viscosity and a momentum exchange term with the baryons and are compensated by the opposite term in the baryon Euler equation

# Viscosity

- Viscosity is generated from radiation streaming from hot to cold regions
- Expect

$$\pi_\gamma \sim v_\gamma \frac{k}{\dot\tau}$$

generated by streaming, suppressed by scattering in a wavelength of the fluctuation. Radiative transfer says

$$\pi_\gamma \approx 2 A_v v_\gamma \frac{k}{\dot\tau}$$

where $A_v = 16/15$

$$\dot v_\gamma = k(\Theta + \Psi) - \frac{k}{3} A_v \frac{k}{\dot\tau} v_\gamma$$

# Oscillator: Penultimate Take

- Adiabatic approximation ( $\omega \gg \dot{a}/a$)

$$\dot{\Theta} \approx -\frac{k}{3} v_\gamma$$

- Oscillator equation contains a $\dot{\Theta}$ damping term

$$c_s^2 \frac{d}{d\eta}(c_s^{-2}\dot{\Theta}) + \frac{k^2 c_s^2}{\dot{\tau}} A_v \dot{\Theta} + k^2 c_s^2 \Theta = -\frac{k^2}{3}\Psi - c_s^2 \frac{d}{d\eta}(c_s^{-2}\dot{\Phi})$$

- Heat conduction term similar in that it is proportional to $v_\gamma$ and is suppressed by scattering $k/\dot{\tau}$. Expansion of Euler equations to leading order in $k/\dot{\tau}$ gives

$$A_h = \frac{R^2}{1+R}$$

since the effects are only significant if the baryons are dynamically important

# Oscillator: Final Take

- Final oscillator equation

$$c_s^2 \frac{d}{d\eta}(c_s^{-2}\dot{\Theta}) + \frac{k^2 c_s^2}{\dot{\tau}}[A_v + A_h]\dot{\Theta} + k^2 c_s^2 \Theta = -\frac{k^2}{3}\Psi - c_s^2 \frac{d}{d\eta}(c_s^{-2}\dot{\Phi})$$

- Solve in the adiabatic approximation

$$\Theta \propto \exp(i \int \omega d\eta)$$

$$-\omega^2 + \frac{k^2 c_s^2}{\dot{\tau}}(A_v + A_h) i\omega + k^2 c_s^2 = 0 \qquad (1)$$

# Dispersion Relation

- Solve

$$\omega^2 = k^2 c_s^2 \left[ 1 + i \frac{\omega}{\dot\tau}(A_v + A_h) \right]$$

$$\omega = \pm k c_s \left[ 1 + \frac{i}{2}\frac{\omega}{\dot\tau}(A_v + A_h) \right]$$

$$= \pm k c_s \left[ 1 \pm \frac{i}{2}\frac{k c_s}{\dot\tau}(A_v + A_h) \right]$$

- Exponentiate

$$\begin{aligned} \exp(i \int \omega d\eta) &= e^{\pm iks} \exp[-k^2 \int d\eta \frac{1}{2} \frac{c_s^2}{\dot\tau}(A_v + A_h)] \\ &= e^{\pm iks} \exp[-(k/k_D)^2] \end{aligned} \tag{2}$$

- Damping is exponential under the scale $k_D$

# Diffusion Scale

- Diffusion wavenumber

$$k_D^{-2} = \int d\eta \frac{1}{\dot{\tau}} \frac{1}{6(1+R)} \left( \frac{16}{15} + \frac{R^2}{(1+R)} \right)$$

- Limiting forms

$$\lim_{R\to 0} k_D^{-2} = \frac{1}{6}\frac{16}{15} \int d\eta \frac{1}{\dot{\tau}}$$
$$\lim_{R\to \infty} k_D^{-2} = \frac{1}{6} \int d\eta \frac{1}{\dot{\tau}}$$

- Geometric mean between horizon and mean free path as expected from a random walk

$$\lambda_D = \frac{2\pi}{k_D} \sim \frac{2\pi}{\sqrt{6}} (\eta \dot{\tau}^{-1})^{1/2}$$

# Astro 282

# Idealized Data Analysis

# Gaussian Statistics

- Statistical isotropy says two-point correlation depends only on the power spectrum

$$\Theta(\hat{\mathbf{n}}) = \sum_{\ell m} \Theta_{\ell m} Y_{\ell m}(\hat{\mathbf{n}})$$

$$\langle \Theta^*_{\ell m} \Theta_{\ell' m'} \rangle = \delta_{\ell\ell'} \delta_{mm'} C_\ell^{\Theta\Theta}$$

- Reality of field says $\Theta_{\ell m} = (-1)^m \Theta_{\ell(-m)}$
- For a Gaussian random field, power spectrum defines all higher order statistics, e.g.

$$\langle \Theta_{\ell_1 m_1} \Theta_{\ell_2 m_2} \Theta_{\ell_3 m_3} \Theta_{\ell_4 m_4} \rangle$$
$$= (-1)^{m_1+m_2} \delta_{\ell_1 \ell_3} \delta_{m_1(-m_3)} \delta_{\ell_2 \ell_4} \delta_{m_2(-m_4)} C_{\ell_1}^{\Theta\Theta} C_{\ell_2}^{\Theta\Theta} + \text{all pairs}$$

# Idealized Statistical Errors

- Take a noisy estimator of the multipoles in the map

$$\hat{\Theta}_{\ell m} = \Theta_{\ell m} + N_{\ell m}$$

and take the noise to be statistically isotropic

$$\langle N^*_{\ell m} N_{\ell' m'} \rangle = \delta_{\ell\ell'}\delta_{mm'} C_\ell^{NN}$$

- Construct an unbiased estimator of the power spectrum $\langle \hat{C}_\ell^{\Theta\Theta} \rangle = C_\ell^{\Theta\Theta}$

$$\hat{C}_\ell^{\Theta\Theta} = \frac{1}{2\ell+1} \sum_{m=-l}^{l} \hat{\Theta}^*_{\ell m} \hat{\Theta}_{\ell m} - C_\ell^{NN}$$

- Variance in estimator

$$\langle \hat{C}_\ell^{\Theta\Theta} \hat{C}_\ell^{\Theta\Theta} \rangle - \langle \hat{C}_\ell^{\Theta\Theta} \rangle^2 = \frac{2}{2\ell+1} (C_\ell^{\Theta\Theta} + C_\ell^{NN})^2$$

# Cosmic and Noise Variance

- RMS in estimator is simply the total power spectrum reduced by $\sqrt{2/N_{\rm modes}}$ where $N_{\rm modes}$ is the number of $m$-mode measurements
- Even a perfect experiment where $C_\ell^{NN} = 0$ has statistical variance due to the Gaussian random realizations of the field. This cosmic variance is the result of having only one realization to measure.
- Noise variance is often approximated as white detector noise. Removing the beam to place the measurement on the sky

$$N_\ell^{\Theta\Theta} = \left(\frac{T}{d_T}\right)^2 e^{\ell(\ell+1)\sigma^2} = \left(\frac{T}{d_T}\right)^2 e^{\ell(\ell+1){\rm FWHM}^2/8\ln 2}$$

where $d_T$ can be thought of as a noise level per steradian of the temperature measurement, $\sigma$ is the Gaussian beam width, FWHM is the full width at half maximum of the beam

# Idealized Parameter Forecasts

- A crude propagation of errors is often useful for estimation purposes.
- Suppose $C_{\alpha\beta}$ describes the covariance matrix of the estimators for a given parameter set $\pi_\alpha$.
- Define $\mathbf{F} = \mathbf{C}^{-1}$ [formalized as the Fisher matrix later]. Making an infinitesimal transformation to a new set of parameters $p_\mu$

$$F_{\mu\nu} = \sum_{\alpha\beta} \frac{\partial \pi_\alpha}{\partial p_\mu} F_{\alpha\beta} \frac{\partial \pi_\beta}{\partial p_\nu}$$

- In our case $\pi_\alpha$ are the $C_\ell$ the covariance is diagonal and $p_\mu$ are cosmological parameters

$$F_{\mu\nu} = \sum_\ell \frac{2\ell+1}{2(C_\ell^{\Theta\Theta} + C_\ell^{NN})^2} \frac{\partial C_\ell^{\Theta\Theta}}{\partial p_\mu} \frac{\partial C_\ell^{\Theta\Theta}}{\partial p_\nu}$$

# Idealized Parameter Forecasts

- Polarization handled in same way (requires covariance)
- Fisher matrix represents a local approximation to the transformation of the covariance and hence is only accurate for well constrained directions in parameter space
- Derivatives evaluated by finite difference
- Fisher matrix identifies parameter degeneracies but only the local direction – i.e. all errors are ellipses not bananas

# Beyond Idealizations: Time Ordered Data

- For the data analyst the starting point is a string of “time ordered” data coming out of the instrument (post removal of systematic errors!)
- Begin with a model of the time ordered data as

$$d_t = P_{ti}\Theta_i + n_t$$

where $i$ denotes pixelized positions indexed by $i$, $d_t$ is the data in a time ordered stream indexed by $t$. Number of time ordered data will be of the order $10^{10}$ for a satellite! number of pixels $10^6 - 10^7$.

