Set 5:

Classical E&M and Plasma Processes

Maxwell Equations

- Classical E&M defined by the Maxwell Equations (fields sourced by matter) and the Lorentz force (matter moved by fields)
- In cgs (gaussian) units

$$\nabla \cdot \mathbf{D} = 4\pi\rho, \qquad \nabla \cdot \mathbf{B} = 0,$$
$$\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}, \qquad \nabla \times \mathbf{H} = \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t} + \frac{4\pi}{c} \mathbf{j}.$$

where $\rho = \text{charge density}$, $\mathbf{j} = \text{current density}$, $\mathbf{D} = \epsilon \mathbf{E}$ and $\mathbf{B} = \mu \mathbf{H}$ with $\epsilon = \text{dielectric constant } \mu = \text{magnetic permeability}$

• In vacuum, $\epsilon = \mu = 1$ and the Maxwell equations simplify to

$$\nabla \cdot \mathbf{E} = 4\pi\rho, \qquad \nabla \cdot \mathbf{B} = 0,$$
$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \qquad \nabla \times \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c}\mathbf{j},$$

Charge Conservation

- Maxwell equations are E and B symmetric aside from the lack of magnetic charges
- Divergence of Ampere's law \rightarrow charge conservation

$$\nabla \cdot \left[\nabla \times \mathbf{B} = \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c} \mathbf{j} \right]$$
$$0 = \frac{1}{c} \frac{\partial \nabla \cdot \mathbf{E}}{\partial t} + \frac{4\pi}{c} \nabla \cdot \mathbf{j}$$
$$0 = \frac{4\pi}{c} \frac{\partial \rho}{\partial t} + \frac{4\pi}{c} \nabla \cdot \mathbf{j}$$
$$0 = \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j}$$

• Source free propagation

$$\nabla \cdot \mathbf{E} = 0, \qquad \nabla \cdot \mathbf{B} = 0,$$
$$\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}, \qquad \nabla \times \mathbf{B} = \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t},$$

invariant under $E \to B$ and $B \to -E,$ so work out equation for E

• Curl of Faraday's law \rightarrow wave equation

$$\nabla \times \left[\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} \right]$$
$$\nabla (\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\frac{1}{c} \frac{\partial \nabla \times \mathbf{B}}{\partial t}$$
$$-\nabla^2 \mathbf{E} = -\frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2}$$
$$\nabla^2 \mathbf{E} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

• Similarly for B

$$\left[\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}\right] \begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix} = 0$$

• Wave solutions $k = 2\pi/\lambda = 2\pi\nu/c$ [real part or superimposed k and -k]

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix} = \begin{pmatrix} E_0 \mathbf{e}_1 \\ B_0 \mathbf{e}_2 \end{pmatrix} e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)}$$

• Wave solution in wave equation provides the "dispersion" relation

$$k^2 - \frac{1}{c^2}\omega^2 = 0, \qquad \omega = kc = 2\pi\nu$$

• E and B fields related by Maxwell equations

$$i\mathbf{k} \cdot \hat{\mathbf{e}}_1 E_0 = 0, \qquad i\mathbf{k} \cdot \hat{\mathbf{e}}_2 B_0 = 0,$$
$$i\mathbf{k} \times \hat{\mathbf{e}}_1 E_0 = i\frac{\omega}{c}\hat{\mathbf{e}}_2 B_0, \qquad i\mathbf{k} \times \hat{\mathbf{e}}_2 B_0 = -i\frac{\omega}{c}\hat{\mathbf{e}}_1 E_0,,$$

so $\hat{\mathbf{e}}_1 \perp \mathbf{k}$, $\hat{\mathbf{e}}_2 \perp \mathbf{k}$, $\mathbf{k} \times \hat{\mathbf{e}}_1 \parallel \hat{\mathbf{e}}_2$. So $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{k}})$ form a orthonormal right handed basis

$$i\mathbf{k} \times \hat{\mathbf{e}}_1 E_0 = i\frac{\omega}{c}\hat{\mathbf{e}}_2 B_0, \quad kE_0 = \frac{\omega}{c}B_0, \quad \to E_0 = B_0$$

