

Set 8:

# Compton Scattering

# Energy-Momentum Conservation

- Thomson scattering  $e_i + \gamma_i \rightarrow e_f + \gamma_f$  where the frequencies  $\omega_i = \omega_f$  (elastic scattering) cannot strictly be true
- Photons carry off  $E/c$  momentum and so to conserve momentum the electron must recoil.
- The electron then carries away some of the available energy and so the Thomson limit requires  $\hbar\omega_i/mc^2 \ll 1$  in the electron rest frame
- General case (arbitrary electron velocity)

$$P_i = \frac{E_i}{c}(1, \hat{\mathbf{n}}_i), \quad P_f = \frac{E_f}{c}(1, \hat{\mathbf{n}}_f), \quad \text{photon}$$
$$Q_i = \gamma_i m(c, \hat{\mathbf{v}}_i), \quad Q_f = \gamma_f m(c, \hat{\mathbf{v}}_f), \quad \text{electron}$$

# Energy-Momentum Conservation

- To work out change in photon energy consider Lorentz invariants:

$$Q_\mu Q^\mu = \gamma^2 m^2 (v^2 - c^2) = m^2 \frac{v^2 - c^2}{1 - v^2/c^2} = -m^2 c^2$$

$$P_\mu P^\mu = 0$$

so

$$(P_i + Q_i)_\mu (P_i + Q_i)^\mu = (P_f + Q_f)_\mu (P_f + Q_f)^\mu$$

$$2P_{i\mu} Q_i^\mu - m^2 c^2 = 2P_{f\mu} Q_f^\mu - m^2 c^2$$

$$P_i \cdot Q_i = P_f \cdot Q_f$$

also  $P_f \cdot [P_i + Q_i = P_f + Q_f]$

$$P_f \cdot P_i + P_f \cdot Q_i = P_f \cdot Q_f = P_i \cdot Q_i$$

# Energy-Momentum Conservation

- In three vector notation

$$\frac{E_i}{c} \frac{E_f}{c} (\hat{\mathbf{n}}_i \cdot \hat{\mathbf{n}}_f - 1) + \frac{E_f}{c} \gamma_i m (\hat{\mathbf{n}}_f \cdot \mathbf{v}_i - c) = \frac{E_i}{c} \gamma_i m (\hat{\mathbf{n}}_i \cdot \mathbf{v}_i - c)$$

- Introducing the scattering angle as  $\hat{\mathbf{n}}_i \cdot \hat{\mathbf{n}}_f = \cos \theta$  and auxiliary angles  $\hat{\mathbf{n}}_f \cdot \mathbf{v}_i = v_i \cos \alpha_f$  and  $\hat{\mathbf{n}}_i \cdot \mathbf{v}_i = v_i \cos \alpha_i$

$$E_i E_f (\cos \theta - 1) + E_f \gamma_i m c^2 (\beta_i \cos \alpha_f - 1) = E_i \gamma_i m c^2 (\beta_i \cos \alpha_i - 1)$$

$$E_f = \frac{E_i \gamma_i m c^2 (\beta_i \cos \alpha_i - 1)}{E_i (\cos \theta - 1) + \gamma_i m c^2 (\beta_i \cos \alpha_i - 1)}$$

- So the change in photon energy is given by

$$\frac{E_f}{E_i} = \frac{1 - \beta_i \cos \alpha_i}{1 - \beta_i \cos \alpha_f + \frac{E_i}{\gamma m c^2} (1 - \cos \theta)}$$

# Recoil Effect

- Two ways of changing the energy: Doppler boost  $\beta_i$  from incoming electron velocity and  $E_i$  comparable to  $\gamma mc^2$
- Take the incoming electron rest frame  $\beta_i = 0$  and  $\gamma = 1$

$$\left. \frac{E_f}{E_i} \right|_{\text{rest}} = \frac{1}{1 + \frac{E_i}{mc^2}(1 - \cos \theta)}$$