- The noise $n_t$ is drawn from a distribution with a known power spectrum

$$\langle n_t n_{t'} \rangle = C_{d,tt'}$$

# Pointing Matrix

- The pointing matrix $\mathbf{P}$ is the mapping between pixel space and the time ordered data
- Simplest incarnation: row with all zeros except one column which just says what point in the sky the telescope is pointing at that time

$$\mathbf{P} = \begin{pmatrix} 0 & 0 & 1 & \ldots & 0 \\ 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 1 & \ldots & 0 \end{pmatrix}$$

- More generally encorporates differencing, beam, rotation (for polarization)

# Maximum Likelihood Mapmaking

- What is the best estimator of the underlying map $\Theta_i$
- Likelihood function: the probability of getting the data given the theory $\mathcal{L} \equiv P[\mathrm{data}|\mathrm{theory}]$. In this case, the *theory* is the set of parameters $\Theta_i$.

$$\mathcal{L}_\Theta(d_t) = \frac{1}{(2\pi)^{N_t/2}\sqrt{\det \mathbf{C}_d}} \exp\left[-\frac{1}{2}\left(d_t - P_{ti}\Theta_i\right) C^{-1}_{d,tt'}\left(d_{t'} - P_{t'j}\Theta_j\right)\right] .$$

- Bayes theorem says that $P[\Theta_i|d_t]$, the probability that the temperatures are equal to $\Theta_i$ given the data, is proportional to the likelihood function times a *prior* $P(\Theta_i)$, taken to be uniform

$$P[\Theta_i|d_t] \propto P[d_t|\Theta_i] \equiv \mathcal{L}_\Theta(d_t)$$

# Maximum Likelihood Mapmaking

- Maximizing the likelihood of $\Theta_i$ is simple since the log-likelihood is quadratic.
- Differentiating the argument of the exponential with respect to $\Theta_i$ and setting to zero leads immediately to the estimator

$$\hat{\Theta}_i = C_{N,ij} P_{jt} C^{-1}_{d,tt'} d_{t'} \ ,$$

where $\mathbf{C}_N \equiv (\mathbf{P}^{\mathrm{tr}} \mathbf{C}_d^{-1} \mathbf{P})^{-1}$ is the covariance of the estimator

- Given the large dimension of the time ordered data, direct matrix manipulation is unfeasible. A key simplifying assumption is the stationarity of the noise, that $C_{d,tt'}$ depends only on $t - t'$ (temporal statistical homogeneity)

# Foregrounds

- Maximum likelihood mapmaking can be applied to the time streams of multiple observations frequencies $N_\nu$ and hence obtain multiple maps
- A cleaned CMB map can be obtained by modeling the maps as

$$\hat{\Theta}_i^\nu = A_i^\nu \Theta_i + n_i^\nu + f_i^\nu$$

  where $A_i^\nu = 1$ if all the maps are at the same resolution (otherwise, embed the beam as in the pointing matrix; $f_i^\nu$ is the noise contributed by the foregrounds
- Again, a map making problem. Given a covariance matrix for foregrounds noise (a prior from other data), same solution. Alternately, can derive weights from stats of the recovered maps
- 5 foregrounds: synchrotron, free-free, radio pt sources, at low frequencies and dust and IR pt sources at high frequencies.

# Power Spectrum

- The next step in the chain of inference is the power spectrum extraction. Here the correlation between pixels is modelled through the power spectrum

$$C_{S,ij} \equiv \langle \Theta_i \Theta_j \rangle = \sum_{\ell} \Delta^2_{T,\ell} W_{\ell,ij}$$

- Here $W_\ell$, the window function, is derived by writing down the expansion of $\Theta(\hat{\mathbf{n}})$ in harmonic space, including smoothing by the beam and pixelization
- For example in the simple case of a gaussian beam of width $\sigma$ it is proportional to the Legendre polynomial $P_\ell(\hat{\mathbf{n}}_i \cdot \hat{\mathbf{n}}_j)$ for the pixel separation multiplied by $b_\ell^2 \propto e^{-\ell(\ell+1)\sigma^2}$

# Band Powers

- In principle the underlying theory to extract from maximum likelihood is the power spectrum at every $\ell$
- However with a finite patch of sky, it is not possible to extract multipoles separated by $\Delta\ell < 2\pi/L$ where $L$ is the dimension of the survey
- So consider instead a theory parameterization of $\Delta^2_{T,\ell}$ constant in bands of $\Delta\ell$ chosen to match the survey forming a set of band powers $B_a$
- The likelihood of the bandpowers given the pixelized data is

$$\mathcal{L}_B(\Theta_i) = \frac{1}{(2\pi)^{N_p/2}\sqrt{\det \mathbf{C}_\Theta}} \exp\left( -\frac{1}{2}\Theta_i C^{-1}_{\Theta,ij}\Theta_j \right)$$

where $\mathbf{C}_\Theta = \mathbf{C}_S + \mathbf{C}_N$ and $N_p$ is the number of pixels in the map.

# Band Power Esitmation

- As before, $\mathcal{L}_B$ is Gaussian in the anisotropies $\Theta_i$, but in this case $\Theta_i$ are *not* the parameters to be determined; the theoretical parameters are the $B_a$, upon which the covariance matrix depends.
- The likelihood function is not Gaussian in the parameters, and there is no simple, analytic way to find the maximum likelihood bandpowers
- Iterative approach to maximizing the likelihood: take a trial point $B_a^{(0)}$ and improve estimate based a Newton-Rhapson approach to finding zeros

$$\hat{B}_a = \hat{B}_a^{(0)} + \hat{F}_{B,ab}^{-1} \frac{\partial \ln \mathcal{L}_B}{\partial B_b}$$

$$= \hat{B}_a^{(0)} + \frac{1}{2} \hat{F}_{B,ab}^{-1} \left( \Theta_i C_{\Theta,ij}^{-1} \frac{\partial C_{\Theta,jk}}{\partial B_b} C_{\Theta,kl}^{-1} \Theta_l - C_{\Theta,ij}^{-1} \frac{\partial C_{\Theta,ji}}{\partial B_b} \right) ,$$

# Fisher Matrix

- The expectation value of the local curvature is the Fisher matrix

$$
\begin{aligned}
F_{B,ab} &\equiv \left\langle -\frac{\partial^2 \ln \mathcal{L}_B}{\partial B_a \partial B_b} \right\rangle \\
&= \frac{1}{2} C^{-1}_{\Theta,ij} \frac{\partial C_{\Theta,jk}}{\partial B_a} C^{-1}_{\Theta,kl} \frac{\partial C_{\Theta,li}}{\partial B_b} \,.
\end{aligned}
$$

- This is a general statement: for a gaussian distribution the Fisher matrix

$$
F_{ab} = \frac{1}{2} \mathrm{Tr}[\mathbf{C}^{-1}\mathbf{C}_{,a}\mathbf{C}^{-1}\mathbf{C}_{,b}]
$$

- Kramer-Rao identity says that the best possible covariance matrix on a set of parameters is $\mathbf{C} = \mathbf{F}^{-1}$
- Thus, the iteration returns an estimate of the covariance matrix of the estimators $\mathbf{C}_B$

# Cosmological Parameters

- The probability distribution of the bandpowers given the cosmological parameters $c_i$ is not Gaussian but it is often an adequate approximation

$$\mathcal{L}_c(\hat{B}_a) \approx \frac{1}{(2\pi)^{N_c/2}\sqrt{\det \mathbf{C}_B}} \exp\left[-\frac{1}{2}(\hat{B}_a - B_a) C^{-1}_{B,ab} (\hat{B}_b - B_b)\right]$$

- Grid based approaches evaluate the likelihood in cosmological parameter space and maximize
- Faster approaches monte carlo the exploration of the likelihood space intelligently ("Monte Carlo Markov Chains")
- Since the number of cosmological parameters in the working model is $N_c \sim 10$ this represents a final radical compression of information in the original timestream which recall has up to $N_t \sim 10^{10}$ data points.

# MCMC

- Monte Carlo Markov Chain: a random walk in parameter space
- Start with a set of cosmological parameters $\mathbf{c}^m$, compute likelihood
- Take a random step in parameter space to $\mathbf{c}^{m+1}$ of size drawn from a multivariate Gaussian (a guess at the parameter covariance matrix) $\mathbf{C}_c$ (e.g. from the crude Fisher approximation. Compute likelihood.
- Draw a random number between 0,1 and if the likelihood ratio exceeds this value take the step (add to Markov chain); if not then do not take the step (add the original point to the Markov chain). Goto 3.

# MCMC

- With a complete chain of $N_M$ elements, compute the mean of the chain and its variance

$$\bar{c}_i = \frac{1}{N_M} \sum_{m=1}^{N_M} c_i^m$$

$$\sigma^2(c_i) = \frac{1}{N_M - 1} \sum_{m=1}^{N_M} (c_i^m - \bar{c}_i)^2$$

- Trick is in assuring burn in (not sensitive to initial point), step size, and convergence
- Usually requires running multiple chains. Typically tens of thousands of elements per chain.

# Radical Compression

- Started with time ordered data $\sim 10^{10}$ numbers for a satellite experiment
- Compressed to a map assuming a CMB spectrum (and time independent fluctuations) $\sim 10^7$ numbers
- Compressed to a power spectrum (Gaussian statistics) independent of $m$ (statistical isotropy) $\sim 10^3$ numbers
- Compressed to cosmological parameters (a cosmological model) $\sim 10^3$
- A factor of $10^9$ reduction in the representation. Nature is very efficient.