Lorentz Force

• Lorentz force

$$\mathbf{F} = q \left[\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right]$$

where q is the charge

For a distribution of charges in a volume, charge and current density j = ρv provide a force density

$$\mathbf{f} = \rho \mathbf{E} + \frac{1}{c} \mathbf{j} \times \mathbf{B}$$

• Work and change in energy density

$$W = \int \mathbf{F} \cdot d\mathbf{x}, \qquad \frac{dW}{dt} = \mathbf{F} \cdot \mathbf{v} = q\mathbf{E} \cdot \mathbf{v}$$
$$\frac{du_{\text{mech}}}{dt} = \rho \mathbf{E} \cdot \mathbf{v} = \mathbf{E} \cdot \mathbf{j}$$

Field Energy

• Use energy conservation to define field and radiation energy

$$\begin{aligned} \frac{4\pi}{c} \mathbf{j} &= \nabla \times \mathbf{H} - \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t} \\ \mathbf{j} &= \frac{c}{4\pi} \left(\nabla \times \mathbf{H} - \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t} \right) \\ \frac{du_{\text{mech}}}{dt} &= \mathbf{E} \cdot \mathbf{j} = \frac{c}{4\pi} \left(\nabla \times \mathbf{H} - \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t} \right) \cdot \mathbf{E} \\ & \left[\mathbf{E} \cdot (\nabla \times \mathbf{H}) = \mathbf{H} \cdot (\nabla \times \mathbf{E}) - \nabla \cdot (\mathbf{E} \times \mathbf{H}) \right] \\ \frac{du_{\text{mech}}}{dt} &= \frac{c}{4\pi} \left[\mathbf{H} \cdot (\nabla \times \mathbf{E}) - \nabla \cdot (\mathbf{E} \times \mathbf{H}) - \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t} \cdot \mathbf{E} \right] \\ & \left[\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}, \quad \mathbf{H} = \mathbf{B}/\mu, \quad \mathbf{D} = \epsilon \mathbf{E} \right] \\ &= \frac{1}{4\pi} \left[-\frac{1}{\mu} \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{B} - \epsilon \frac{\partial \mathbf{E}}{\partial t} \cdot \mathbf{E} - c \nabla \cdot (\mathbf{E} \times \mathbf{H}) \right] \end{aligned}$$

Field Energy

• Rewrite field terms as a total derivative

$$\frac{\partial}{\partial t}(\mathbf{B} \cdot \mathbf{B}) = 2\mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t}, \quad \frac{\partial}{\partial t}(\mathbf{E} \cdot \mathbf{E}) = 2\mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t}$$

• Equation forms a conservation law

$$\frac{\partial}{\partial t}(u_{\rm mech} + u_{\rm field}) + \nabla \cdot \mathbf{S} = 0$$

with

$$u_{\text{field}} = \frac{1}{8\pi} \left(\epsilon E^2 + \frac{1}{\mu} B^2 \right)$$

and the energy flux carried by the radiation, or Poynting vector

$$\mathbf{S} = \frac{c}{4\pi} \mathbf{E} \times \mathbf{H}$$

Field Energy

• For vacuum $\mu = \epsilon = 1$ and so field energy in a monocromatic wave, time averaged over the oscillation

$$\langle u_{\text{field}} \rangle = \frac{1}{8\pi} \frac{1}{2} (E_0^2 + B_0^2) = \frac{1}{8\pi} E_0^2$$
$$\langle S \rangle = \frac{c}{4\pi} \frac{1}{2} E_0 B_0 = \frac{c}{8\pi} E_0^2$$

which says that $\langle S \rangle / \langle u_{\text{field}} \rangle = c$, (recall $I_{\nu}/u_{\nu} = c$)

