Astro 305

Lecture Notes

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Set 1:

Radiative Transfer

Radiation Observables

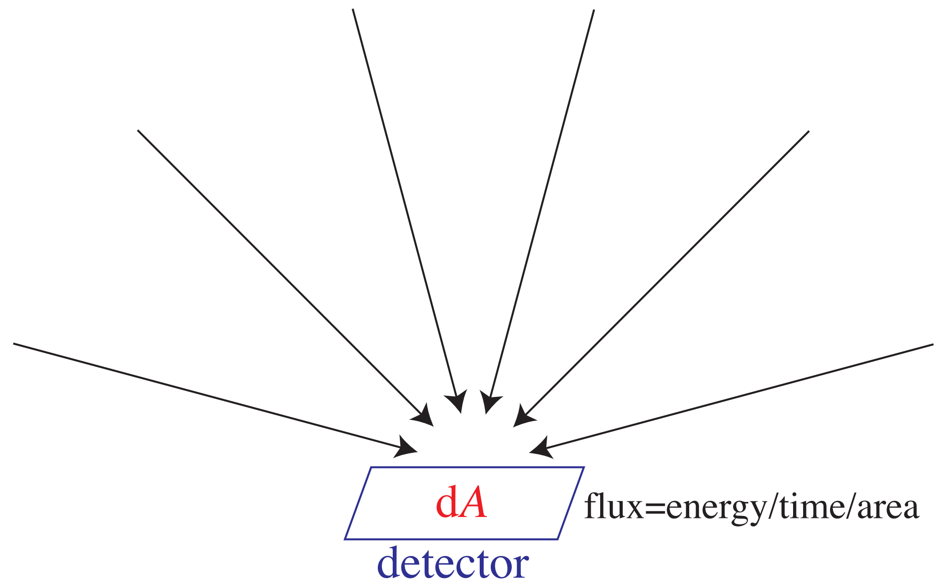
- From an empiricist's point of view there are 4 observables for radiation
  - Energy Flux
  - Direction
  - Frequency
  - Polarization

Observables: Flux

- Energy Flux

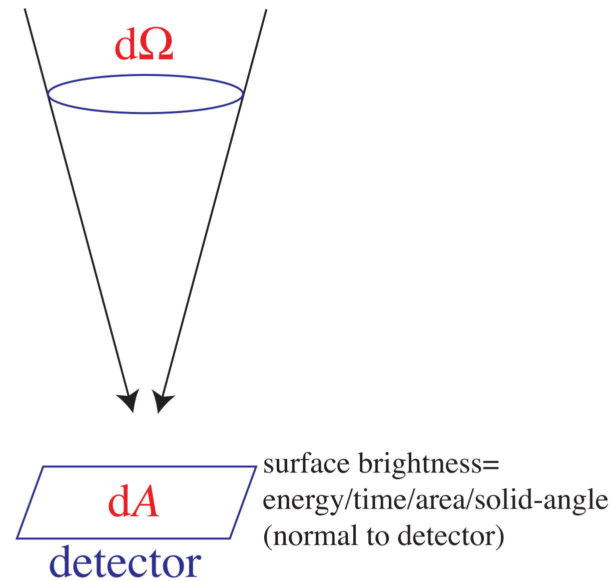
$$F = \frac{dE}{dt\,dA}$$

- Units: erg s^{-1} cm^{-2}
- Radiation can
  hit detector from all angles



Observables: Surface Brightness

- Direction: columate (e.g. pinhole) in an acceptance angle $d\Omega$ normal to $dA \rightarrow$ surface brightness



$$S(\Omega) = \frac{dE}{dt dA d\Omega}$$

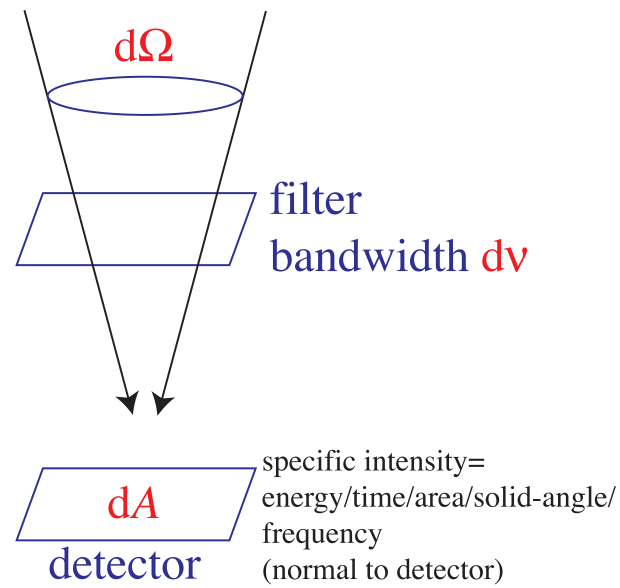
- Units: erg s^{-1} cm^{-2} sr^{-1}

Observables: Specific intensity

- Frequency: filter
  in a band of frequency
  $d\nu \rightarrow$ specific intensity

