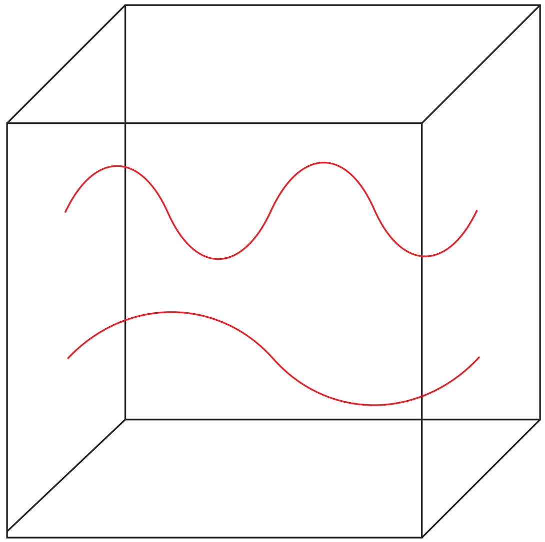
Set 2:

Statistical Mechanics

How Many Particles Fit in a Box?



- Counting momentum states due to the wave nature of particles with momentum q and de Broglie wavelength

$$\lambda = \frac{h}{q} = \frac{2\pi\hbar}{q}$$

- In a discrete volume L^3 there is a discrete set of states that satisfy periodic boundary conditions

How Many Particles Fit in a Box?

- As in Fourier analysis: $e^{2\pi i x/\lambda} = e^{i(q/\hbar)x} = e^{i(q/\hbar)(x+L)}$ yields a discrete set of allowed states

$$\frac{Lq}{\hbar} = 2\pi m_i, \quad m_i = 1, 2, 3...$$

$$q_i = m_i \frac{2\pi\hbar}{L}$$

- In each of 3 directions: $\sum_{m_{xi} m_{yj} m_{zk}} \rightarrow \int d^3m$
- The differential number of allowed momenta in the volume

$$d^3m = \left(\frac{L}{2\pi\hbar}\right)^3 d^3q$$

Density of States

- The total number of states allows for a number of internal degrees of freedom, e.g. spin, quantified by the degeneracy factor g: total density of states:

$$\frac{dN_s}{V} = \frac{g}{V} d^3 m = \frac{g}{(2\pi\hbar)^3} d^3 q$$

- If all states were occupied by a single particle, then the particle density

$$n_s = \frac{N_s}{V} = \frac{1}{V} \int dN_s = \int \frac{g}{(2\pi\hbar)^3} d^3 q$$

Distribution Function

- The distribution function f quantifies the occupation of the allowed momentum states

$$n = \frac{N}{V} = \frac{1}{V}\int f dN_s = \int \frac{g}{(2\pi\hbar)^3} f d^3q$$

- f, aka phase space occupation number, also quantifies the density of particles per unit phase space $dN/(\Delta x)^3(\Delta q)^3$
- For photons, the spin degeneracy $g = 2$ accounting for the 2 polarization states
- Energy $E(q) = (q^2c^2 + m^2c^4)^{1/2}$
- Momentum $\rightarrow$ frequency $q = h/\lambda = h\nu/c = E/c$ (where $m = 0$ and $\lambda\nu = c$)

Number Density

- Momentum state defines the direction of the radiation

$$
\begin{aligned}
n &= g \int \frac{d^3 q}{(2\pi\hbar)^3} f \\
&= 2 \int \frac{d\Omega q^2 dq}{(2\pi\hbar)^3} f \\
&= 2 \int d\Omega \left(\frac{h}{c}\right)^3 \frac{1}{h^3} \int \nu^2 d\nu f \\
&= 2 \int d\Omega \frac{1}{c^3} \int \nu^2 d\nu f
\end{aligned}
$$

- Gives number density in a given direction and frequency band

Energy Density

- In general the energy density is

$$u = g \int \frac{d^3q}{(2\pi\hbar)^3} E(q) f$$

- For radiation

$$u = g \int \frac{d^3q}{(2\pi\hbar)^3} E(q) f = 2 \int d\Omega \frac{1}{c^3} \int \nu^2 d\nu h\nu f$$

- So specific energy density

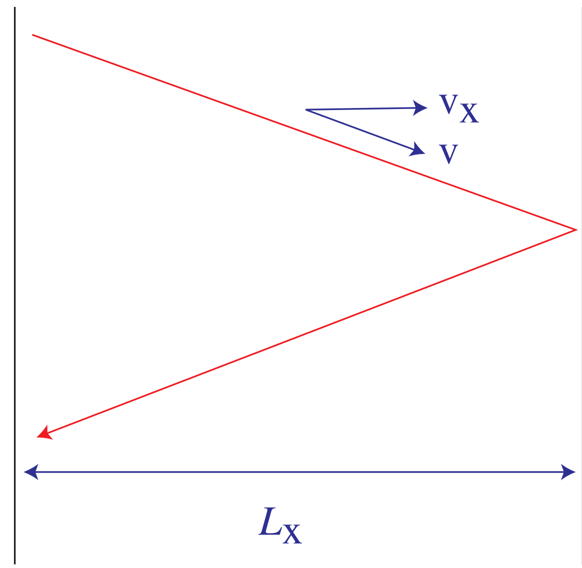
$$u_\nu(\Omega) = \frac{d^2u}{d\Omega d\nu} = \frac{2\nu^3 h}{c^3} f$$