• Energy Flux

$$\langle S \rangle = \frac{c}{8\pi} E_0^2 = \int d\nu \int d\Omega \cos \theta I_{\nu}$$

so the specific intensity of a monocromatic plane wave is a delta function in frequency and angle

Specific Intensity

• Actual processes will not be monocromatic but have some waveform associated with the acceleration of the emitter: superposition of plane waves

$$E(t) = \int_{-\infty}^{\infty} E(\omega)e^{-i\omega t}d\omega$$
$$E(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E(t)e^{i\omega t}dt$$

Total emission will be an incoherent superposition of these single particle sources

$$E(t)$$

Specific Intensity

• Energy flux normal to propagation direction

$$\frac{dW}{dtdA} = S = \frac{c}{4\pi}E^2(t)$$

• Total energy passing through dA

$$\frac{dW}{dA} = \frac{c}{4\pi} \int_{-\infty}^{\infty} E^2(t) dt$$
$$\begin{bmatrix} \int_{-\infty}^{\infty} E^2(t) dt = 2\pi \int_{-\infty}^{\infty} |E(\omega)|^2 d\omega \end{bmatrix}$$
$$= \frac{c}{4\pi} \left[4\pi \int_0^{\infty} |E(\omega)|^2 d\omega \right]$$
$$\frac{dW}{dAd\omega} = c |E(\omega)|^2$$

Specific Intensity

• Now given a timescale T for single particle processes

$$\frac{dW}{dAd\omega dt} = \frac{1}{T} \frac{dW}{dAd\omega} = \frac{c}{T} |E(\omega)|^2$$

which given $\omega = 2\pi\nu$ is the specific flux. Given a resolved source divide by the source solid angle to get I_{ν}



Stokes Parameters

- Specific intensity is related to quadratic combinations of the field. Define the intensity matrix (time averaged over oscillations) $\langle \mathbf{E} \mathbf{E}^{\dagger} \rangle$
- Hermitian matrix can be decomposed into Pauli matrices

$$\mathbf{P} = \left\langle \mathbf{E} \, \mathbf{E}^{\dagger} \right\rangle = \frac{1}{2} \left(I \boldsymbol{\sigma}_{0} + Q \, \boldsymbol{\sigma}_{3} + U \, \boldsymbol{\sigma}_{1} - V \, \boldsymbol{\sigma}_{2} \right) \,,$$

where

$$\boldsymbol{\sigma}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \boldsymbol{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \boldsymbol{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \boldsymbol{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

• Stokes parameters recovered as $Tr(\sigma_i \mathbf{P})$

Stokes Parameters

• Consider a general plane wave solution

$$\mathbf{E}(t,z) = E_1(t,z)\hat{\mathbf{e}}_1 + E_2(t,z)\hat{\mathbf{e}}_2$$
$$E_1(t,z) = A_1 e^{i\phi_1} e^{i(kz-\omega t)}$$
$$E_2(t,z) = A_2 e^{i\phi_2} e^{i(kz-\omega t)}$$

• Explicitly:

$$I = \langle E_1 E_1^* + E_2 E_2^* \rangle = A_1^2 + A_2^2$$
$$Q = \langle E_1 E_1^* - E_2 E_2^* \rangle = A_1^2 - A_2^2$$
$$U = \langle E_1 E_2^* + E_2 E_1^* \rangle = 2A_1 A_2 \cos(\phi_2 - \phi_1)$$
$$V = -i \langle E_1 E_2^* - E_2 E_1^* \rangle = 2A_1 A_2 \sin(\phi_2 - \phi_1)$$

so that the Stokes parameters define the state up to an unobservable overall phase of the wave