$$I_\nu = \frac{dE}{dt dA d\Omega d\nu}$$

  which
  is the fundamental quantity
  for radiative processes



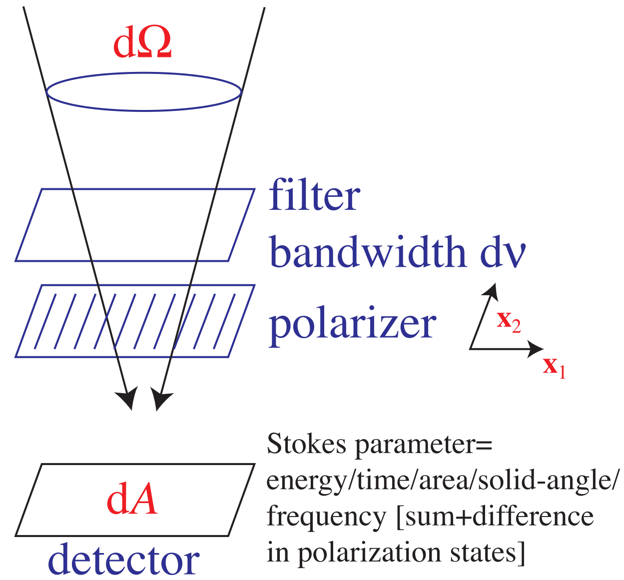
- Units:
  erg s^{-1} cm^{-2} sr^{-1} Hz^{-1}
- Astro-lingo: color is the difference between frequency bands

Observables: Polarization

- Polarization:
  filter in linear (1,2) [or 45 degrees rotated (1',2') or circular (+,-) polarization states] – Stokes parameters

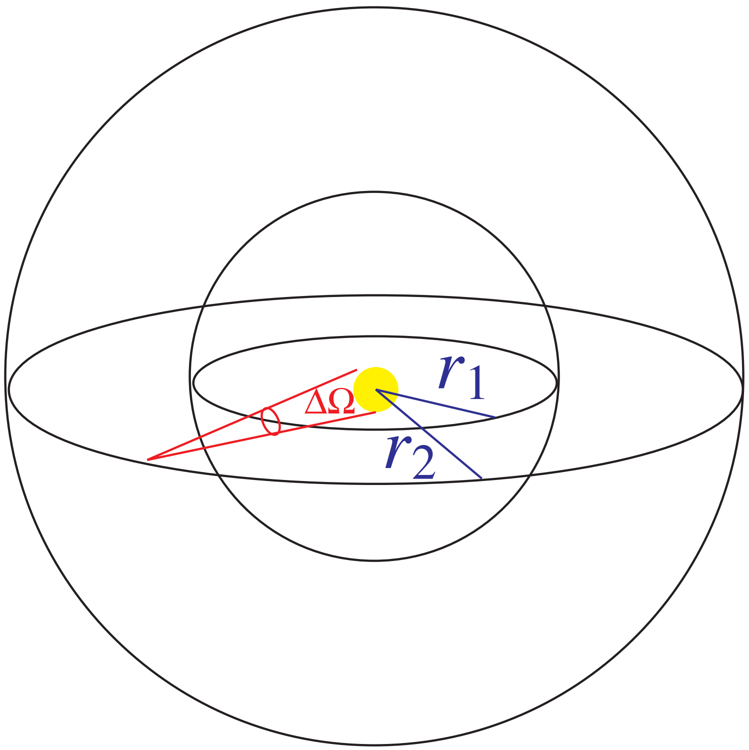
$$I_\nu = I_{\nu 1} + I_{\nu 2}$$
$$Q_\nu = I_{\nu 1} - I_{\nu 2}$$
$$U_\nu = I_{\nu 1'} - I_{\nu 2'}$$
$$V_\nu = I_{\nu +} - I_{\nu -}$$