- Since  $-1 \leq \cos \theta \leq 1$ ,  $E_f \leq E_i$ , energy is lost from the recoil except for purely forward scattering via comparing the incoming photon energy  $E_i$  and the electron rest mass  $mc^2$
- The backwards scattering limit is easy to derive

$$|\mathbf{q}_f| = m|\mathbf{v}_f| = 2\frac{E_i}{c}, \quad \Delta E = \frac{1}{2}mv_f^2 = \frac{1}{2}m\left(\frac{2E_i}{mc}\right)^2 = \frac{2E_i}{mc^2}E_i$$
$$E_f = E_i - \Delta E = \left(1 - \frac{2E_i}{mc^2}\right)E_i \approx \frac{E_i}{1 + \frac{2E_i}{mc^2}}$$

# Compton Wavelength

- Alternate phrasing in terms of length scales  $E = h\nu = hc/\lambda$  so

$$\left. \frac{\lambda_i}{\lambda_f} \right|_{\text{rest}} = \frac{1}{1 + \frac{hc}{\lambda_i mc^2} (1 - \cos \theta)}$$

$$\left. \frac{\lambda_f}{\lambda_i} - 1 \right|_{\text{rest}} = \frac{h}{mc\lambda_i} (1 - \cos \theta) \equiv \frac{\lambda_c}{\lambda_i} (1 - \cos \theta)$$

where the Compton wavelength is  $\lambda_c = h/mc \approx 2.4 \times 10^{-10}$

- Doppler effect: consider the limit of  $\beta_i \ll 1$  then expand to first order

$$\frac{E_f}{E_i} = 1 - \beta_i \cos \alpha_i + \beta_i \cos \alpha_f - \frac{E_i}{mc^2} (1 - \cos \theta)$$

however averaging over angles the Doppler shifts don't change the energies

# Second Order Doppler Shift

- To second order in the velocities, the Doppler shift transfers energy from the electron to the photon in opposition to the recoil

$$\frac{E_f}{E_i} = 1 - \beta_i \cos \alpha_i + \beta_i \cos \alpha_f + \beta_i^2 \cos^2 \alpha_f - \frac{E_i}{mc^2}$$
$$\left\langle \frac{E_f}{E_i} \right\rangle \approx 1 + \frac{1}{3} \beta_i^2 - \frac{E_i}{mc^2}$$

- For a thermal distribution of velocities

$$\frac{1}{2} m \langle v^2 \rangle = \frac{3kT}{2} \quad \beta_i^2 \approx \frac{3kT}{mc^2} \rightarrow \left\langle \frac{E_f}{E_i} - 1 \right\rangle \sim \frac{kT - E_i}{mc^2}$$

- so that if  $E_i \ll kT$  the photon gains energy and  $E_i \gg kT$  it loses energy  $\rightarrow$  this is a thermalization process

# Energy Transfer

- A more proper treatment of the radiative transfer equation (below) for the distribution gives the Kompaneets equation and

$$\frac{E_f}{E_i} - 1 = \frac{4kT - E_i}{mc^2}$$

where the coefficient reflects the fact that averaged over the Wien tail  $\langle E_i \rangle = 3kT$  and  $\langle E_i^2 \rangle = 12(kT)^2$  so that

$$E_f - E_i \propto 4kT E_i - E_i^2 = 0$$

in thermodynamic equilibrium. For multiple scattering events one finds that the energy transfer goes as

$$4 \frac{k(T_e - T_\gamma)}{mc^2} \tau$$

# Inverse Compton Scattering

- Relativistic electrons boost energy of low energy photons by a potentially enormous amount - a way of getting  $\gamma$  rays in astrophysics

$$\frac{E_f}{E_i} = \frac{1 - \beta_i \cos \alpha_i}{1 - \beta_i \cos \alpha_f + \frac{E_i}{\gamma_i mc^2} (1 - \cos \theta)}$$

- Take  $\beta_i \rightarrow 1$  and  $\gamma \gg 1$  but  $E_i < \gamma_i mc^2$  so that the scattering is Thomson in the rest frame
- Qualitative:  $\alpha_i$  incoming angle wrt electron velocity can be anything; BUT outgoing  $\alpha_f$  is strongly beamed in direction of velocity with  $\alpha_f \sim 1/\gamma \ll 1$  or  $\cos \alpha_f \approx 1$  so

$$\begin{aligned} \frac{E_f}{E_i} &\approx \frac{1 - \beta_i \cos \alpha_i}{1 - \beta_i} \left[ \frac{1}{\gamma_i^2} = (1 - \beta)(1 + \beta) \approx 2(1 - \beta) \right] \\ &\approx (1 - \beta_i \cos \alpha_i) 2\gamma_i^2 \end{aligned}$$