# Parameter Forecasts

- The Fisher matrix of the cosmological parameters becomes

$$F_{c,ij} = \frac{\partial B_a}{\partial c_i} C^{-1}_{B,ab} \frac{\partial B_b}{\partial c_j} \,.$$

which is the error propagation formula discussed above

- The Fisher matrix can be more accurately defined for an experiment by taking the pixel covariance and using the general formula for the Fisher matrix of gaussian data
- Corrects for edge effects with the approximate effect of

$$F_{ij} = \sum_\ell \frac{(2\ell+1) f_{\rm sky}}{2(C_\ell^{\Theta\Theta} + C_\ell^{NN})^2} \frac{\partial C_\ell^{\Theta\Theta}}{\partial c_i} \frac{\partial C_\ell^{\Theta\Theta}}{\partial c_j}$$

where the sky fraction $f_{\rm sky}$ quantifies the loss of independent modes due to the sky cut

# Astro 282

# Polarization

# Stokes Parameters

- Polarization state of radiation in direction $\hat{\mathbf{n}}$ described by the intensity matrix $\langle E_i(\hat{\mathbf{n}}) E_j^*(\hat{\mathbf{n}}) \rangle$, where $\mathbf{E}$ is the electric field vector and the brackets denote time averaging.
- As a hermitian matrix, it can be decomposed into the Pauli basis

$$
\begin{aligned}
\mathbf{P} &= C \left\langle \mathbf{E}(\hat{\mathbf{n}})\, \mathbf{E}^\dagger(\hat{\mathbf{n}}) \right\rangle \\
&= \Theta(\hat{\mathbf{n}}) \boldsymbol{\sigma}_0 + Q(\hat{\mathbf{n}})\, \boldsymbol{\sigma}_3 + U(\hat{\mathbf{n}})\, \boldsymbol{\sigma}_1 + V(\hat{\mathbf{n}})\, \boldsymbol{\sigma}_2\,,
\end{aligned}
$$

where

$$
\boldsymbol{\sigma}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\boldsymbol{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\boldsymbol{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}
\boldsymbol{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
$$

- Stokes parameters recovered as $\mathrm{Tr}(\sigma_i \mathbf{P})/2$

# Monochromatic Wave

- A pure monochromatic wave is fully polarized

$$\mathbf{E} = E_1\mathbf{e}_1 + E_2\mathbf{e}_2$$

where

$$E_{1,2} = \mathrm{Re}[A_{1,2}e^{i\phi_{1,2}}e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}]$$

- Implies $\Theta^2 = Q^2 + U^2 + V^2$
- However a finite bandwidth leads to a sum of components

$$\mathbf{E} = \sum_\alpha \mathbf{E}^\alpha$$

# Partial Polarization

- A signal of finite bandwidth is only partially polarized since the time averaging will destroy the correlation between the frequency components

$$\left\langle \mathbf{E}\,\mathbf{E}^\dagger \right\rangle = \sum_\alpha \left\langle \mathbf{E}^\alpha\,\mathbf{E}^{\alpha\dagger} \right\rangle$$

- Stokes parameters then add

$$\Theta = \sum_\alpha \Theta^\alpha, \quad Q = \sum_\alpha Q^\alpha, \quad U = \sum_\alpha U^\alpha, \quad V = \sum_\alpha V^\alpha$$

- Result is $\Theta^2 > Q^2 + U^2 + V^2$ (since $Q, U, V$ have either sign) or partially polarized radiation - like a mixed state in quantum mechanics)

# Linear Polarization

- $Q \propto \langle E_1 E_1^* \rangle - \langle E_2 E_2^* \rangle$, $U \propto \langle E_1 E_2^* \rangle + \langle E_2 E_1^* \rangle$.
- Counterclockwise rotation of axes by $\theta = 45°$

$$E_1 = (E_1' - E_2')/\sqrt{2}\,, \quad E_2 = (E_1' + E_2')/\sqrt{2}$$

- $U \propto \langle E_1' E_1'^* \rangle - \langle E_2' E_2'^* \rangle$, difference of intensities at $45°$ or $Q'$
- More generally, $\mathbf{P}$ transforms as a tensor under rotations and

$$Q' = \cos(2\theta) Q + \sin(2\theta) U$$
$$U' = -\sin(2\theta) Q + \cos(2\theta) U$$

- or

$$Q' \pm iU' = e^{\mp 2i\theta}[Q \pm iU]$$

acquires a phase under rotation and is a spin $\pm 2$ object

# Coordinate Independent Representation

- Two directions: orientation of polarization and change in amplitude, i.e. $Q$ and $U$ in the basis of the Fourier wavevector (pointing with angle $\phi_l$) for small sections of sky are called $E$ and $B$ components

$$E(\mathbf{l}) \pm iB(\mathbf{l}) = -\int d\hat{\mathbf{n}}[Q'(\hat{\mathbf{n}}) \pm iU'(\hat{\mathbf{n}})]e^{-i\mathbf{l}\cdot\hat{\mathbf{n}}}$$

$$= -e^{\mp 2i\phi_l} \int d\hat{\mathbf{n}}[Q(\hat{\mathbf{n}}) \pm iU(\hat{\mathbf{n}})]e^{-i\mathbf{l}\cdot\hat{\mathbf{n}}}$$

- For the $B$-mode to not vanish, the polarization must point in a direction not related to the wavevector - not possible for density fluctuations in linear theory

- Generalize to all-sky: plane waves are eigenmodes of the Laplace operator on the tensor $\mathbf{P}$.

# Spin Harmonics

- Laplace Eigenfunctions

$$\nabla^2 {}_{\pm 2}Y_{\ell m}[\boldsymbol{\sigma}_3 \mp i\boldsymbol{\sigma}_1] = -[l(l+1) - 4]_{\pm 2}Y_{\ell m}[\boldsymbol{\sigma}_3 \mp i\boldsymbol{\sigma}_1]$$

- Spin $s$ spherical harmonics: orthogonal and complete

$$\int d\hat{\mathbf{n}}\, {}_sY_{\ell m}^*(\hat{\mathbf{n}})\, {}_sY_{\ell m}(\hat{\mathbf{n}}) = \delta_{\ell\ell'}\delta_{mm'}$$

$$\sum_{\ell m} {}_sY_{\ell m}^*(\hat{\mathbf{n}})\, {}_sY_{\ell m}(\hat{\mathbf{n}}') = \delta(\phi - \phi')\delta(\cos\theta - \cos\theta')$$

where the ordinary spherical harmonics are $Y_{\ell m} = {}_0Y_{\ell m}$

- Given in terms of the rotation matrix

$${}_sY_{\ell m}(\beta\alpha) = (-1)^m \sqrt{\frac{2\ell+1}{4\pi}} D^{\ell}_{-ms}(\alpha\beta 0)$$

# Statistical Representation

- All-sky decomposition

$$[Q(\hat{\mathbf{n}}) \pm iU(\hat{\mathbf{n}})] = \sum_{\ell m} [E_{\ell m} \pm iB_{\ell m}]_{\pm 2}Y_{\ell m}(\hat{\mathbf{n}})$$

- Power spectra

$$\langle E^*_{\ell m} E_{\ell m} \rangle = \delta_{\ell \ell'} \delta_{m m'} C_\ell^{EE}$$

$$\langle B^*_{\ell m} B_{\ell m} \rangle = \delta_{\ell \ell'} \delta_{m m'} C_\ell^{BB}$$

- Cross correlation

$$\langle E^*_{\ell m} E_{\ell m} \rangle = \delta_{\ell \ell'} \delta_{m m'} C_\ell^{\Theta E}$$

others vanish if parity is conserved

# Thomson Scattering

- Differential cross section

$$\frac{d\sigma}{d\Omega} = \frac{3}{8\pi} |\hat{\mathbf{E}}' \cdot \hat{\mathbf{E}}|^2 \sigma_T \ ,$$

where $\sigma_T = 8\pi\alpha^2/3m_e$ is the Thomson cross section, $\hat{\mathbf{E}}'$ and $\hat{\mathbf{E}}$ denote the incoming and outgoing directions of the electric field or polarization vector.