- Radiation field involves a directed quantity, the electric field vector, which defines the polarization
- Consider a general plane wave solution

$$\mathbf{E}(t, z) = E_1(t, z)\hat{\mathbf{e}}_1 + E_2(t, z)\hat{\mathbf{e}}_2$$
$$E_1(t, z) = \operatorname{Re}A_1 e^{i\phi_1} e^{i(kz - \omega t)}$$
$$E_2(t, z) = \operatorname{Re}A_2 e^{i\phi_2} e^{i(kz - \omega t)}$$

or at z = 0 the field vector traces out an ellipse

$$\mathbf{E}(t,0) = A_1 \cos(\omega t - \phi_1)\hat{\mathbf{e}}_1 + A_2 \cos(\omega t - \phi_2)\hat{\mathbf{e}}_2$$

with principal axes defined by

$$\mathbf{E}(t,0) = A'_1 \cos(\omega t) \hat{\mathbf{e}}'_1 - A'_2 \sin(\omega t) \hat{\mathbf{e}}'_2$$

so as to trace out a clockwise rotation for $A'_1, A'_2 > 0$

• Define polarization angle

$$\hat{\mathbf{e}}_1' = \cos \chi \hat{\mathbf{e}}_1 + \sin \chi \hat{\mathbf{e}}_2$$
$$\hat{\mathbf{e}}_2' = -\sin \chi \hat{\mathbf{e}}_1 + \cos \chi \hat{\mathbf{e}}_2$$

• Match

$$\mathbf{E}(t,0) = A'_1 \cos \omega t [\cos \chi \hat{\mathbf{e}}_1 + \sin \chi \hat{\mathbf{e}}_2] - A'_2 \cos \omega t [-\sin \chi \hat{\mathbf{e}}_1 + \cos \chi \hat{\mathbf{e}}_2] = A_1 [\cos \phi_1 \cos \omega t + \sin \phi_1 \sin \omega t] \hat{\mathbf{e}}_1 + A_2 [\cos \phi_2 \cos \omega t + \sin \phi_2 \sin \omega t] \hat{\mathbf{e}}_2$$



• Define relative strength of two principal states

 $A_1' = E_0 \cos\beta \quad A_2' = E_0 \sin\beta$

• Characterize the polarization by two angles

$$A_1 \cos \phi_1 = E_0 \cos \beta \cos \chi, \qquad A_1 \sin \phi_1 = E_0 \sin \beta \sin \chi,$$
$$A_2 \cos \phi_2 = E_0 \cos \beta \sin \chi, \qquad A_2 \sin \phi_2 = -E_0 \sin \beta \cos \chi$$

Or Stokes parameters by

$$I = E_0^2, \quad Q = E_0^2 \cos 2\beta \cos 2\chi$$
$$U = E_0^2 \cos 2\beta \sin 2\chi, \quad V = E_0^2 \sin 2\beta$$

• So $I^2 = Q^2 + U^2 + V^2$, double angles reflect the spin 2 field or headless vector nature of polarization

Special cases

If β = 0, π/2, π then only one principal axis, ellipse collapses to a line and V = 0 → linear polarization oriented at angle χ

If $\chi = 0, \pi/2, \pi$ then $I = \pm Q$ and U = 0If $\chi = \pi/4, 3\pi/4...$ then $I = \pm U$ and Q = 0 - so U is Q in a frame rotated by 45 degrees

- If β = π/4, 3π/4, then principal components have equal strength and E field rotates on a circle: I = ±V and Q = U = 0 → circular polarization
- $U/Q = \tan 2\chi$ defines angle of linear polarization and $V/I = \sin 2\beta$ defines degree of circular polarization

Natural Light

- A monochromatic plane wave is completely polarized $I^2 = Q^2 + U^2 + V^2$
- Polarization matrix is like a density matrix in quantum mechanics and allows for pure (coherent) states and mixed states
- Suppose the total $E_{\rm tot}$ field is composed of different (frequency) components

$$\mathbf{E}_{ ext{tot}} = \sum_i \mathbf{E}_i$$

• Then components decorrelate in time average

$$\left\langle \mathbf{E}_{\mathrm{tot}} \mathbf{E}_{\mathrm{tot}}^{\dagger} \right\rangle = \sum_{ij} \left\langle \mathbf{E}_{i} \mathbf{E}_{j}^{\dagger} \right\rangle = \sum_{i} \left\langle \mathbf{E}_{i} \mathbf{E}_{i}^{\dagger} \right\rangle$$