- Units:
  erg s^{-1} cm^{-2} sr^{-1} Hz^{-1}



Radiative Transfer

- Radiative Transfer = change in I_ν as radiation propagates
- Simple example: how does the specific intensity of sunlight change as it propagates to the earth
- Energy conservation says



$$F(r_1)4\pi r_1^2 = F(r_2)4\pi r_2^2$$

$$F \propto r^{-2}$$

Radiative Transfer

- But the angular size subtended by the sun

$$\Delta\Omega \propto r^{-2}$$

$$S = \frac{F}{\Delta\Omega} = \text{const}$$

- And frequency doesn't change so that

$$I_\nu = \text{const}$$

Radiative Transfer

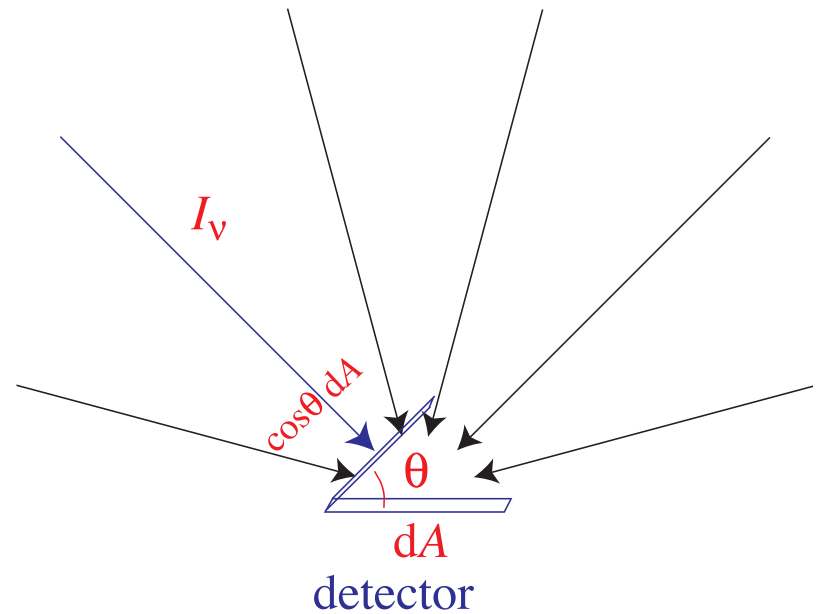
- More generally: I_ν changes due to
  - Scattering (directional change)
  - Doppler or Red-shift (frequency change)
  - Absorption
  - Emission
- If frequency changes due to redshift $\nu \propto (1+z)^{-1}$, photon conservation implies

  $$I_\nu/\nu^3 = \text{const}$$

  and so surface brightness $S = \int I_\nu d\nu$ scales as the redshift of $(1+z)^{-4}$ (cosmological surface brightness dimming, conversely relativistic Doppler boost)
- Liouville's theorem (conservation of I_ν/ν^3) in absence of interactions; Boltzmann equation (radiative transfer equation)

Auxiliary Quantities

- Specific intensity
  is defined as energy flux per unit area, per unit solid angle (normal to the area), per unit frequency
- So specific flux
  through a detector not oriented normal to source has a lowered cross section due to projection



$$F_\nu = \int I_\nu \cos\theta d\Omega, \quad F = \int d\nu F_\nu$$

if I_ν is constant in angle, no net energy flux $F_\nu = 0 = F$

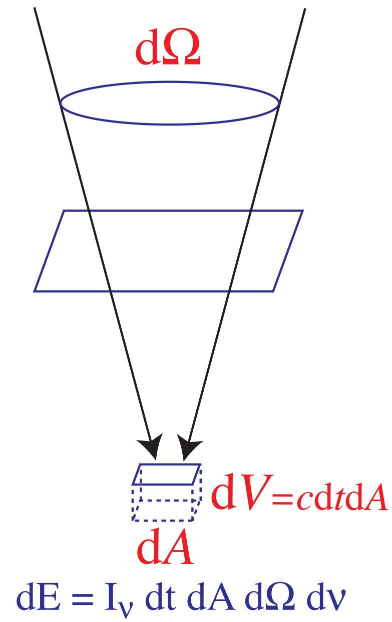
Auxiliary Quantities

- Angle averaged intensity

$$J_\nu = \int \frac{d\Omega}{4\pi} I_\nu$$

- Specific energy density:
  start with energy detected

$$dE = I_\nu dt dA d\Omega d\nu$$



Auxiliary Quantities

- Light travels at c,
  dE is the energy in the volume

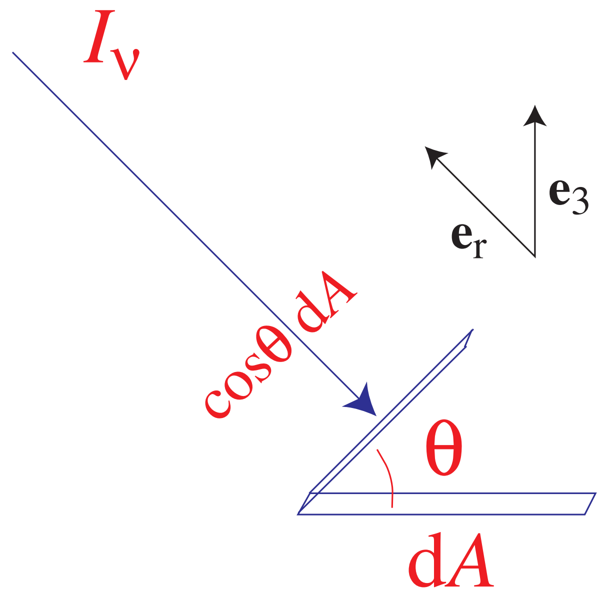
$$dV = c\,dt\,dA$$

$$dE = \frac{I_\nu}{c} dV\, d\Omega d\nu$$

- Specific energy density $u_\nu(\Omega) \equiv I_\nu / c$
- Energy density:

$$u = \int d\nu \int d\Omega I_\nu / c = \int d\nu 4\pi J_\nu / c$$

Auxiliary Quantities



- Momentum $\mathbf{q} = E/c\,\hat{\mathbf{e}}_r$
- Change in momentum per area:

$$\frac{d\mathbf{q}}{dtdA} = \frac{1}{c} I_\nu \cos\theta d\Omega d\nu \hat{\mathbf{e}}_r$$

- Pressure: the component normal to surface $\hat{\mathbf{e}}_3 \cdot \hat{\mathbf{e}}_r = \cos\theta$
- If surface absorbs radiation

$$p_{\nu,\mathrm{abs}} = \frac{1}{c}\int F_\nu \cos\theta d\Omega = \frac{1}{c}\int I_\nu \cos^2\theta d\Omega$$

Auxiliary Quantities

- If surface reflects radiation

$$
\begin{aligned}
p_\nu &= 2p_{\nu,\mathrm{abs}} \\
&= \frac{2}{c}\int I_\nu \cos^2\theta d\Omega
\end{aligned}
$$

- Equation of state of isotropic radiation $I_\nu = J_\nu$

$$
p_\nu = \frac{2}{c} J_\nu \int \cos^2\theta d\Omega = \frac{4\pi}{3c} J_\nu
$$

$$
p = \frac{4\pi}{3c}\int d\nu J_\nu = \frac{1}{3}u
$$

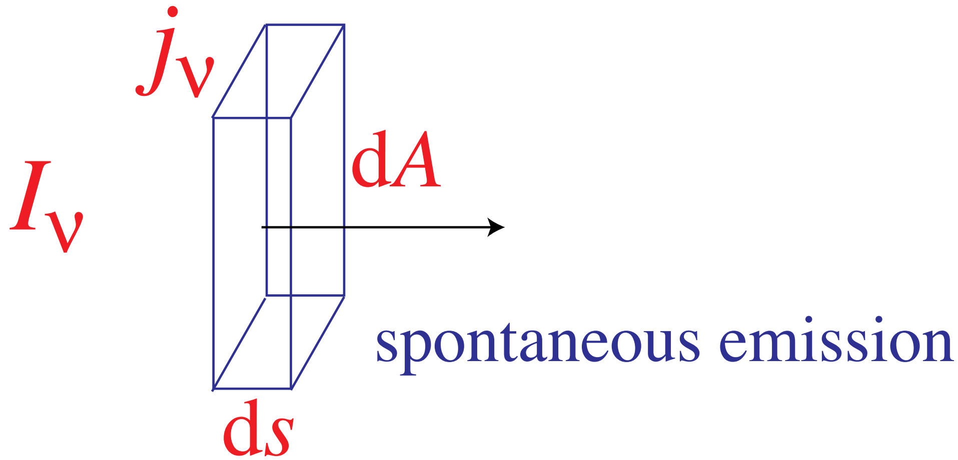
Radiative Processes

- Spontaneous Emission: matter spontaneously emits a photon/radiation
- Absorption: radiation absorbed by matter
- Stimulated Emission: passing radiation stimulates matter to emit in the same frequency and direction. Stimulated emission is mathematically the same as negative absorption - both proportional to the incoming radiation
- Scattering: absorption followed immediately by emission

  Coherent or elastic: emission at the same frequency

  Isotropic: radiates equally in all directions
- All can be related to the matrix element for interaction - interrelated by Einstein relations

Spontaneous Emission



- Given a source isotropically radiating power into a band $d\nu$ at a rate

$$\frac{\text{power}}{\text{volume}} = d\nu P_\nu$$

- The amount of energy radiated per unit solid angle is

$$dE_{\text{em}} = P_\nu d\nu dV dt \frac{d\Omega}{4\pi}$$

Spontaneous Emission

- Generalize to a non-isotropically emitting source

$$\frac{P_\nu}{4\pi} \to j_\nu(\Omega)$$

called the monocromatic emission coefficient

- Emission coefficient is alternately given per unit mass $\epsilon_\nu = 4\pi j_\nu/\rho$ where ρ is the mass density