# Inverse Compton Scattering

- Since the incoming angle is random, a typical photon gains energy by a factor of  $\gamma_i^2$ !
- From the electron's perspective: see a bunch of (beamed) photons coming head on boosted in energy by a factor of  $\gamma_i$  and scatter out roughly isotropic *preserving energy* in the rest frame. Reverse the boost and pick up another factor of  $\gamma_i$  in the lab frame beamed into the direction of motion.
- Limit to energy transfer - total energy of the electron in lab frame

$$E_f - E_i \approx E_f < \gamma_i mc^2$$

$$\gamma_i^2 E_i < \gamma_i mc^2$$

$$\gamma_i E_i < mc^2$$

# Inverse Compton Scattering

- Since  $\gamma_i E_i$  is also the typical photon energy in the rest frame, the validity of the  $\gamma_i^2$  energy transfer is simply that recoil can be neglected in the rest frame  $E_i|_{\text{rest}} = \gamma_i E_i \ll mc^2$  so that scattering is Thomson in the rest frame
- Maximum energy boost

$$E_f = \gamma_i(\gamma_i E_i) < \gamma_i mc^2 = \gamma_i(511\text{keV})$$

so that  $E_f$  can be an enormous energy and the scattering is still Thomson in the rest frame

- Is it consistent to neglect recoil compared with  $1/\gamma_i^2$  in the lab frame

$$\text{Recoil} = \frac{E_i}{\gamma_i mc^2} \ll \frac{mc^2}{\gamma_i} \frac{1}{\gamma_i mc^2} = \frac{1}{\gamma_i^2}$$

# Inverse Compton Scattering

- With  $\gamma_i = 10^3$

radio     $1\text{GHz} = 10^9\text{Hz} \rightarrow 10^{15}\text{Hz} \approx 300\text{nm}$     UV

optical     $4 \times 10^{14}\text{Hz} \rightarrow 4 \times 10^{20}\text{Hz} \approx 1.6\text{MeV}$     gamma ray

# Single Scattering

- Consider a distribution of photons (and electrons) in the lab frame. Characterize the radiative transfer in the optically thin single scattering regime
- Consider the energy density ( $A$  is a constant)

$$u = 2 \int \frac{d^3p}{(2\pi\hbar)^2} E f = A \int f p^3 dp d\Omega$$

- In the electron rest frame the energy density transforms with the aid of the Lorentz transformations

$$\begin{aligned} p' &= p\gamma(1 - \beta\mu) \\ d\Omega' &= \frac{d\Omega}{\gamma^2(1 - \beta\mu)^2} \quad \rightarrow \quad p'^2 d\Omega' = p^2 d\Omega \\ f' &= f \end{aligned}$$

# Single Scattering

- Therefore energy density transforms as

$$\begin{aligned}u' &= A \int f' p'^3 dp' d\Omega' = A \int f p' p^2 dp' d\Omega \\ &= A \int \gamma^2 (1 - \beta\mu)^2 f p^3 dp d\Omega\end{aligned}$$

- Assume that  $f$  is isotropic (in lab frame) then the energy densities are related as

$$\begin{aligned}u' &= \int d\Omega \gamma^2 (1 - \beta\mu)^2 A \int f p^3 dp \\ &= \int d\Omega \gamma^2 (1 - \beta\mu)^2 \frac{u}{4\pi} \\ &= \gamma^2 \left(1 + \frac{1}{3}\beta^2\right) u\end{aligned}$$

# Single Scattering

- In the rest frame the emitted power is given by the Thomson scattering strength  $\sigma_T = P/S$  and power is a Lorentz invariant

$$\begin{aligned}\frac{dE_{\text{rad}}}{dt} &= \frac{dE'}{dt'} = c\sigma_T u' \\ &= c\sigma_T \gamma^2 \left[1 + \frac{1}{3}\beta^2\right] u\end{aligned}$$

- The total power accounts for the “absorbed” energy of the incoming photons

$$\begin{aligned}\frac{dE}{dt} &= c\sigma_T \left[\gamma^2 \left(1 + \frac{1}{3}\beta^2\right) - 1\right] u, & \gamma^2 - 1 &= \gamma^2 \beta^2 \\ &= \frac{4}{3} \sigma_T c \gamma^2 \beta^2 u\end{aligned}$$