Natural Light

• So Stokes parameters of incoherent contributions add

$$I = \sum_{i} I_{i} \quad Q = \sum_{i} Q_{i} \quad U = \sum_{i} U_{i} \quad V = \sum_{i} V_{i}$$

and since individual Q, U and V can have either sign: $I^2 \ge Q^2 + U^2 + V^2$, all 4 Stokes parameters needed

Polarized Radiative Transfer

• Define a specific intensity "vector": $\mathbf{I}_{\nu} = (I_{\nu 1}, I_{\nu 2}, U, V)$ where $I = I_{\nu 1} + I_{\nu 2}, Q = I_{\nu 1} - I_{\nu 2}$

$$\frac{d\mathbf{I}_{\nu}}{ds} = \alpha_{\nu} (\mathbf{S}_{\nu} - \mathbf{I}_{\nu})$$

- Source vector in practice can be complicated
- Thomson collision
 based on differential cross section

$$\frac{d\sigma_T}{d\Omega} = \frac{3}{8\pi} |\hat{\mathbf{E}}' \cdot \hat{\mathbf{E}}|^2 \sigma_T \,,$$



Polarized Radiative Transfer

- Ê' and Ê denote the incoming and outgoing directions of the electric field or polarization vector.
- Thomson scattering into $e_1: I_{\nu 2} \to I_{\nu 2}$ but $I_{\nu 1}$ does not scatter
- More generally if Θ is the scattering angle then referenced to the plane of the scattering $\alpha_{\nu} = n_e \sigma_T$ and

$$\mathbf{S}_{\nu} = \frac{3}{8\pi} \int d\Omega' \begin{pmatrix} \cos^2 \Theta & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos \Theta & 0 \\ 0 & 0 & 0 & \cos \Theta \end{pmatrix} \mathbf{I}'_{\nu}$$

• But to calculate Stokes parameters in a fixed coordinate system must rotate into the scattering basis, scatter and rotate back out to the fixed coordinate system

Plasma Effects

- Astrophysical media are typically ionized so that radiation does not propagate in a vacuum but through an ionized plasma.
- However the plasma is typically so rarified that only the very lowest frequency radiation is affected
- Maxwell equations for plane wave radiation $\exp[i({\bf k}\cdot{\bf r}-\omega t)]$ with sources

$$i\mathbf{k} \cdot \mathbf{E} = 4\pi\rho, \qquad i\mathbf{k} \cdot \mathbf{B} = 0,$$

 $i\mathbf{k} \times \mathbf{E} = i\frac{\omega}{c}\mathbf{B}, \qquad i\mathbf{k} \times \mathbf{B} = -i\frac{\omega}{c}\mathbf{E} + \frac{4\pi}{c}\mathbf{j},$

 Medium is globally neutral but electric field of the radiation cause a high frequency electron drift → current → charge via continuity

Plasma Sources

• Lorentz force

$$m\dot{\mathbf{v}} = e\mathbf{E} \quad \rightarrow \mathbf{v} = -\frac{e\mathbf{E}}{i\omega m}$$

• Current density carried by electrons of number density n

$$\mathbf{j} = ne\mathbf{v} = -\frac{ne^2\mathbf{E}}{i\omega m} \equiv \sigma\mathbf{E}$$
$$\sigma = \frac{ine^2}{\omega m} \quad \text{conductivity}$$

• Charge conservation

$$-i\omega\rho + i\mathbf{k}\cdot\mathbf{j} = 0$$
$$\rho = \frac{\mathbf{k}\cdot\mathbf{j}}{\omega} = \frac{\sigma}{\omega}\mathbf{k}\cdot\mathbf{E}$$