- Summed over angle and incoming polarization

$$\sum_{i=1,2} \int d\hat{\mathbf{n}}' \frac{d\sigma}{d\Omega} = \sigma_T$$

# Polarization Generation

- Heuristic: incoming radiation shakes an electron in direction of electric field vector $\hat{\mathbf{E}}'$
- Radiates photon with polarization also in direction $\hat{\mathbf{E}}'$
- But photon cannot be longitudinally polarized so that scattering into $90^\circ$ can only pass one polarization
- Linearly polarized radiation like polarization by reflection
- Unlike reflection of sunlight, incoming radiation is nearly isotropic
- Missing linear polarization supplied by scattering from direction orthogonal to original incoming direction
- Only quadrupole anisotropy generates polarization by Thomson scattering

# Acoustic Polarization

- Break down of tight-coupling leads to quadrupole anisotropy of

$$\pi_\gamma \approx \frac{k}{\dot\tau} v_\gamma$$

- Scaling $k_D = (\dot\tau/\eta_*)^{1/2} \rightarrow \dot\tau = k_D^2 \eta_*$
- Know: $k_D s_* \approx k_D \eta_* \approx 10$
- So:

$$\pi_\gamma \approx \frac{k}{k_D}\frac{1}{10} v_\gamma$$

$$\Delta_P \approx \frac{\ell}{\ell_D}\frac{1}{10}\Delta_T$$

# Acoustic Polarization

- Gradient of velocity is along direction of wavevector, so polarization is pure $E$-mode
- Velocity is $90°$ out of phase with temperature – turning points of oscillator are zero points of velocity:

$$\Theta + \Psi \propto \cos(ks); \quad v_\gamma \propto \sin(ks)$$

- Polarization peaks are at troughs of temperature power

# Cross Correlation

- Cross correlation of temperature and polarization

$$(\Theta + \Psi)(v_\gamma) \propto \cos(ks)\sin(ks) \propto \sin(2ks)$$

- Oscillation at twice the frequency
- Correlation: radial or tangential around hot spots
- Partial correlation: easier to measure if polarization data is noisy, harder to measure if polarization data is high $S/N$ or if bands do not resolve oscillations
- Good check for systematics and foregrounds
- Comparison of temperature and polarization is proof against features in initial conditions mimicking acoustic features

# Astro 282

# Formal CMB Theory & CMBFAST

# Boltzmann Equation

- CMB radiation is generally described by the phase space distribution function for each polarization state $f_a(\mathbf{x}, \mathbf{q}, \eta)$, where $\mathbf{x}$ is the comoving position and $\mathbf{q}$ is the photon momentum
- Boltzmann equation describes the evolution of the distribution function under gravity and collisions
- Low order moments of the Boltzmann equation are simply the covariant conservation equations
- Higher moments provide the closure condition to the conservation law (specification of stress tensor) and the CMB observable – fine scale anisotropy
- Higher moments mainly describe the simple geometry of source projection

# Liouville Equation

- In absence of scattering, the phase space distribution of photons is conserved along the propagation path
- Rewrite variables in terms of the photon propagation direction $\mathbf{q} = q\hat{\mathbf{n}}$, so $f_a(\mathbf{x}, \hat{\mathbf{n}}, q, \eta)$ and

$$\begin{aligned}
\frac{d}{d\eta} f_a(\mathbf{x}, \hat{\mathbf{n}}, q, \eta) &= 0 \\
&= \left( \frac{\partial}{\partial \eta} + \frac{d\mathbf{x}}{d\eta} \cdot \frac{\partial}{\partial \mathbf{x}} + \frac{d\hat{\mathbf{n}}}{d\eta} \cdot \frac{\partial}{\partial \hat{\mathbf{n}}} + \frac{dq}{d\eta} \cdot \frac{\partial}{\partial q} \right) f_a
\end{aligned}$$

- For simplicity, assume spatially flat universe $K = 0$ then $d\hat{\mathbf{n}}/d\eta = 0$ and $d\mathbf{x} = \hat{\mathbf{n}} d\eta$

$$\dot{f}_a + \hat{\mathbf{n}} \cdot \nabla f_a + \dot{q} \frac{\partial}{\partial q} f_a = 0$$

# Correspondence to Einstein Eqn.

- Geodesic equation gives the redshifting term

$$\frac{\dot{q}}{q} = -\frac{\dot{a}}{a} - \frac{1}{2} n^i n^j \dot{H}_{Tij} - \dot{H}_L + n^i \dot{B}_i - \hat{\mathbf{n}} \cdot \nabla A$$

- which is incorporated in the conservation and gauge transformation equations
- Stress energy tensor involves integrals over the distribution function the two polarization states

$$T^{\mu\nu} = \int \frac{d^3 q}{(2\pi)^3} \frac{q^\mu q^\nu}{E} (f_a + f_b)$$

- Components are simply the low order angular moments of the distribution function

# Angular Moments

- Define the angularly dependent temperature perturbation

$$\Theta(\mathbf{x}, \hat{\mathbf{n}}, \eta) = \frac{1}{4\rho_\gamma} \int \frac{q^3 dq}{2\pi^2} (f_a + f_b) - 1$$

and likewise for the linear polarization states $Q$ and $U$

- Decompose into normal modes: plane waves for spatial part and spherical harmonics for angular part

$$G_\ell^m(\mathbf{k}, \mathbf{x}, \hat{\mathbf{n}}) \equiv (-i)^\ell \sqrt{\frac{4\pi}{2\ell+1}} Y_\ell^m(\hat{\mathbf{n}}) \exp(i\mathbf{k}\cdot\mathbf{x})$$

$${}_{\pm 2}G_\ell^m(\mathbf{k}, \mathbf{x}, \hat{\mathbf{n}}) \equiv (-i)^\ell \sqrt{\frac{4\pi}{2\ell+1}} {}_{\pm 2}Y_\ell^m(\hat{\mathbf{n}}) \exp(i\mathbf{k}\cdot\mathbf{x})$$

- In a spatially curved universe generalize the plane wave part

# Normal Modes

- Temperature and polarization fields

$$\Theta(\mathbf{x}, \hat{\mathbf{n}}, \eta) = \int \frac{d^3k}{(2\pi)^3} \sum_{\ell m} \Theta_\ell^{(m)} G_\ell^m$$

$$[Q \pm iU](\mathbf{x}, \hat{\mathbf{n}}, \eta) = \int \frac{d^3k}{(2\pi)^3} \sum_{\ell m} [E_\ell^{(m)} \pm iB_\ell^{(m)}]_{\pm 2} G_\ell^m$$

- For each $\mathbf{k}$ mode, work in coordinates where $\mathbf{k} \parallel \mathbf{z}$ and so $m = 0$ represents scalar modes, $m = \pm 1$ vector modes, $m = \pm 2$ tensor modes, $|m| > 2$ vanishes. Since modes add incoherently and $Q \pm iU$ is invariant up to a phase, rotation back to a fixed coordinate system is trivial.

# Scalar, Vector, Tensor

- Normalization of modes is chosen so that the lowest angular mode for scalars, vectors and tensors are normalized in the same way as the mode function

$$G_0^0 = Q^{(0)} \quad G_1^0 = n^i Q_i^{(0)} \quad G_2^0 \propto n^i n^j Q_{ij}^{(0)}$$

$$G_1^{\pm 1} = n^i Q_i^{(\pm 1)} \quad G_2^{\pm 1} \propto n^i n^j Q_{ij}^{(\pm 1)}$$

$$G_2^{\pm 2} = n^i n^j Q_{ij}^{(\pm 2)}$$

where recall

$$\begin{aligned}
Q^{(0)} &= \exp(i\mathbf{k}\cdot\mathbf{x}) \\
Q_i^{(\pm 1)} &= \frac{-i}{\sqrt{2}}(\hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2)_i \exp(i\mathbf{k}\cdot\mathbf{x}) \\
Q_{ij}^{(\pm 2)} &= -\sqrt{\frac{3}{8}}(\hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2)_i(\hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2)_j \exp(i\mathbf{k}\cdot\mathbf{x})
\end{aligned}$$

# Geometrical Projection

- Main content of Liouville equation is purely geometrical and describes the projection of inhomogeneities into anisotropies
- Spatial gradient term hits plane wave:

$$\hat{\mathbf{n}} \cdot \nabla e^{i\mathbf{k}\cdot\mathbf{x}} = i\hat{\mathbf{n}} \cdot \mathbf{k} e^{i\mathbf{k}\cdot\mathbf{x}} = i\sqrt{\frac{4\pi}{3}} k Y_1^0(\hat{\mathbf{n}}) e^{i\mathbf{k}\cdot\mathbf{x}}$$

- Dipole term adds to angular dependence through the addition of angular momentum

$$\sqrt{\frac{4\pi}{3}} Y_1^0 Y_\ell^m = \frac{\kappa_\ell^m}{\sqrt{(2\ell+1)(2\ell-1)}} Y_{\ell-1}^m + \frac{\kappa_{\ell+1}^m}{\sqrt{(2\ell+1)(2\ell+3)}} Y_{\ell+1}^m$$

where $\kappa_\ell^m = \sqrt{\ell^2 - m^2}$ is given by Clebsch-Gordon coefficients.

# Temperature Hierarchy

- Absorb recoupling of angular momentum into evolution equation for normal modes

$$\dot{\Theta}_\ell^{(m)} = k \left[ \frac{\kappa_\ell^m}{2\ell+1} \Theta_{\ell-1}^{(m)} - \frac{\kappa_{\ell+1}^m}{2\ell+3} \Theta_{\ell+1}^{(m)} \right] - \dot{\tau} \Theta_\ell^{(m)} + S_\ell^{(m)}$$

  where $S_\ell^{(m)}$ are the gravitational (and later scattering sources; added scattering suppression of anisotropy)

- An originally isotropic $\ell = 0$ temperature perturbation will eventually become a high order anisotropy by “free streaming” or simple projection

- Original CMB codes solved the full hierarchy equations out to the $\ell$ of interest.