- And specific intensity

$$I_\nu(\Omega) = u_\nu(\Omega) c = \frac{2\nu^3 h}{c^2} f$$

Pressure

- Pressure: particles bouncing off a surface of area A in a volume spanned by $V = AL_x$: per momentum state



$$p_q = \frac{F}{A} = \frac{N_{\rm part}}{A}\frac{\Delta q_x}{\Delta t}$$

$$(\Delta q_x = 2|q_x|, \quad \Delta t = 2L_x/v_x, \quad q/E = v/c^2)$$

$$= \frac{N_{\rm part}}{V}|q_x||v_x| = \frac{N_{\rm part}}{V}\frac{|q||v|}{3} = \frac{N_{\rm part}}{V}\frac{q^2c^2}{3E}$$

($\cos^2$ term in radiative pressure calc.)

Moments

- Occupation number defines the $N_{\rm part}/V$ per momentum state so that summed over states

$$p = g \int \frac{d^3q}{(2\pi\hbar)^3} \frac{|q|^2 c^2}{3E(q)} f$$

- Radiation

$$p = g \int \frac{d^3q}{(2\pi\hbar)^3} \frac{E(q)}{3} f = \frac{1}{3} u$$

- Energy and pressure are part of the angular moments of the distribution function – the isotropic ones
- First order anisotropy is the bulk momentum density or dipole of the distribution:

$$(u+p)\mathbf{v}/c = g \int \frac{d^3q}{(2\pi\hbar)^3} \mathbf{q} c f$$

Fluid Approximation Redux

- Continue with the second moments: radiative viscosity or anisotropic stress

$$\pi_{ij} = g \int \frac{d^3q}{(2\pi\hbar)^3} \frac{3q_i q_j - q^2\delta_{ij}}{3E(q)} f$$

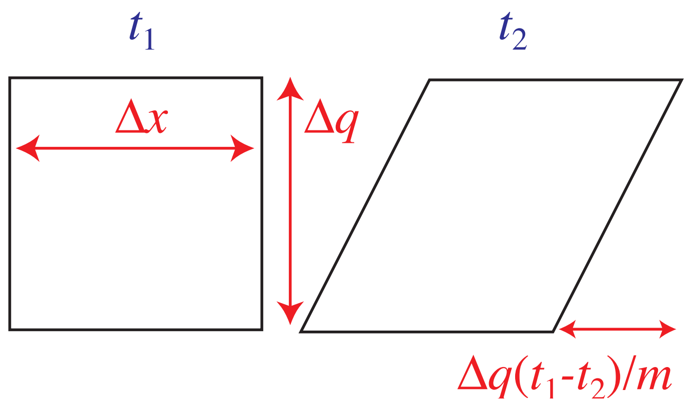
- Eddington approximation is that all the higher order moments from the radiative viscosity onward vanishes
- Since particle kinetics must obey energy and momentum conservation, in the fluid limit there are two equations of motion: continuity and Euler equations
- Three quantities of interest: energy density, pressure, bulk velocity means that a third relation is needed: $p(\rho)$ the equation of state

Astro-Particle Dictionary

Astrophysicists and physicist use different words to describe same thing:

- Specific intensity $I_\nu \leftrightarrow$ phase space distribution f
- Surface brightness conservation $\leftrightarrow$ Liouville equation
- Absorption, emission, scattering $\leftrightarrow$ Collision term
- Einstein relations $\leftrightarrow$ Single matrix element
- Radiative transfer equation $\leftrightarrow$ Boltzmann equation
- Eddington approximation $\leftrightarrow$ Fluid approximation
- Moments of $I_\nu \leftrightarrow$ Radiative viscosity
- Rosseland Approximation $\leftrightarrow$ Tight coupling approximation

Liouville Equation



- In absence of interactions, particle conservation implies that the phase space distribution is invariant along the trajectory of the particles
- Follow an element in Δx with spread Δq. For example for non relativistic particles a spread in velocity of $\Delta v = \Delta q/m$.
- After a time δt the low velocity tail will lag the high velocity tail by $\delta x = \Delta v \delta t = \Delta q \delta t/m$
- For ultrarelativistic particles $v = c$ and $\Delta v = 0$, so obviously true

Liouville Equation

- The phase space element can shear but preserves area $\Delta x \Delta q$
- This remains true under Lorentz and even a general coordinate transform
- Therefore $df/dt = 0$ or f is conserved when evaluated along the path of the particles
- Liouville Equation: $f \propto I_\nu/\nu^3$ and $ds = cdt$

$$\frac{df}{dt} = 0 \to \frac{dI}{ds} = 0$$

if frequency is also conserved on the path

Liouville Equation

- In general, expand out the total derivative

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \sum_i \left(\frac{dx_i}{dt} \frac{\partial f}{\partial x_i} + \frac{dq_i}{dt} \frac{\partial f}{\partial q_i} \right) = 0$$