# Single Scattering

- If we assume a thermal distribution of electrons  $\langle \beta^2 \rangle = 3kT/mc^2$  we get

$$\frac{du}{dt} = \frac{4kT}{mc^2} c\sigma_T n_e u \quad \rightarrow \quad \frac{du}{d\tau} = \frac{4kT}{mc^2} u$$

- Assume a power law distribution of electron energies  $dE = mc^2 d\gamma$  confined to a range  $\gamma_{\min} \leq \gamma \leq \gamma_{\max}$

$$n_e = \int n_{e,\gamma} d\gamma \quad n_{e,\gamma} = K \gamma^{-p}$$

- Change in the radiation energy density given  $\gamma \gg 1$  ( $\beta \approx 1$ )

$$\begin{aligned} \frac{du}{dt} &= \frac{d^2 E}{dt dV} = \int \frac{dE}{dt} n_{e,\gamma} d\gamma \\ &= \frac{4}{3} \sigma_T c u \frac{K}{3-p} (\gamma_{\max}^{3-p} - \gamma_{\min}^{3-p}) \end{aligned}$$

# Single Scattering

- Energy spectrum in the power law case

$$\frac{d^2 E}{dt dV} \propto \int \gamma^{2-p} d\gamma$$

and the scattered photon energy  $E_f \propto \gamma^2 E_i$

$$\frac{d^2 E}{dt dV dE_f} = \frac{d\gamma}{dE_f} \frac{d^2 E}{dt dV d\gamma} \propto \frac{1}{\gamma} \gamma^{2-p} = \gamma^{1-p} \propto E_f^{(1-p)/2}$$

so the scattered spectrum is also a power law  $E_f^{-s}$  but with a power law index  $s = (p - 1)/2$ .

# Radiative Transfer Equation

- For full radiative transfer, we must go beyond the single scattering, optically thin limit. Generally (set  $\hbar = c = k = 1$  and neglect Pauli blocking and polarization)

$$\begin{aligned} \frac{\partial f}{\partial t} = & \frac{1}{2E(p_f)} \int \frac{d^3 p_i}{(2\pi)^3} \frac{1}{2E(p_i)} \int \frac{d^3 q_f}{(2\pi)^3} \frac{1}{2E(q_f)} \int \frac{d^3 q_i}{(2\pi)^3} \frac{1}{2E(q_i)} \\ & \times (2\pi)^4 \delta(p_f + q_f - p_i - q_i) |M|^2 \\ & \times \{f_e(q_i) f(p_i) [1 + f(p_f)] - f_e(q_f) f(p_f) [1 + f(p_i)]\} \end{aligned}$$

- where the matrix element is calculated in field theory and is Lorentz invariant. In terms of the rest frame  $\alpha = e^2/\hbar c$  (c.f. Klein Nishina Cross Section)

$$|M|^2 = 2(4\pi)^2 \alpha^2 \left[ \frac{E(p_i)}{E(p_f)} + \frac{E(p_f)}{E(p_i)} - \sin^2 \beta \right]$$

with  $\beta$  as the rest frame scattering angle

# Kompaneets Equation

- The Kompaneets equation is the radiative transfer equation in the limit that electrons are thermal

$$f_e = e^{-(m-\mu)/T_e} e^{-q^2/2mT_e} \quad \left[ n_e = e^{-(m-\mu)/T_e} \left( \frac{mT_e}{2\pi} \right)^{3/2} \right]$$
$$= \left( \frac{2\pi}{mT_e} \right)^{3/2} n_e e^{-q^2/2mT_e}$$

and assume that the energy transfer is small (non-relativistic electrons,  $E_i \ll m$ )

$$\frac{E_f - E_i}{E_i} \ll 1 \quad [\mathcal{O}(T_e/m, E_i/m)]$$

- Kompaneets equation (restoring  $\hbar, c, k$ )

$$\frac{\partial f}{\partial t} = n_e \sigma_T c \left( \frac{kT_e}{mc^2} \right) \frac{1}{x^2} \frac{\partial}{\partial x} \left[ x^4 \left( \frac{\partial f}{\partial x} + f(1+f) \right) \right] \quad x = \hbar\omega/kT_e$$