# Integral Solution

- Hierarchy equation simply represents geometric projection, exactly as we have seen before in the projection of temperature perturbations on the last scattering surface
- In general, the solution describes the decomposition of the source $S_\ell^{(m)}$ with its local angular dependence as seen at a distance $\mathbf{x} = D\hat{\mathbf{n}}$.
- Proceed by decomposing the angular dependence of the plane wave

$$e^{i\mathbf{k}\cdot\mathbf{x}} = \sum_\ell (-i)^\ell \sqrt{4\pi(2\ell+1)} j_\ell(kD) Y_\ell^0(\hat{\mathbf{n}})$$

- Recouple to the local angular dependence of $G_\ell^m$

$$G_{\ell_s}^m = \sum_\ell (-i)^\ell \sqrt{4\pi(2\ell+1)} \alpha_{\ell_s \ell}^{(m)}(kD) Y_\ell^m(\hat{\mathbf{n}})$$

# Integral Solution

- Projection kernels:

$$\ell_s = 0, \quad m = 0 \qquad\qquad \alpha_{0\ell}^{(0)} \equiv j_\ell$$

$$\ell_s = 1, \quad m = 0 \qquad\qquad \alpha_{1\ell}^{(0)} \equiv j'_\ell$$

- Integral solution:

$$\frac{\Theta_\ell^{(m)}(k,\eta_0)}{2\ell+1} = \int_0^{\eta_0} d\eta e^{-\tau} \sum_{\ell_s} S_{\ell_s}^{(m)} \alpha_{\ell_s \ell}^{(m)}(k(\eta_0 - \eta))$$

- Power spectrum:

$$C_\ell = \frac{2}{\pi} \int \frac{dk}{k} \sum_m \frac{k^3 \langle \Theta_\ell^{(m)*} \Theta_\ell^{(m)} \rangle}{(2\ell+1)^2}$$

- Solving for $C_\ell$ reduces to solving for the behavior of a handful of sources

# Polarization Hiearchy

- In the same way, the coupling of a gradient or dipole angular momentum to the spin harmonics leads to the polarization hiearchy:

$$\dot{E}_\ell^{(m)} = k\left[\frac{{}_2\kappa_\ell^m}{2\ell-1}E_{\ell-1}^{(m)} - \frac{2m}{\ell(\ell+1)}B_\ell^{(m)} - \frac{{}_2\kappa_{\ell+1}^m}{2\ell+3}\right] - \dot{\tau}E_\ell^{(m)} + \mathcal{E}_\ell^{(m)}$$

$$\dot{B}_\ell^{(m)} = k\left[\frac{{}_2\kappa_\ell^m}{2\ell-1}B_{\ell-1}^{(m)} + \frac{2m}{\ell(\ell+1)}B_\ell^{(m)} - \frac{{}_2\kappa_{\ell+1}^m}{2\ell+3}\right] - \dot{\tau}E_\ell^{(m)} + \mathcal{B}_\ell^{(m)}$$

  where ${}_2\kappa_\ell^m = \sqrt{(\ell^2 - m^2)(\ell^2-4)/\ell^2}$ is given by the Clebsch-Gordon coefficients and $\mathcal{E}$, $\mathcal{B}$ are the sources (scattering only).

- Note that for vectors and tensors $|m| > 0$ and $B$ modes may be generated from $E$ modes by projection. Cosmologically $\mathcal{B}_\ell^{(m)} = 0$

# Polarization Integral Solution

- Again, we can recouple the plane wave angular momentum of the source inhomogeneity to its local angular dependence directly

$$\frac{E_\ell^{(m)}(k,\eta_0)}{2\ell+1} = \int_0^{\eta_0} d\eta e^{-\tau} \mathcal{E}_{\ell_s}^{(m)} \epsilon_{\ell_s \ell}^{(m)}(k(\eta_0-\eta))$$

$$\frac{B_\ell^{(m)}(k,\eta_0)}{2\ell+1} = \int_0^{\eta_0} d\eta e^{-\tau} \mathcal{E}_{\ell_s}^{(m)} \beta_{\ell_s \ell}^{(m)}(k(\eta_0-\eta))$$

- The only source to the polarization is from the quadrupole anisotropy so we only need $\ell_s = 2$, e.g. for scalars

$$\epsilon_{2\ell}^{(0)}(x) = \sqrt{\frac{3}{8}\frac{(\ell+2)!}{(\ell-2)!}}\frac{j_\ell(x)}{x^2} \qquad \beta_{2\ell}^{(0)} = 0$$

# Truncated Hierarchy

- CMBFast uses the integral solution and relies on a fast $j_\ell$ generator
- However sources are not external to system and are defined through the Boltzmann hierarchy itself
- Solution: recall that we used this technique in the tight coupling regime by applying a closure condition from tight coupling
- CMBFast extends this idea by solving a truncated hierarchy of equations, e.g. out to $\ell = 25$ with non-reflecting boundary conditions

# Thomson Collision Term

- Full Boltzmann equation

$$\frac{d}{d\eta} f_{a,b} = C[f_a, f_b]$$

- Collision term describes the scattering out of and into a phase space element
- Thomson collision based on differential cross section

$$\frac{d\sigma}{d\Omega} = \frac{3}{8\pi} |\hat{\mathbf{E}}' \cdot \hat{\mathbf{E}}|^2 \sigma_T \, ,$$

where $\hat{\mathbf{E}}'$ and $\hat{\mathbf{E}}$ denote the incoming and outgoing directions of the electric field or polarization vector.

# Scattering Calculation

- Start in the electron rest frame and in a coordinate system fixed by the scattering plane, spanned by incoming and outgoing directional vectors $-\hat{\mathbf{n}}' \cdot \hat{\mathbf{n}} = \cos\beta$, where $\beta$ is the scattering angle
- $\Theta_{||}$: in-plane polarization state; $\Theta_{\perp}$: $\perp$-plane polarization state
- Transfer probability (constant set by $\dot{\tau}$)

$$\Theta_{||} \propto \cos^2\beta\, \Theta'_{||}, \qquad \Theta_{\perp} \propto \Theta'_{\perp}$$

- and with the $45°$ axes as

$$\hat{\mathbf{E}}_1 = \frac{1}{\sqrt{2}}(\hat{\mathbf{E}}_{||} + \hat{\mathbf{E}}_{\perp}), \qquad \hat{\mathbf{E}}_2 = \frac{1}{\sqrt{2}}(\hat{\mathbf{E}}_{||} - \hat{\mathbf{E}}_{\perp})$$

# Stokes Parameters

- Define the temperature in this basis

$$
\begin{aligned}
\Theta_1 &\propto |\hat{\mathbf{E}}_1 \cdot \hat{\mathbf{E}}_1|^2 \Theta'_1 + |\hat{\mathbf{E}}_1 \cdot \hat{\mathbf{E}}_2|^2 \Theta'_2 \\
&\propto \frac{1}{4}(\cos\beta + 1)^2 \Theta'_1 + \frac{1}{4}(\cos\beta - 1)^2 \Theta'_2 \\
\Theta_2 &\propto |\hat{\mathbf{E}}_2 \cdot \hat{\mathbf{E}}_2|^2 \Theta'_2 + |\hat{\mathbf{E}}_2 \cdot \hat{\mathbf{E}}_1|^2 \Theta'_1 \\
&\propto \frac{1}{4}(\cos\beta + 1)^2 \Theta'_2 + \frac{1}{4}(\cos\beta - 1)^2 \Theta'_1
\end{aligned}
$$

or $\Theta_1 - \Theta_2 \propto \cos\beta(\Theta'_1 - \Theta'_2)$

- Define $\Theta$, $Q$, $U$ in the scattering coordinates

$$
\Theta \equiv \frac{1}{2}(\Theta_\parallel + \Theta_\perp), \quad Q \equiv \frac{1}{2}(\Theta_\parallel - \Theta_\perp), \quad U \equiv \frac{1}{2}(\Theta_1 - \Theta_2)
$$

# Scattering Matrix

- Transfer of Stokes states, e.g.

$$\Theta = \frac{1}{2}(\Theta_{\parallel} + \Theta_{\perp}) \propto \frac{1}{4}(\cos^2\beta + 1)\Theta' + \frac{1}{4}(\cos^2\beta - 1)Q'$$

- Transfer matrix of Stokes state $\mathbf{T} \equiv (\Theta, Q + iU, Q - iU)$

$$\mathbf{T} \propto \mathbf{S}(\beta)\mathbf{T}'$$

$$\mathbf{S}(\beta) = \frac{3}{4}\begin{pmatrix} \cos^2\beta + 1 & -\frac{1}{2}\sin^2\beta & -\frac{1}{2}\sin^2\beta \\ -\frac{1}{2}\sin^2\beta & \frac{1}{2}(\cos\beta + 1)^2 & \frac{1}{2}(\cos\beta - 1)^2 \\ -\frac{1}{2}\sin^2\beta & \frac{1}{2}(\cos\beta - 1)^2 & \frac{1}{2}(\cos\beta + 1)^2 \end{pmatrix}$$

normalization factor of 3 is set by photon conservation in scattering

# Scattering Matrix

- Transform to a fixed basis, by a rotation of the incoming and outgoing states $\mathbf{T} = \mathbf{R}(\psi)\mathbf{T}$ where

$$\mathbf{R}(\psi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-2i\psi} & 0 \\ 0 & 0 & e^{2i\psi} \end{pmatrix}$$

giving the scattering matrix

$$\mathbf{R}(-\gamma)\mathbf{S}(\beta)\mathbf{R}(\alpha) =$$

$$\frac{1}{2}\sqrt{\frac{4\pi}{5}} \begin{pmatrix} Y_2^0(\beta,\alpha) + 2\sqrt{5}Y_0^0(\beta,\alpha) & -\sqrt{\frac{3}{2}}Y_2^{-2}(\beta,\alpha) & -\sqrt{\frac{3}{2}}Y_2^2(\beta,\alpha) \\ -\sqrt{6}\,{}_2Y_2^0(\beta,\alpha)e^{2i\gamma} & 3\,{}_2Y_2^{-2}(\beta,\alpha)e^{2i\gamma} & 3\,{}_2Y_2^2(\beta,\alpha)e^{2i\gamma} \\ -\sqrt{6}\,{}_{-2}Y_2^0(\beta,\alpha)e^{-2i\gamma} & 3\,{}_{-2}Y_2^{-2}(\beta,\alpha)e^{-2i\gamma} & 3\,{}_{-2}Y_2^2(\beta,\alpha)e^{-2i\gamma} \end{pmatrix}$$