Plasma Frequency

• Maxwell equation with ρ

$$i\mathbf{k} \cdot \mathbf{E} = 4\pi \frac{\sigma}{\omega} \mathbf{k} \cdot \mathbf{E}$$
$$i\left(1 - \frac{4\pi\sigma}{\omega i}\right) \mathbf{k} \cdot \mathbf{E} = 0$$
$$i\mathbf{k} \cdot \epsilon \mathbf{E} = 0$$

with the dielectric constant

$$\epsilon = 1 - \frac{4\pi\sigma}{\omega i} = 1 - \frac{4\pi n e^2}{m\omega^2}$$
$$= 1 - \frac{\omega_p^2}{\omega^2} \qquad \left[\omega_p^2 = \frac{4\pi n e^2}{m}\right]$$

Plasma Frequency

• Likewise the Maxwell equation with **j**

$$i\mathbf{k} \times \mathbf{B} = \frac{4\pi}{c}\mathbf{j} - i\frac{\omega}{c}\mathbf{E}$$
$$= \left(\frac{4\pi}{c}\sigma - i\frac{\omega}{c}\right)\mathbf{E}$$
$$i\mathbf{k} \times \mathbf{B} = -i\frac{\omega}{c}\epsilon\mathbf{E}$$

• So that the Maxwell equations become source free equations

$$i\mathbf{k} \cdot \epsilon \mathbf{E} = 0, \qquad i\mathbf{k} \cdot \mathbf{B} = 0,$$

 $i\mathbf{k} \times \mathbf{E} = i\frac{\omega}{c}\mathbf{B}, \qquad i\mathbf{k} \times \mathbf{B} = -i\frac{\omega}{c}\epsilon\mathbf{E},$

• Wave equation becomes (similarly for $\mathbf{B} \perp \mathbf{E}$)

$$i[\mathbf{k} \times (\mathbf{k} \times \mathbf{E})] = -ik^2 \mathbf{E} = i\frac{\omega}{c}\mathbf{k} \times \mathbf{B} = -i\frac{\omega^2}{c^2}\epsilon\mathbf{E}$$

• Modified dispersion relation

$$k^{2} = \frac{\omega^{2}}{c^{2}}\epsilon = \frac{\omega^{2}}{c^{2}}\left(1 - \frac{\omega_{p}^{2}}{\omega^{2}}\right)$$
$$k = \frac{1}{c}\sqrt{\omega^{2} - \omega_{p}^{2}}$$

 If ω < ω_p then k is imaginary and the wave function has an exponential suppression - waves don't propagate below the plasma frequency

$$\nu_p = \frac{\omega_p}{2\pi} = 0.01 \,\mathrm{MHz} \left(\frac{n}{1 \mathrm{cm}^{-3}}\right)^{1/2}$$

Plasma Cutoff & Refraction

- For the ionosphere $n\sim 10^4~{\rm cm}^{-3}$ and radio waves at $<1{\rm MHz}$ cannot propagate
- For ISM $n < 1 \text{ cm}^{-3}$ and the cut off is a much smaller < 0.01 MHz
- The phase velocity defines the index of refaction

$$v_p = \frac{\omega}{k} \equiv \frac{c}{n_r} \quad \rightarrow n_r \equiv \sqrt{\epsilon} = \sqrt{1 - \frac{\omega_p^2}{\omega^2}}$$

• Radio waves can be refracted according to Snell's law and change their direction of propagation along the path *s*

$$\frac{dn_r\hat{\mathbf{k}}}{ds} = \nabla n_r$$

Dispersion Measure

• For wave packet propagation the relevant quantity is the group velocity defined by demanding that the phase remain stationary for constructive interference

$$\begin{split} \phi(k) &= kz - \omega(k)t \\ \frac{\partial \phi}{\partial k} &= 0 = z - \frac{\partial \omega}{\partial k}t = z - v_g t \\ v_g &= \frac{\partial \omega}{\partial k} = c\sqrt{1 - \frac{\omega_p^2}{\omega^2}} \le c \\ &\approx c(1 - \frac{1}{2}\frac{\omega_p^2}{\omega^2}) \qquad [\omega \gg \omega_p] \end{split}$$