# Kompaneets Equation

- Equilibrium solution must be a Bose-Einstein distribution

$$\partial f / \partial t = 0$$

$$\left[ x^4 \left( \frac{\partial f}{\partial x} + f(1 + f) \right) \right] = K$$

$$\frac{\partial f}{\partial x} + f(1 + f) = \frac{K}{x^4}$$

assume that as  $x \rightarrow 0$ ,  $f \rightarrow 0$  then  $K = 0$  and

$$\frac{df}{dx} = -f(1 + f) \quad \rightarrow \quad \frac{df}{f(1 + f)} = dx$$

$$\ln \frac{f}{1 + f} = -x + c \quad \rightarrow \quad \frac{f}{1 + f} = e^{-x+c}$$

$$f = \frac{e^{-x+c}}{1 - e^{-x+c}} = \frac{1}{e^{x-c} - 1}$$

# Kompaneets Equation

- More generally, no evolution in the number density

$$n_\gamma \propto \int d^3p f \propto \int dx x^2 f$$

$$\frac{\partial n_\gamma}{\partial t} \propto \int dx x^2 \frac{1}{x^2} \frac{\partial}{\partial x} \left[ x^4 \left( \frac{\partial f}{\partial x} + f(1+f) \right) \right]$$

$$\propto x^4 \left[ \frac{\partial f}{\partial x} + f(1+f) \right]_0^\infty = 0$$

- Energy evolution  $R \equiv n_e \sigma_T c (kT_e / mc^2)$

$$u = 2 \int \frac{d^3p}{(2\pi\hbar)^3} E f = 2 \int \frac{p^3 dp c}{2\pi^2 \hbar^3} f = \left[ \frac{(kT_e)^4}{c^4 \hbar^3} \frac{1}{\pi^2} \equiv A \right] \int x^3 dx f$$

$$\frac{\partial u}{\partial t} = AR \int dx x \frac{\partial}{\partial x} \left[ x^4 \left( \frac{\partial f}{\partial x} + f(1+f) \right) \right]$$

# Kompaneets Equation

$$\begin{aligned}\frac{\partial u}{\partial t} &= -AR \int dx x^4 \left( \frac{\partial f}{\partial x} + f(1+f) \right) \\ &= AR \int dx 4x^3 f - AR \int dx x^4 f(1+f) \\ &= 4n_e \sigma_T c \frac{kT_e}{mc^2} u - AR \int dx x^4 f(1+f)\end{aligned}$$

Change in energy is difference between Doppler and recoil

- If  $f$  is a Bose-Einstein distribution at temperature  $T_\gamma$

$$\frac{\partial f}{\partial x_\gamma} = -f(1+f) \quad x_\gamma = \frac{pc}{kT_\gamma}$$

$$AR \int dx x^4 f(1+f) = -AR \int dx x^4 \frac{\partial f}{\partial x_\gamma} = AR \int dx 4x^3 \frac{dx}{dx_\gamma} f$$

# Kompaneets Equation

- Radiative transfer equation for energy density

$$\frac{\partial u}{\partial t} = 4n_e \sigma_T c \frac{kT_e}{mc^2} \left[ 1 - \frac{T_\gamma}{T_e} \right] u$$
$$\frac{1}{u} \frac{\partial u}{\partial t} = 4n_e \sigma_T c \frac{k(T_e - T_\gamma)}{mc^2}$$

which is our original form from the energy-momentum conservation argument

- The analogue to the optical depth for energy transfer is the Compton  $y$  parameter

$$d\tau = n_e \sigma_T ds = n_e \sigma_t c dt$$
$$dy = \frac{k(T_e - T_\gamma)}{mc^2} d\tau$$

# Kompaneets Equation

- Example: hot  $X$ -ray cluster with  $kT \sim \text{keV}$  and the CMB:

$$T_e \gg T_\gamma$$

- Inverse Compton scattering transfers energy to the photons while conserving the photon number
- Optically thin conditions: low energy photons boosted to high energy leaving a deficit in the number density in the RJ tail and an enhancement in the Wien tail called a Compton- $y$  distortion — see problem set
- Compton scattering off high energy electrons can give low energy photons a large boost in energy but cannot create the photons in the first place