# Addition Theorem for Spin Harmonics

- Spin harmonics are related to rotation matrices as

$$
{}_sY_\ell^m(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}}\mathcal{D}^\ell_{-ms}(\phi,\theta,0)
$$

Note: for explicit evaluation sign convention differs from usual (e.g. Jackson) by $(-1)^m$

- Multiplication of rotations

$$
\sum_{m''}\mathcal{D}^\ell_{mm''}(\alpha_2,\beta_2,\gamma_2)\mathcal{D}^\ell_{m''m}(\alpha_1,\beta_1,\gamma_1) = \mathcal{D}^\ell_{mm'}(\alpha,\beta,\gamma)
$$

- Implies

$$
\sum_m {}_{s_1}Y_\ell^{m*}(\theta',\phi')\ {}_{s_2}Y_\ell^m(\theta,\phi) = (-1)^{s_1-s_2}\sqrt{\frac{2\ell+1}{4\pi}}\ {}_{s_2}Y_\ell^{-s_1}(\beta,\alpha)e^{is_2\gamma}
$$

# Sky Basis

- Scattering into the state (rest frame)

$$C_{\rm in}[\mathbf{T}] = \dot\tau \int \frac{d\hat{\mathbf{n}}'}{4\pi} \mathbf{R}(-\gamma)\mathbf{S}(\beta)\mathbf{R}(\alpha)\mathbf{T}(\hat{\mathbf{n}}') \,,$$

$$= \dot\tau \int \frac{d\hat{\mathbf{n}}'}{4\pi} (\Theta', 0, 0) + \frac{1}{10}\dot\tau \int d\hat{\mathbf{n}}' \sum_{m=-2}^{2} \mathbf{P}^{(m)}(\hat{\mathbf{n}}, \hat{\mathbf{n}}')\mathbf{T}(\hat{\mathbf{n}}') \,.$$

where the quadrupole coupling term is $\mathbf{P}^{(m)}(\hat{\mathbf{n}}, \hat{\mathbf{n}}') =$

$$\begin{pmatrix} Y_2^{m*}(\hat{\mathbf{n}}')\, Y_2^m(\hat{\mathbf{n}}) & -\sqrt{\frac{3}{2}}\, {}_2Y_2^{m*}(\hat{\mathbf{n}}')\, Y_2^m(\hat{\mathbf{n}}) & -\sqrt{\frac{3}{2}}\, {}_{-2}Y_2^{m*}(\hat{\mathbf{n}}')\, Y_2^m(\hat{\mathbf{n}}) \\ -\sqrt{6} Y_2^{m*}(\hat{\mathbf{n}}')\, {}_2Y_2^m(\hat{\mathbf{n}}) & 3\, {}_2Y_2^{m*}(\hat{\mathbf{n}}')\, {}_2Y_2^m(\hat{\mathbf{n}}) & 3\, {}_{-2}Y_2^{m*}(\hat{\mathbf{n}}')\, {}_2Y_2^m(\hat{\mathbf{n}}) \\ -\sqrt{6} Y_2^{m*}(\hat{\mathbf{n}}')\, {}_{-2}Y_2^m(\hat{\mathbf{n}}) & 3\, {}_2Y_2^{m*}(\hat{\mathbf{n}}')\, {}_{-2}Y_2^m(\hat{\mathbf{n}}) & 3\, {}_{-2}Y_2^{m*}(\hat{\mathbf{n}}')\, {}_{-2}Y_2^m(\hat{\mathbf{n}}) \end{pmatrix},$$

expression uses angle addition relation above. We call this term $C_Q$.

# Scattering Matrix

- Full scattering matrix involves difference of scattering into and out of state

$$C[\mathbf{T}] = C_{\rm in}[\mathbf{T}] - C_{\rm out}[\mathbf{T}]$$

- In the electron rest frame

$$C[\mathbf{T}] = \dot\tau \int \frac{d\hat{\mathbf{n}}'}{4\pi}(\Theta', 0, 0) - \dot\tau \mathbf{T} + C_Q[\mathbf{T}]$$

which describes isotropization in the rest frame. All moments have $e^{-\tau}$ suppression except for isotropic temperature $\Theta_0$. Transformation into the background frame simply induces a dipole term

$$C[\mathbf{T}] = \dot\tau \left( \hat{\mathbf{n}} \cdot \mathbf{v}_b + \int \frac{d\hat{\mathbf{n}}'}{4\pi}\Theta', 0, 0 \right) - \dot\tau \mathbf{T} + C_Q[\mathbf{T}]$$

# Source Terms

- Temperature source terms $S_l^{(m)}$ (rows $\pm|m|$; flat assumption

$$\begin{pmatrix} \dot\tau\Theta_0^{(0)} - \dot H_L^{(0)} & \dot\tau v_b^{(0)} + \dot B^{(0)} & \dot\tau P^{(0)} - \frac{2}{3}\dot H_T^{(0)} \\ 0 & \dot\tau v_b^{(\pm1)} + \dot B^{(\pm1)} & \dot\tau P^{(\pm1)} - \frac{\sqrt{3}}{3}\dot H_T^{(\pm1)} \\ 0 & 0 & \dot\tau P^{(\pm2)} - \dot H_T^{(\pm2)} \end{pmatrix}$$

where

$$P^{(m)} \equiv \frac{1}{10}(\Theta_2^{(m)} - \sqrt{6}E_2^{(m)})$$

- Polarization source term

$$\mathcal{E}_\ell^{(m)} = -\dot\tau\sqrt{6}P^{(m)}\delta_{\ell,2}$$

$$\mathcal{B}_\ell^{(m)} = 0$$

# Astro 282

# Secondaries

# Reionization

- Ionization depth during reionization

$$\tau(z) = \int d\eta n_e \sigma_T a = \int d\ln a \frac{n_e \sigma_T}{H(a)} \propto (\Omega_b h^2)(\Omega_m h^2)^{-1/2}(1+z)^{3/2}$$

$$= \left(\frac{\Omega_b h^2}{0.02}\right) \left(\frac{\Omega_m h^2}{0.15}\right)^{-1/2} \left(\frac{1+z}{61}\right)^{3/2}$$

- Quasars say $z_{ri} \geq 7$ so $\tau > 0.04$
- During reionization, cosmic quadrupole of $\sim 30\mu$K from the Sachs-Wolfe effect scatters into $E$-polarization
- Few percent optical depth leads to fraction of a $\mu$K signal
- Peaks at horizon scale at recombination: quadrupole source $j_2(kD_*)$ maximal at $kD_* \approx k\eta \approx 2$

# Breaking degeneracies

- First objects, breaking degeneracy of initial amplitude vs optical depth in the peak heights

$$C_\ell \propto e^{-2\tau}$$

  only below horizon scale at reionization

- Breaks degeneracies in angular diameter distance by removing an ambiguity for ISW-dark energy measure, helps in $\Omega_{DE} - w_{DE}$ plane

# Gravitational Wave

- Gravitational waves produce a quadrupolar distortion in the temperature of the CMB like effect on a ring of test particles
- Like ISW effect, source is a metric perturbation with time dependent amplitude
- After recombination, is a source of observable temperature anisotropy – but is therefore confined to low order multipoles
- Generated during inflation by quandum fluctuations

# Gravitational Wave Polarization

- In the tight coupling regime, quadrupole anisotropy suppressed by scattering

$$\pi_\gamma \approx \frac{\dot h}{\dot \tau}$$

- Since gravitational waves oscillate and decay at horizon crossing, the polarization peaks at the horizon scale at recombination not the damping scale
- More distinct signature in the $B$-mode polarization since symmetry of plane wave is broken by the transverse nature of gravity wave polarization

# Secondary Anisotropy

- CMB photons traverse the large-scale structure of the universe from $z = 1000$ to the present.
- With the nearly scale-invariant adiabatic fluctuations observed in the CMB, structures form from the bottom up, i.e. small scales first, a.k.a. hierarchical structure formation.
- First objects reionize the universe between $z \sim 7 - 30$
- Main sources of secondary anisotropy
- Gravitational: Integrated Sachs-Wolfe effect (gravitational redshift) and gravitational lensing
- Scattering: peak suppression, large-angle polarization, Doppler effect(s), inverse Compton scattering

# Transfer Function

- Transfer function transfers the initial Newtonian curvature to its value today (linear response theory)

$$T(k) = \frac{\Phi(k, a=1)}{\Phi(k, a_{\rm init})} \frac{\Phi(k_{\rm norm}, a_{\rm init})}{\Phi(k_{\rm norm}, a=1)}$$

