Set 2:
Statistical Mechanics
How Many Particles Fit in a Box?

- Counting momentum states due to the wave nature of particles with momentum $q$ and de Broglie wavelength

$$\lambda = \frac{h}{q} = \frac{2\pi \hbar}{q}$$

- In a discrete volume $L^3$ there is a discrete set of states that satisfy periodic boundary conditions.
How Many Particles Fit in a Box?

- As in Fourier analysis: $e^{2\pi i x / \lambda} = e^{i(q/\hbar)x} = e^{i(q/\hbar)(x+L)}$ yields a discrete set of allowed states

  $$\frac{Lq}{\hbar} = 2\pi m_i, \quad m_i = 1, 2, 3...$$

  $$q_i = m_i \frac{2\pi \hbar}{L}$$

- In each of 3 directions: $\sum_{m_xi m_yj m_zk} \rightarrow \int d^3m$

- The differential number of allowed momenta in the volume

  $$d^3m = \left( \frac{L}{2\pi \hbar} \right)^3 d^3q$$
Density of States

- The total number of states allows for a number of internal degrees of freedom, e.g. spin, quantified by the degeneracy factor \( g \): total density of states:

\[
\frac{dN_s}{V} = \frac{g}{V} d^3m = \frac{g}{(2\pi\hbar)^3} d^3q
\]

- If all states were occupied by a single particle, then the particle density

\[
n_s = \frac{N_s}{V} = \frac{1}{V} \int dN_s = \int \frac{g}{(2\pi\hbar)^3} d^3q
\]
Distribution Function

- The distribution function $f$ quantifies the occupation of the allowed momentum states

$$n = \frac{N}{V} = \frac{1}{V} \int f dN_s = \int \frac{g}{(2\pi\hbar)^3} f d^3q$$

- $f$, aka phase space occupation number, also quantifies the density of particles per unit phase space $dN/(\Delta x)^3(\Delta q)^3$

- For photons, the spin degeneracy $g = 2$ accounting for the 2 polarization states

- Energy $E(q) = (q^2c^2 + m^2c^4)^{1/2}$

- Momentum → frequency $q = h/\lambda = h\nu/c = E/c$ (where $m = 0$ and $\lambda\nu = c$)
Number Density

- Momentum state defines the direction of the radiation

\[ n = g \int \frac{d^3 q}{(2\pi\hbar)^3} f \]

\[ = 2 \int d\Omega q^2 dq \frac{d^3 q}{(2\pi\hbar)^3} f \]

\[ = 2 \int d\Omega \left( \frac{\hbar}{c} \right)^3 \frac{1}{\hbar^3} \int \nu^2 d\nu f \]

\[ = 2 \int d\Omega \frac{1}{c^3} \int \nu^2 d\nu f \]

- Gives number density in a given direction and frequency band
Energy Density

- In general the energy density is

\[ u = g \int \frac{d^3q}{(2\pi \hbar)^3} E(q)f \]

- For radiation

\[ u = g \int \frac{d^3q}{(2\pi \hbar)^3} E(q)f = 2 \int d\Omega \frac{1}{c^3} \int \nu^2 d\nu h\nu f \]

- So specific energy density

\[ u_\nu(\Omega) = \frac{d^2u}{d\Omega d\nu} = \frac{2\nu^3 h}{c^3} f \]

- And specific intensity

\[ I_\nu(\Omega) = u_\nu(\Omega)c = \frac{2\nu^3 h}{c^2} f \]
Pressure

- Pressure: particles bouncing off a surface of area $A$ in a volume spanned by $V = AL_x$: per momentum state

\[
p_q = \frac{F}{A} = \frac{N_{\text{part}}}{A} \frac{\Delta q_x}{\Delta t}
\]

\[
(\Delta q_x = 2|q_x|, \quad \Delta t = 2L_x/v_x, \quad q/E = v/c^2)
\]

\[
= \frac{N_{\text{part}}}{V} |q_x| |v_x| = \frac{N_{\text{part}}}{V} \frac{|q||v|}{3} = \frac{N_{\text{part}}}{V} \frac{q^2 c^2}{3E}
\]

(cos$^2$ term in radiative pressure calc.)
Moments

- Occupation number defines the $N_{\text{part}}/V$ per momentum state so that summed over states

$$p = g \int \frac{d^3q}{(2\pi\hbar)^3} \frac{|q|^2 c^2}{3E(q)} f$$

- Radiation

$$p = g \int \frac{d^3q}{(2\pi\hbar)^3} \frac{E(q)}{3} f = \frac{1}{3} u$$

- Energy and pressure are part of the angular moments of the distribution function – the isotropic ones

- First order anisotropy is the bulk momentum density or dipole of the distribution:

$$(u + p)v/c = g \int \frac{d^3q}{(2\pi\hbar)^3} q cf$$
Fluid Approximation Redux

- Continue with the second moments: radiative viscosity or anisotropic stress

\[
\pi_{ij} = g \int \frac{d^3 q}{(2\pi \hbar)^3} \frac{3q_i q_j - q^2 \delta_{ij}}{3E(q)} f
\]

- Eddington approximation is that all the higher order moments from the radiative viscosity onward vanishes

- Since particle kinetics must obey energy and momentum conservation, in the fluid limit there are two equations of motion: continuity and Euler equations

- Three quantities of interest: energy density, pressure, bulk velocity means that a third relation is needed: \( p(\rho) \) the equation of state
Astro-Particle Dictionary

Astrophysicists and physicist use different words to describe same thing:

- Specific intensity $I_\nu$ ↔ phase space distribution $f$
- Surface brightness conservation ↔ Liouville equation
- Absorption, emission, scattering ↔ Collision term
- Einstein relations ↔ Single matrix element
- Radiative transfer equation ↔ Boltzmann equation
- Eddington approximation ↔ Fluid approximation
- Moments of $I_\nu$ ↔ Radiative viscosity
- Rosseland Approximation ↔ Tight coupling approximation
Liouville Equation

- In absence of interactions, particle conservation implies that the phase space distribution is invariant along the trajectory of the particles.