$$dE_{\rm em} = \frac{\epsilon_\nu}{4\pi} d\nu dm dt, \quad dm = \rho dV$$

Spontaneous Emission

- Effect on specific intensity across a path length ds

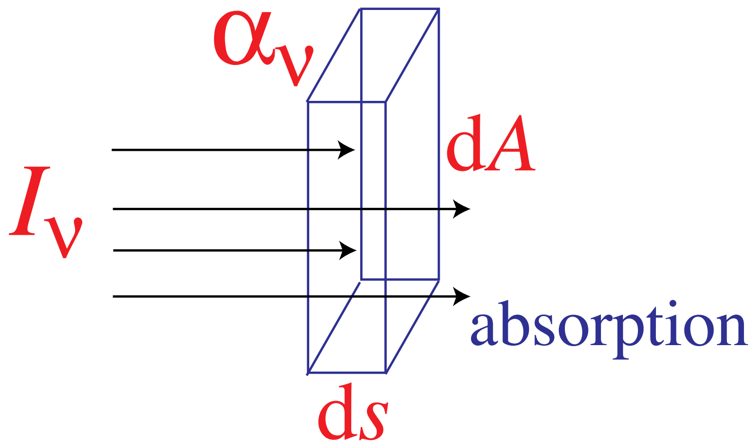
$$
\begin{aligned}
dI_\nu dt dA d\Omega d\nu &= dE_{\mathrm{em}} \\
&= j_\nu dV d\Omega d\nu dt
\end{aligned}
$$

$$dI_\nu = j_\nu ds, \quad dV = ds dA$$

- Radiative transfer equation

$$\frac{dI_\nu}{ds} = j_\nu, \quad I_\nu(s) = I_\nu(s_0) + \int_{s_0}^{s} j_\nu(s) ds$$

Absorption



- Travelling through a material a fraction α_ν of the radiation is absorbed per unit length

$$\frac{dI_\nu}{I_\nu} = -\alpha_\nu ds$$

where α_ν is called the absorption coefficient

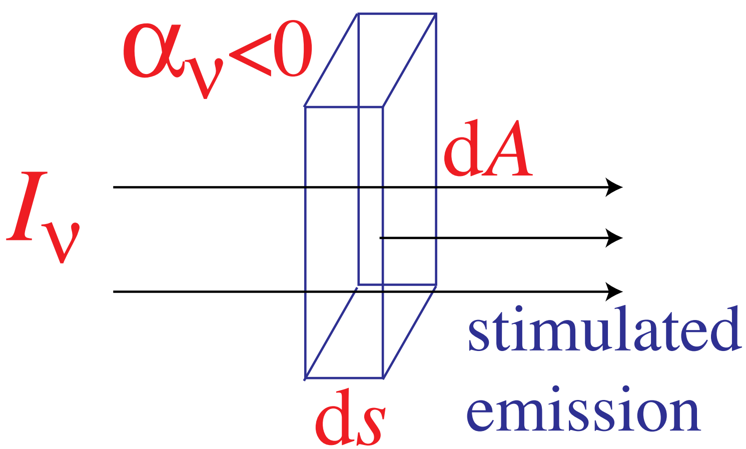
Absorption

- If material does not emit, solution is an exponential suppression

$$\ln I_\nu = -\int ds\alpha_\nu + C$$

$$I_\nu(s) = I_\nu(s_0)e^{-\int \alpha_\nu ds}$$

- If $\alpha_\nu < 0$ (stimulated emission) then there is an exponential growth



Optical Depth

- Useful to measure length in units of the typical path length to absorption or interaction

$$L = \frac{1}{\alpha_\nu}$$

so that the total path length in units of L becomes

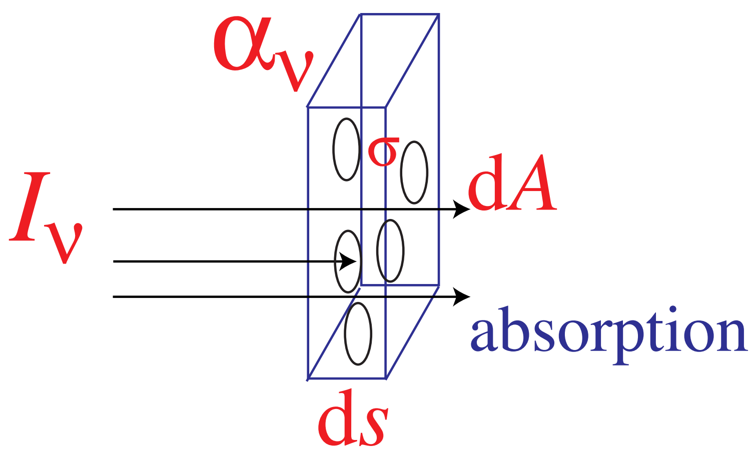
$$\int \alpha_\nu ds = \int \frac{ds}{L} \equiv \tau_\nu, \quad I_\nu(s) = I_\nu(s_0)e^{-\tau_\nu}$$

- Radiative transfer equation

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu + j_\nu$$
$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + S_\nu, \quad S_\nu \equiv j_\nu/\alpha_\nu$$

where S_ν is the source function.