- Conservation of Bardeen curvature: Newtonian curvature is a constant when stress perturbations are negligible: above the horizon during radiation and dark energy domination, on all scales during matter domination
- When stress fluctuations dominate, perturbations are stabilized by the Jeans mechanism
- Hybrid Poisson equation: Newtonian curvature, comoving density perturbation $\Delta \equiv (\delta\rho/\rho)_{\rm com}$ implies $\Phi$ decays

$$(k^2 - 3K)\Phi = 4\pi G\rho\Delta \sim \eta^{-2}\Delta$$

# Transfer Function

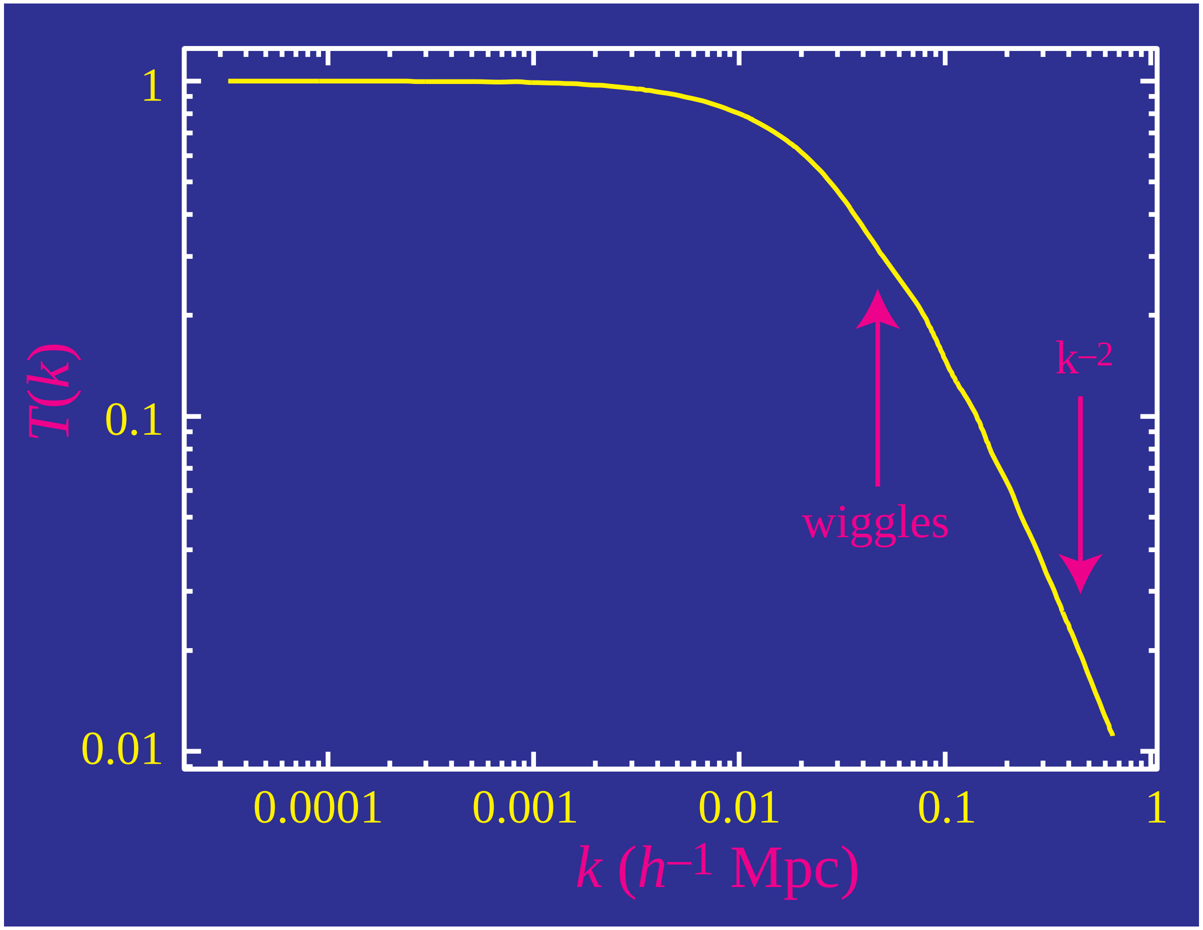
- Matter-radiation example: Jeans scale is horizon scale and $\Delta$ freezes into its value at horizon crossing $\Delta_H \approx \Phi_{\rm init}$
- Freezing of $\Delta$ stops at $\eta_{\rm eq}$

$$\Phi \sim (k\eta_{\rm eq})^{-2}\Delta_H \sim (k\eta_{\rm eq})^{-2}\Phi_{\rm init}$$

- Conventionally $k_{\rm norm}$ is chosen as a scale between the horizon at matter radiation equality and dark energy domination.
- Small correction since growth with a smooth radiation component is logarithmic not frozen
- Run CMBfast to get transfer function or use fits

# Transfer Function

- Transfer function has a $k^{-2}$ fall-off beyond $k_{\rm eq} \sim \eta_{\rm eq}^{-1}$




- Additional baryon wiggles are due to acoustic oscillations at recombination – an interesting means of measuring distances

# Growth Function

- Same physics applies to the dark energy dominated universe
- Under the dark energy sound horizon, dark energy density frozen. Potential decays at the same rate for all scales

$$g(a) = \frac{\Phi(k_{\rm norm}, a)}{\Phi(k_{\rm norm}, a_{\rm init})}$$

- Pressure growth suppression: $\delta \equiv \delta\rho_m/\rho_m \propto ag$

$$\frac{d^2 g}{d\ln a^2} + \left[\frac{5}{2} - \frac{3}{2} w(z)\Omega_{DE}(z)\right] \frac{dg}{d\ln a} + \frac{3}{2}[1 - w(z)]\Omega_{DE}(z) g = 0\,,$$

  where $w \equiv p_{DE}/\rho_{DE}$ and $\Omega_{DE} \equiv \rho_{DE}/(\rho_m + \rho_{DE})$ with initial conditions $g = 1$, $dg/d\ln a = 0$

- As $\Omega_{DE} \to 0$ $g =$const. is a solution. The other solution is the decaying mode, elimated by initial conditions

# ISW effect

- Potential decay leads to gravitational redshifts through the integrated Sachs-Wolfe effect
- Intrinsically a large effect since $2\Delta\Phi = 6\Psi_{\rm init}/3$
- But net redshift is integral along along line of sight

$$\frac{\Theta_\ell(k,\eta_0)}{2\ell+1} = \int_0^{\eta_0} d\eta e^{-\tau}[2\dot\Phi(k,\eta)] j_\ell(k(\eta_0-\eta))$$
$$= 2\Phi(k,\eta_{MD}) \int_0^{\eta_0} d\eta e^{-\tau} \dot g(D) j_\ell(kD)$$

- On small scales where $k \gg \dot g/g$, can pull source out of the integral

$$\int_0^{\eta_0} d\eta \dot g(D) j_\ell(kD) \approx \dot g(D=\ell/k)\frac{1}{k}\sqrt{\frac{\pi}{2\ell}}$$

evaluated at peak, where we have used $\int dx j_\ell(x) = \sqrt{\pi/2\ell}$

# ISW effect

- Power spectrum

$$C_\ell = \frac{2}{\pi}\int \frac{dk}{k}\frac{k^3\langle\Theta_\ell^*(k,\eta_0)\Theta_\ell(k,\eta_0)\rangle}{(2\ell+1)^2}$$
$$= \frac{2\pi^2}{l^3}\int d\eta D\dot{g}^2(\eta)\Delta_\Phi^2(\ell/D,\eta_{MD})$$

- Or $l^2C_l/2\pi \propto 1/\ell$ for scale invariant potential. This is the Limber equation in spherical coordinates. Projection of $3D$ power retains only the transverse piece. For a general dark energy model, add in the scale dependence of growth rate on large scales.
- Cancellation of redshifts and blueshifts as the photon traverses many crests and troughs of a small scale fluctuation during decay. Enhancement of the $\ell < 10$ multipoles. Difficult to extract from cosmic variance and galaxy. Current ideas: cross correlation with other tracers of structure

# Gravitational Lensing

- Lensing is a surface brightness conserving remapping of source to image planes by the gradient of the projected potential

$$\phi(\hat{\mathbf{n}}) = 2 \int_{\eta_*}^{\eta_0} d\eta \, \frac{(D_* - D)}{D \, D_*} \Phi(D\hat{\mathbf{n}}, \eta) \, .$$

  such that the fields are remapped as

$$x(\hat{\mathbf{n}}) \to x(\hat{\mathbf{n}} + \nabla\phi) \, ,$$

  where $x \in \{\Theta, Q, U\}$ temperature and polarization.