- Follow an element in $\Delta x$ with spread $\Delta q$. For example for non relativistic particles a spread in velocity of $\Delta v = \Delta q/m$.

- After a time $\delta t$ the low velocity tail will lag the high velocity tail by $\delta x = \Delta v \delta t = \Delta q \delta t / m$.

- For ultrarelativistic particles $v = c$ and $\Delta v = 0$, so obviously true.
Liouville Equation

• The phase space element can shear but preserves area $\Delta x \Delta q$

• This remains true under Lorentz and even a general coordinate transform

• Therefore $df/dt = 0$ or $f$ is conserved when evaluated along the path of the particles

• Liouville Equation: $f \propto I_\nu / \nu^3$ and $ds = cdt$

\[ \frac{df}{dt} = 0 \rightarrow \frac{dI}{ds} = 0 \]

if frequency is also conserved on the path
Liouville Equation

- In general, expand out the total derivative

\[
\frac{df}{dt} = \frac{\partial f}{\partial t} + \sum_i \left( \frac{dx_i}{dt} \frac{\partial f}{\partial x_i} + \frac{dq_i}{dt} \frac{\partial f}{\partial q_i} \right) = 0
\]

- The spatial gradient terms are responsible for flow of particles in and out of a fixed volume

- The momentum derivative terms are responsible for redshift effects
Boltzmann Equation

- Heuristically

\[ \frac{df}{dt} = \text{particle sources - sinks} \]
\[ \frac{dI_\nu}{ds} = \text{emission - absorption} \]

the r.h.s. is called the collision term and given as \( C[f] \)

- Collision term: integrate over the phase space of incoming particles, connect to outgoing state with the matrix element of the transition \( M \)

- Form:

\[
C[f] = \int d(\text{phase space})[ \text{energy-momentum conservation} ] \times |M|^2[\text{emission} - \text{absorption}]
\]
Boltzmann Equation

- (Lorentz invariant) phase space element (here $\hbar = c = 1$) over the other particles ($\gamma + i \leftrightarrow \mu$)

$$\int d(\text{phase space}) = \Pi_i \frac{g_i}{(2\pi)^3} \int \frac{d^3 q_i}{2E_i}$$

and likewise for $\mu$ particles – note that $\mu$ can involve a photon in another momentum state, e.g. in scattering

- Energy conservation: $(2\pi)^4 \delta^{(4)}(q_1 + q_2 + ...)$

- [emission-absorption] $+ = \text{boson}; \ - = \text{fermion}$

$$\Pi_i \Pi_\mu f_\mu (1 \pm f_i)(1 \pm f) - \Pi_i \Pi_\mu (1 \pm f_\mu)f_i f$$
Boltzmann Equation

- **Photon Emission:** $f_\mu (1 \pm f_i)(1 + f)$
  
  $f_\mu$: proportional to number of emitters
  
  $(1 \pm f_i)$: if final state is occupied and a fermion, process blocked; if boson the process enhanced
  
  $(1 + f)$: final state factor for photons: “1”: spontaneous emission (remains if $f = 0$); “$+f$”: stimulated and proportional to the occupation of final photon

- **Photon Absorption:** $-(1 \pm f_\mu)f_if$
  
  $(1 \pm f_\mu)$: if final state is occupied and fermion, process blocked; if boson the process enhanced
  
  $f_i$: proportional to number of absorbers
  
  $f$: proportional to incoming photons
Boltzmann Equation

- The matrix $|M|^2$ or analogously the cross section for absorption defines all processes (the physical content of the Einstein relations)
- Expect that $\sigma \propto |M|^2$
- Integration over momentum state converts $f$’s to $n$’s
- Example: a line transition from single lower $i = 1$ state to upper $\mu = 2$ state assuming that outgoing states are not occupied
- Absorption: $-(1 \pm f_\mu)f_i f \rightarrow -n_1 f, |M|^2 \rightarrow \sigma, 2h\nu^3 f / c^2 \rightarrow I_\nu$
  so that $\alpha_\nu |_{\text{true absorption}} = n_1 \sigma$
- Emission: $f_\mu (1 \pm f_i)(1 + f) \rightarrow n_2 (1 + f) = n_2 + n_2 f$ so that spontaneous emission $j_\nu \sim n_2 \sigma \cdot 2\nu^3 h / c^2$ and stimulated emission is negative absorption with $\alpha_\nu |_{\text{stim emiss}} \sim -n_2 \sigma$
Boltzmann Equation

- Implies a source function

\[ S_\nu = j_\nu / \alpha_\nu \sim \frac{1}{n_1/n_2 - 1} \frac{2h\nu^3}{c^2} \]

- We will find that the full Einstein relationship is

\[ S_\nu = j_\nu / \alpha_\nu = \frac{1}{(n_1g_2/n_2g_1 - 1)} \frac{2h\nu^3}{c^2} \]

where degeneracy factors appear for levels that have multiple states

- Interactions drive \( I_\nu \) to \( S_\nu \) which nulls the rhs radiative trans. eqn.

- Likewise collisions drive \( f \) to some equilibrium distribution and then remains constant thereafter in spite of further collisions \( \rightarrow \) black body distribution