• Photons effectively gain a mass leading to a delay in arrival times

Dispersion Measure

• For a pulse of radiation from a pulsar

$$t_p = \int_0^d \frac{ds}{v_g} \approx \frac{d}{c} + \frac{1}{2c\omega^2} \int_0^d \omega_p^2 ds$$

$$t_p = \frac{d}{c} + \frac{2\pi e^2}{mc\omega^2} \left[\int_0^d n ds \equiv D \right]$$
$$\frac{\partial t_p}{\partial \omega} = -\frac{4\pi e^2}{mc\omega^3} D$$

Change in arrival time with frequency \rightarrow dispersion measure \rightarrow distance given a mean n

• In an external magnetic field $\mathbf{B}_0 = B_0 \mathbf{e}_3$ the electron responds to the magnetic field as well as the electric field of the radiation

$$m\frac{d\mathbf{v}}{dt} = e\mathbf{E} + \frac{e}{c}\mathbf{v} \times \mathbf{B}_0$$

• Examine the propagation of circularly polarized states

$$\mathbf{E}_{\pm}(t) = E_{\pm}e^{-i\omega t}[\mathbf{e}_1 \pm i\mathbf{e}_2]$$

• Take a trial solution

$$\mathbf{v}_{\pm}(t) = v_{\pm}e^{-i\omega t}[\mathbf{e}_{1} \pm i\mathbf{e}_{2}]$$
$$-im\omega v_{\pm}[\mathbf{e}_{1} \pm i\mathbf{e}_{2}] = eE_{\pm}[\mathbf{e}_{1} \pm i\mathbf{e}_{2}] + \frac{e}{c}v_{\pm}B_{0}[-\mathbf{e}_{2} \pm i\mathbf{e}_{1}]$$
$$= [eE_{\pm} \pm i\frac{e}{c}B_{0}v_{\pm}][\mathbf{e}_{1} \pm i\mathbf{e}_{2}]$$

$$-i(\omega m \pm \frac{e}{c}B_0)v_{\pm} = eE_{\pm}$$
$$v_{\pm} = \frac{ieE_{\pm}}{m(\omega \pm \omega_B)} \qquad \left[\omega_B = \frac{eB_0}{mc}\right]$$

• Conductivity

$$\sigma = \frac{\mathbf{j}_{\pm}}{\mathbf{E}_{\pm}} = \frac{env_{\pm}}{E_{\pm}} = \frac{ie^2n}{m(\omega \pm \omega_B)}$$
$$\epsilon_{\pm} = 1 - \frac{4\pi\sigma}{\omega i}$$
$$= 1 - \frac{4\pi ne^2}{m(\omega \pm \omega_B)\omega}$$
$$= 1 - \frac{\omega_p^2}{\omega(\omega \pm \omega_B)}$$

 Right and left polarizations travel at different velocities: disperson relation for ω ≫ ω_B and ω ≫ ω_p

$$k_{\pm} = \frac{\omega}{c} \sqrt{\epsilon_{\pm}} \approx \frac{\omega}{c} \left[1 - \frac{1}{2} \frac{\omega_p^2}{\omega^2} \left(1 \pm \frac{\omega_B}{\omega} \right) \right]$$



Considering linear polarization

 as a superposition of right and left circular
 polarization, the difference in propagation
 speeds will lead to a Faraday rotation of the linear polarization

• Phase

$$\phi_{\pm} = \int_0^d k_{\pm} ds$$
$$\frac{\Delta \phi}{2} = \frac{1}{2} \int_0^d (k_+ - k_-) ds = \frac{1}{2c} \int_0^d \frac{\omega_p^2}{\omega^2} \omega_B ds$$