- Taylor expansion leads to product of fields and Fourier mode-coupling

# Flat-sky Treatment

- Talyor expand

$$\Theta(\hat{\mathbf{n}}) = \tilde{\Theta}(\hat{\mathbf{n}} + \nabla\phi)$$
$$= \tilde{\Theta}(\hat{\mathbf{n}}) + \nabla_i\phi(\hat{\mathbf{n}})\nabla^i\tilde{\Theta}(\hat{\mathbf{n}}) + \frac{1}{2}\nabla_i\phi(\hat{\mathbf{n}})\nabla_j\phi(\hat{\mathbf{n}})\nabla^i\nabla^j\tilde{\Theta}(\hat{\mathbf{n}}) + \ldots$$

- Fourier decomposition

$$\phi(\hat{\mathbf{n}}) = \int \frac{d^2l}{(2\pi)^2}\phi(\mathbf{l})e^{i\mathbf{l}\cdot\hat{\mathbf{n}}}$$
$$\tilde{\Theta}(\hat{\mathbf{n}}) = \int \frac{d^2l}{(2\pi)^2}\tilde{\Theta}(\mathbf{l})e^{i\mathbf{l}\cdot\hat{\mathbf{n}}}$$

# Flat-sky Treatment

- Mode coupling of harmonics

$$\Theta(\mathbf{l}) = \int d\hat{\mathbf{n}}\, \Theta(\hat{\mathbf{n}}) e^{-il\cdot\hat{\mathbf{n}}}$$
$$= \tilde{\Theta}(\mathbf{l}) - \int \frac{d^2\mathbf{l}_1}{(2\pi)^2} \tilde{\Theta}(\mathbf{l}_1) L(\mathbf{l}, \mathbf{l}_1)\,,$$

where

$$L(\mathbf{l}, \mathbf{l}_1) = \phi(\mathbf{l} - \mathbf{l}_1)\,(\mathbf{l} - \mathbf{l}_1)\cdot \mathbf{l}_1$$
$$+ \frac{1}{2}\int \frac{d^2\mathbf{l}_2}{(2\pi)^2} \phi(\mathbf{l}_2)\phi^*(\mathbf{l}_2 + \mathbf{l}_1 - \mathbf{l})\,(\mathbf{l}_2\cdot\mathbf{l}_1)(\mathbf{l}_2 + \mathbf{l}_1 - \mathbf{l})\cdot\mathbf{l}_1\,.$$

- Represents a coupling of harmonics separated by $L \approx 60$ peak of deflection power

# Power Spectrum

- Power spectra

$$\langle \Theta^*(\mathbf{l})\Theta(\mathbf{l}')\rangle = (2\pi)^2\delta(\mathbf{l}-\mathbf{l}')\, C_l^{\Theta\Theta}\,,$$

$$\langle \phi^*(\mathbf{l})\phi(\mathbf{l}')\rangle = (2\pi)^2\delta(\mathbf{l}-\mathbf{l}')\, C_l^{\phi\phi}\,,$$

becomes

$$C_l^{\Theta\Theta} = \left(1 - l^2 R\right)\tilde{C}_l^{\Theta\Theta} + \int \frac{d^2\mathbf{l}_1}{(2\pi)^2}\tilde{C}_{|\mathbf{l}-\mathbf{l}_1|}^{\Theta\Theta} C_{l_1}^{\phi\phi}[(\mathbf{l}-\mathbf{l}_1)\cdot\mathbf{l}_1]^2\,,$$

where

$$R = \frac{1}{4\pi}\int \frac{dl}{l}\, l^4 C_l^{\phi\phi}\,. \tag{3}$$

# Smoothing Power Spectrum

- If $\tilde{C}_l^{\Theta\Theta}$ slowly varying then two term cancel

$$\tilde{C}_l^{\Theta\Theta} \int \frac{d^2\mathbf{l}_1}{(2\pi)^2} C_l^{\phi\phi} (\mathbf{l} \cdot \mathbf{l}_1)^2 \approx l^2 R \tilde{C}_l^{\Theta\Theta} \, .$$

- So lensing acts to smooth features in the power spectrum. Smoothing kernel is $L \sim 60$ the peak of deflection power spectrum
- Because acoustic feature appear on a scale $l_A \sim 300$, smoothing is a subtle effect in the power spectrum.
- Lensing generates power below the damping scale which directly reflect power in deflections on the same scale

# Polarization Lensing

- Polarization field harmonics lensed similarly

$$[Q \pm iU](\hat{\mathbf{n}}) = -\int \frac{d^2 l}{(2\pi)^2} [E \pm iB](\mathbf{l}) e^{\pm 2i\phi_{\mathbf{l}}} e^{\mathbf{l}\cdot\hat{\mathbf{n}}}$$

so that

$$\begin{aligned}
[Q \pm iU](\hat{\mathbf{n}}) &= [\tilde{Q} \pm i\tilde{U}](\hat{\mathbf{n}} + \nabla\phi) \\
&\approx [\tilde{Q} \pm i\tilde{U}](\hat{\mathbf{n}}) + \nabla_i \phi(\hat{\mathbf{n}}) \nabla^i [\tilde{Q} \pm i\tilde{U}](\hat{\mathbf{n}}) \\
&\quad + \frac{1}{2} \nabla_i \phi(\hat{\mathbf{n}}) \nabla_j \phi(\hat{\mathbf{n}}) \nabla^i \nabla^j [\tilde{Q} \pm i\tilde{U}](\hat{\mathbf{n}})
\end{aligned}$$

# Polarization Power Spectra

- Carrying through the algebra

$$C_l^{EE} = (1 - l^2 R)\,\tilde{C}_l^{EE} + \frac{1}{2}\int \frac{d^2\mathbf{l}_1}{(2\pi)^2}[(\mathbf{l}-\mathbf{l}_1)\cdot\mathbf{l}_1]^2 C^{\phi\phi}_{|\mathbf{l}-\mathbf{l}_1|}$$
$$\times\,[(\tilde{C}_{l_1}^{EE} + \tilde{C}_{l_1}^{BB}) + \cos(4\varphi_{l_1})(\tilde{C}_{l_1}^{EE} - \tilde{C}_{l_1}^{BB})]\,,$$
$$C_l^{BB} = (1 - l^2 R)\,\tilde{C}_l^{BB} + \frac{1}{2}\int \frac{d^2\mathbf{l}_1}{(2\pi)^2}[(\mathbf{l}-\mathbf{l}_1)\cdot\mathbf{l}_1]^2 C^{\phi\phi}_{|\mathbf{l}-\mathbf{l}_1|}$$
$$\times\,[(\tilde{C}_{l_1}^{EE} + \tilde{C}_{l_1}^{BB}) - \cos(4\varphi_{l_1})(\tilde{C}_{l_1}^{EE} - \tilde{C}_{l_1}^{BB})]\,,$$
$$C_l^{\Theta E} = (1 - l^2 R)\,\tilde{C}_l^{\Theta E} + \int \frac{d^2\mathbf{l}_1}{(2\pi)^2}[(\mathbf{l}-\mathbf{l}_1)\cdot\mathbf{l}_1]^2 C^{\phi\phi}_{|\mathbf{l}-\mathbf{l}_1|}$$
$$\times\,\tilde{C}_{l_1}^{\Theta E}\cos(2\varphi_{l_1})\,,$$

- Lensing generates $B$-modes out of the acoustic polaraization $E$-modes contaminates gravitational wave signature if $E_i < 10^{16}$GeV.

# Reconstruction from the CMB

- Correlation between Fourier moments reflect lensing potential

$$\langle x(\mathbf{l})x'(\mathbf{l}')\rangle_{\rm CMB} = f_\alpha(\mathbf{l},\mathbf{l}')\phi(\mathbf{l}+\mathbf{l}')\ ,$$

  where $x \in$ temperature, polarization fields and $f_\alpha$ is a fixed weight that reflects geometry

- Each pair forms a noisy estimate of the potential or projected mass - just like a pair of galaxy shears
- Minimum variance weight all pairs to form an estimator of the lensing mass

# Scattering Secondaries

- Optical depth during reionization

$$\tau \approx 0.066 \left(\frac{\Omega_b h^2}{0.02}\right)\left(\frac{\Omega_m h^2}{0.15}\right)^{-1/2}\left(\frac{1+z}{10}\right)^{3/2}$$

- Anisotropy suppressed as $e^{-\tau}$. Integral solution

$$\frac{\Theta_\ell(k,\eta_0)}{2\ell+1} = \int_0^{\eta_0} d\eta e^{-\tau} S_0^{(0)} j_\ell(k(\eta_0-\eta)) + \ldots$$

- Isotropic (lare scale) fluctuations not supressed since suppression represents isotropization by scattering
- Quadrupole from the Sachs-Wolfe effect scatters into a large angle polarization bump

# Doppler Effects

- Velocity fields of $10^{-3}$ and optical depths of $10^{-2}$ would imply large Doppler effect due to reionization
- Limber approximation says only fluctuations transverse to line of sight survive
- In linear theory, transverse fluctuations have no line of sight velocity and so Doppler effect is highly suppressed.
- Beyond linear theory: modulate the optical depth in the transverse direction using density fluctuations or ionization fraction fluctuations. Generate a modulated Doppler effect
- Linear fluctuations: Vishniac effect; Clusters: kinetic SZ effect; ionization patches: inhomogeneous reionization effect

# Thermal SZ Effect

- Thermal velocities also lead to Doppler effect but first order contribution cancels because of random directions
- Residual effect is of order $v^2\tau \approx T_e/m_e\,\tau$ and can reach a sizeable level for clusters with $T_e \approx 10$keV.
- Raleigh-Jeans decrement and Wien enhancement described by second order collision term in Boltzmann equation: Kompaneets equation
- Clusters are rare objects so contribution to power spectrum suppressed, but may have been detected by CBI/BIMA: extremely sensitive to power spectrum normalization $\sigma_8$
- White noise on large-scales $(l < 2000)$, turnover as cluster profile is resolved