- The spatial gradient terms are responsible for flow of particles in and out of a fixed volume
- The momentum derivative terms are responsible for redshift effects

Boltzmann Equation

- Heuristically

$$\frac{df}{dt} = \text{particle sources - sinks}$$
$$\frac{dI_\nu}{ds} = \text{emission - absorption}$$

the r.h.s. is called the collision term and given as $C[f]$

- Collision term: integrate over the phase space of incoming particles, connect to outgoing state with the matrix element of the transition M

- Form:

$$C[f] = \int d(\text{phase space})[\text{ energy-momentum conservation}] \times |M|^2[\text{emission} - \text{absorption}]$$

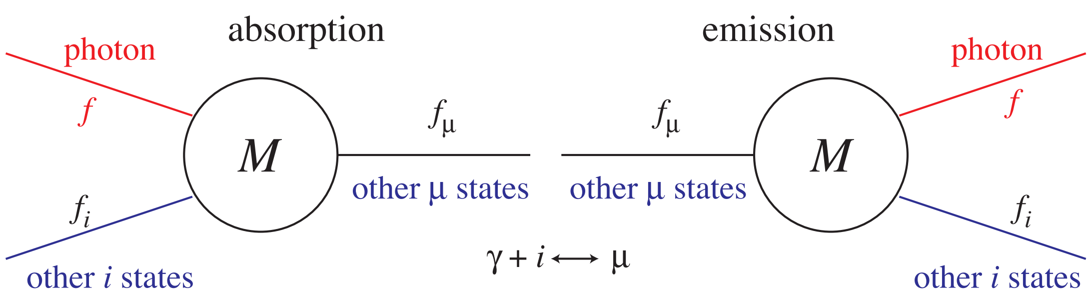
Boltzmann Equation

- (Lorentz invariant) phase space element (here $\hbar = c = 1$) over the other particles ($\gamma + i \leftrightarrow \mu$)

$$\int d(\text{phase space}) = \Pi_i \frac{g_i}{(2\pi)^3} \int \frac{d^3 q_i}{2E_i}$$

and likewise for μ particles – note that μ can involve a photon in another momentum state, e.g. in scattering

- Energy conservation: $(2\pi)^4 \delta^{(4)}(q_1 + q_2 + ...)$



- [emission-absorption] $+ =$ boson; $- =$ fermion

$$\Pi_i \Pi_\mu f_\mu (1 \pm f_i)(1 \pm f) - \Pi_i \Pi_\mu (1 \pm f_\mu) f_i f$$

Boltzmann Equation

- Photon Emission: $f_\mu(1 \pm f_i)(1 + f)$

  f_μ: proportional to number of emitters

  $(1 \pm f_i)$: if final state is occupied and a fermion, process blocked; if boson the process enhanced

  $(1 + f)$: final state factor for photons: "1": spontaneous emission (remains if $f = 0$); "$+f$": stimulated and proportional to the occupation of final photon

- Photon Absorption: $-(1 \pm f_\mu) f_i f$

  $(1 \pm f_\mu)$: if final state is occupied and fermion, process blocked; if boson the process enhanced

  f_i: proportional to number of absorbers

  f: proportional to incoming photons

Boltzmann Equation

- The matrix $|M|^2$ or analogously the cross section for absorption defines all processes (the physical content of the Einstein relations)
- Expect that $\sigma \propto |M|^2$
- Integration over momentum state converts f's to n's
- Example: a line transition from single lower $i = 1$ state to upper $\mu = 2$ state assuming that outgoing states are not occupied
- Absorption: $-(1 \pm f_\mu) f_i f \to -n_1 f$, $|M|^2 \to \sigma$, $2h\nu^3 f/c^2 \to I_\nu$ so that $\alpha_\nu|_{\rm true\ absorption} = n_1 \sigma$
- Emission: $f_\mu(1 \pm f_i)(1 + f) \to n_2(1 + f) = n_2 + n_2 f$ so that spontaneous emission $j_\nu \sim n_2 \sigma \cdot 2\nu^3 h/c^2$ and stimulated emission is negative absorption with $\alpha_\nu|_{\rm stim\ emiss} \sim -n_2 \sigma$

Boltzmann Equation

- Implies a source function

$$S_\nu = j_\nu/\alpha_\nu \sim \frac{1}{n_1/n_2 - 1}\frac{2h\nu^3}{c^2}$$

- We will find that the full Einstein relationship is

$$S_\nu = j_\nu/\alpha_\nu = \frac{1}{(n_1 g_2/n_2 g_1 - 1)}\frac{2h\nu^3}{c^2}$$

  where degeneracy factors appear for levels that have multiple states

- Interactions drive I_ν to S_ν which nulls the rhs radiative trans. eqn.
- Likewise collisions drive f to some equilibrium distribution and then remains constant thereafter in spite of further collisions $\rightarrow$ black body distribution