Cross Section



- Particle description of absorbers and light
- Absorption coefficient related to cross section for interaction
- Example: if medium is full of opaque disks each of area σ
- Given number density n, the covering fraction

$$dA_{\rm abs} = \sigma dN = \sigma n dV = \sigma n dA ds$$

Cross Section

- Radiative transfer equation

$$dA_{\rm abs} = \sigma dN = \sigma n dV = \sigma n dA ds$$

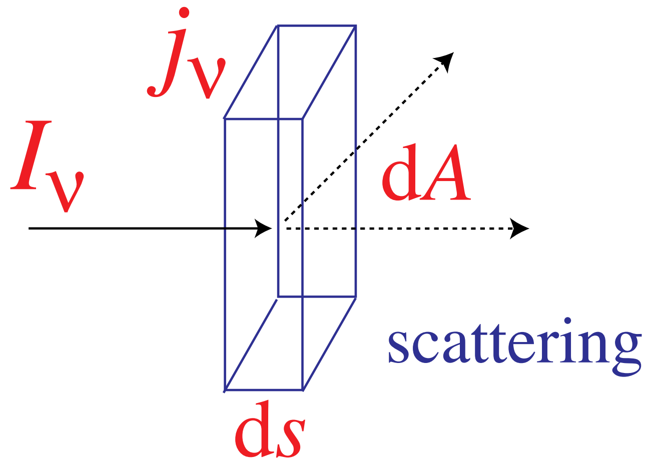
$$\frac{dI_\nu}{I_\nu} = -\sigma n ds$$

- So for a particle description of the absorption coefficient $\alpha_\nu = n\sigma$.

Cross Section

- Applies to a generalized version of σ (e.g. free electrons are point particles but Thomson cross section is finite), cross section for interaction, can depend on frequency.
- Mean free path of a photon $L = 1/\alpha_\nu = 1/n\sigma$, where n is the spatial number density of *interacting* particles (c.f. consider Thomson scattering in air)
- But the total distance travelled by a photon is typically much greater than $s = \tau L$ - an individual photon propagates through the medium via scattering as a random walk

Scattering



- Scattering can be viewed as absorption and emission where the emission is directly proportional to the absorption
- Isotropic scattering: fraction $\alpha_\nu I_\nu$ absorbed and reradiated into 4π

$$j_\nu = \alpha_\nu \int \frac{d\Omega}{4\pi} I_\nu = \alpha_\nu J_\nu$$

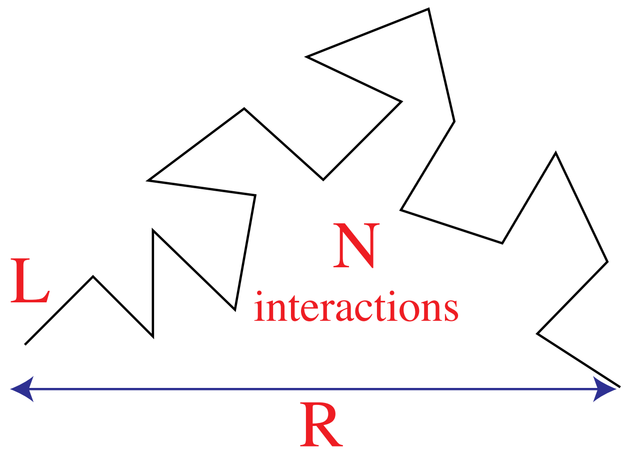
Scattering

- Thus the source function $S_\nu = j_\nu/\alpha_\nu = J_\nu$ and at high optical depth, the specific intensity approaches its angle averaged value J_ν
- But despite this simple interpretation, the radiative transfer equation is an integro-differential equation

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + J_\nu$$
$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + \int \frac{d\Omega}{4\pi} I_\nu$$

- that depends on I_ν not only in the observation direction but *all* directions.

Random Walk



- Each of N steps has length L in a random direction $\mathbf{r}_i = L\hat{\mathbf{r}}_i$

$$\mathbf{R} = \mathbf{r}_1 + \mathbf{r}_2 \ldots + \mathbf{r}_N$$

- Total distance

$$\langle R^2 \rangle = \langle \mathbf{R} \cdot \mathbf{R} \rangle = \langle \mathbf{r}_1 \cdot \mathbf{r}_1 \rangle \ldots + \langle \mathbf{r}_N \cdot \mathbf{r}_N \rangle + 2\langle \mathbf{r}_1 \cdot \mathbf{r}_2 \rangle \ldots$$

- Uncorrelated cross terms $\langle \mathbf{r}_i \cdot \mathbf{r}_j \rangle = \delta_{ij} L^2$

Random Walk

- Average distance $\langle R^2 \rangle = NL^2$ or $R_{\rm rms} = \sqrt{N}L$
- How many scatterings before escaping a distance R?
  $N = R^2/L^2 = \tau_\nu^2$ for optically thick
- For optically thin $\tau_\nu \ll 1$, a typical photon does not scatter and so by definition a fraction τ_ν will interact once, the rest zero and so the average $N = \tau_\nu$
- Quick estimate $N = \max(\tau_\nu, \tau_\nu^2)$

Multiple Processes

- Combining processes: differential elements add, e.g. total opacity is sum of individual opacities – so highest opacity process is most important for blocking I_ν
- But given multiple frequency or spatial channels, energy escapes in the channel with the lowest opacity

  Example: a transition line vs continuum scattering – photons will wander in frequency out of line and escape through lower opacity scattering $\rightarrow$ lines are often dark (sun)
- Scattering ($\alpha_{\nu s}$) and absorption ($\alpha_{\nu a}$)

$$\frac{dI_\nu}{ds} = -\alpha_{\nu a}(I_\nu - S_{\nu a}) - \alpha_{\nu s}(I_\nu - J_\nu)$$

Multiple Processes

- Collect terms

$$\frac{dI_\nu}{ds} = -(\alpha_{\nu a} + \alpha_{\nu s})I_\nu + (\alpha_{\nu a} S_{\nu a} + \alpha_{\nu s} J_\nu)$$

- Combined source function and absorption

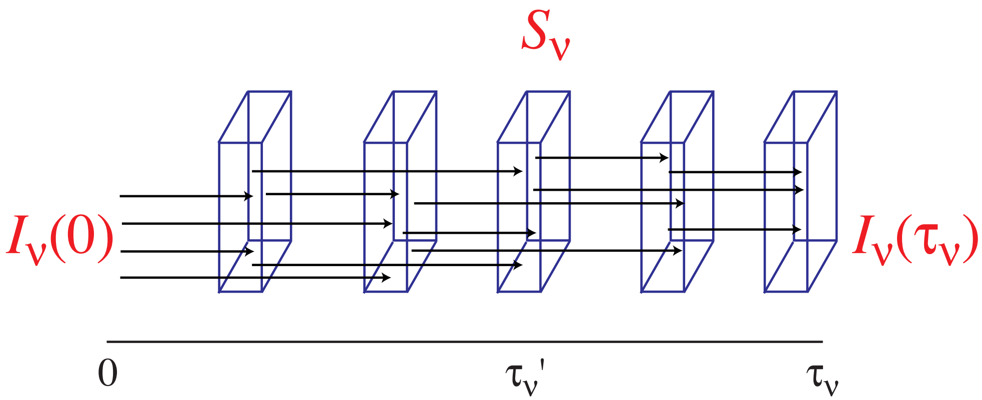
$$\alpha_\nu = \alpha_{\nu a} + \alpha_{\nu s}, \quad S_\nu = \frac{j_\nu}{\alpha_\nu} = \frac{\alpha_{\nu a} S_{\nu a} + \alpha_{\nu s} J_\nu}{\alpha_{\nu a} + \alpha_{\nu s}}$$

- Mean free path $L = 1/\alpha_\nu = 1/(\alpha_{\nu a} + \alpha_{\nu s})$, typical length before absorption *or* scattering. Fraction that ends in absorption

$$\epsilon_\nu = \frac{\alpha_{\nu a}}{\alpha_{\nu a} + \alpha_{\nu s}}, \quad 1 - \epsilon_\nu = \frac{\alpha_{\nu s}}{\alpha_{\nu a} + \alpha_{\nu s}} \ \text{single scattering albedo}$$

$$S_\nu = \epsilon_\nu S_{\nu a} + (1 - \epsilon_\nu) J_\nu$$

Formal Solution to Radiative Transfer



- Formal solution

$$I_\nu(\tau_\nu) = I_\nu(0)e^{-\tau_\nu} + \int_0^{\tau_\nu} d\tau'_\nu S_\nu(\tau'_\nu)e^{-(\tau_\nu - \tau'_\nu)}$$

- Interpretation: initial specific intensity I_ν attenuated by absorption and replaced by source function, attenuated by absorption from foreground matter

Formal Solution to Radiative Transfer

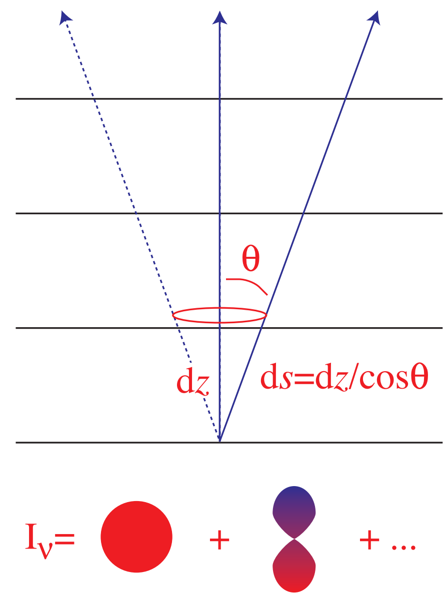
- Special case S_ν independent of τ_ν take out of integral

$$
\begin{aligned}
I_\nu(\tau_\nu) &= I_\nu(0)e^{-\tau_\nu} + S_\nu \int_0^{\tau_\nu} d\tau'_\nu e^{-(\tau_\nu - \tau'_\nu)} \\
&= I_\nu(0)e^{-\tau_\nu} + S_\nu[1 - e^{-\tau_\nu}]
\end{aligned}
$$

  at low optical depth I_ν unchanged, high optical depth $I_\nu \to S_\nu$

- Integration is along the path of the radiation (direction dependent); source function can depend on I_ν in a different direction!

Fluid or Eddington Approximation



- Often a good approximation that radiation is nearly isotropic: consider the fact that scattering, absorption and emission randomizes the direction of radiation
- Eddington (or fluid) approximation: radiation has a dipole structure at most
- Plane parallel symmetry (e.g. star, Fourier expansion): specific intensity depends only on polar angle from normal to plane and vertical spatial coordinate z. Path length $ds = dz/\cos\theta = dz/\mu$.

Fluid or Eddington Approximation

- Now approximate I_ν as a *linear* function of μ (isotropic + dipole $\rightarrow$ energy density + bulk momentum density: $I_\nu = a + b\mu$
- [More generally: a Legendre polynomial expansion of I_ν - e.g. CMB typically keeps 25-50 moments to solve for S_ν and then 3000-6000 moments to describe I_ν]
- Angular moments of specific intensity:

$$J_\nu = \frac{1}{2}\int_{-1}^{+1} I_\nu d\mu \quad \text{(energy density)}$$

$$H_\nu = \frac{1}{2}\int_{-1}^{+1} \mu I_\nu d\mu \quad \text{(momentum density)}$$

$$K_\nu = \frac{1}{2}\int_{-1}^{+1} \mu^2 I_\nu d\mu = \frac{1}{3} J_\nu \quad \text{(pressure)}$$

Fluid or Eddington Approximation

- Radiative transfer equation:

$$\frac{dI_\nu}{ds} = \mu \frac{dI_\nu}{dz} = -\alpha_\nu (I_\nu - S_\nu)$$
$$\mu \frac{dI_\nu}{d\tau} = -(I_\nu - S_\nu), \quad d\tau = \alpha_\nu dz$$

- Angular moments of specific intensity: zeroth moment $\frac{1}{2}\int d\mu \ldots$ (for simplicity assume S_ν is isotropic)

$$\frac{dH_\nu}{d\tau} = -(J_\nu - S_\nu)$$

In fluid mechanics this is the Euler equation: local imbalance generates a flow

Fluid or Eddington Approximation

- First moment $\frac{1}{2}\int \mu d\mu \ldots$

$$\frac{dK_\nu}{d\tau} = \frac{1}{3}\frac{dJ_\nu}{d\tau} = -H_\nu$$

  This is the continuity equation, a flow generates a change in energy density

- Take derivative and combine

$$\frac{1}{3}\frac{d^2 J_\nu}{d\tau^2} = -\frac{dH_\nu}{d\tau} = J_\nu - S_\nu$$

- Explicit equation for $J_\nu(z)$ if S_ν considered an external source
- Consider scattering + absorption/emission

$$\begin{aligned}\frac{1}{3}\frac{d^2 J_\nu}{d\tau^2} &= J_\nu - [\epsilon_\nu S_{\nu a} + (1-\epsilon_\nu)J_\nu] \\ &= -\epsilon_\nu(S_{\nu a} - J_\nu)\end{aligned}$$

Fluid or Eddington Approximation

- Explicitly solve with two boundary conditions: e.g. $J_\nu(0) = J_{\nu 0}$ and $J_\nu(\infty) = S_{\nu a}$. If $S_{\nu a}$ constant

$$J_\nu = S_{\nu a} + [J_{\nu 0} - S_{\nu a}] e^{-\tau\sqrt{3\epsilon_\nu}}$$

- So relaxation to source at $\tau \sim 1/\sqrt{3\epsilon}$ - modified since part of the optical depth is due to scattering