• $\Delta \phi/2$ gives the rotation of linear polarization

$$\Delta \theta = \frac{\Delta \phi}{2} = \frac{1}{2c\omega^2} \frac{4\pi e^2}{m} \frac{e}{mc} \int_0^d B_0 n ds$$
$$= \frac{2\pi e^3}{m^2 c^2 \omega^2} \int B_0 n ds$$

• More generally $B_0 \rightarrow B_{\parallel}$ the line of sight component

 Given an average n measure B – e.g. magnetic field of ISM, cluster

E & M Potentials

- Introduce the vector and scalar potential to simplify source calculation
- $\nabla \cdot \mathbf{B} = 0$ implies that $\mathbf{B} = \nabla \times \mathbf{A}$, where \mathbf{A} is the vector potential
- So Faraday's law becomes

$$\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial}{\partial t} (\nabla \times \mathbf{A})$$
$$\nabla \times [\mathbf{E} + \frac{1}{c} \frac{\partial}{\partial t} \mathbf{A}] = 0$$

implying a scalar potential ϕ

$$\mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} = -\nabla \phi$$

Gauge

• Potentials (ϕ, \mathbf{A}) allow for gauge freedom. Given a change in the potentials through an arbitrary field ψ

$$\mathbf{A}' = \mathbf{A} + \nabla \psi$$
$$\phi' = \phi - \frac{1}{c} \frac{\partial \psi}{\partial t}$$

the observable ${\bf E}$ and ${\bf B}$ fields invariant

$$\begin{split} \mathbf{E} &= -\frac{1}{c} \frac{\partial \mathbf{A}'}{\partial t} - \nabla \phi' \\ &= -\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} - \frac{1}{c} \frac{\partial}{\partial t} \nabla \psi - \nabla \phi + \frac{1}{c} \frac{\partial}{\partial t} \nabla \psi \\ &= -\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} - \nabla \phi \\ \mathbf{B} &= \nabla \times \mathbf{A}' = \nabla \times \mathbf{A} \end{split}$$

Lorentz Gauge

- Gauge freedom allows one to choose a convenient gauge to simplify equations
- Choose a gauge where the relationship between the potentials is

$$\nabla \cdot \mathbf{A} + \frac{1}{c} \frac{\partial \phi}{\partial t} = 0$$

• Maxwell equations simplify to

$$\nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = -4\pi\rho$$
$$\nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\frac{4\pi}{c} \mathbf{j}$$

Retarded Potentials

- Green function solutions (propagate a δ function disturbance; superimpose to get arbitrary source. [See Jackson]
- Looks like electrostatics but accounts for the finite propagation time of light

$$\phi(\mathbf{r},t) = \int \frac{[\rho]d^3r'}{|\mathbf{r} - \mathbf{r}'|}$$
$$\mathbf{A}(\mathbf{r},t) = \frac{1}{c} \int \frac{[\mathbf{j}]d^3r'}{|\mathbf{r} - \mathbf{r}'|}$$

where the [] denotes evaluation at the retarded time

$$[f](\mathbf{r}',t) = f(\mathbf{r}',t - \frac{1}{c}|\mathbf{r} - \mathbf{r}'|)$$

Lienard-Wiechart Potential

• Consider a single charge on a trajectory $\mathbf{r}_0(t)$ with velocity $\mathbf{u} = \dot{\mathbf{r}}_0(t)$

$$\rho(\mathbf{r}, t) = q\delta(\mathbf{r} - \mathbf{r}_0(t))$$
$$\mathbf{j}(\mathbf{r}, t) = q\mathbf{u}\delta(\mathbf{r} - \mathbf{r}_0(t))$$

• Scalar potential



$$\begin{split} \phi(\mathbf{r},t) &= \int d^3r' \int dt' \frac{\rho(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|} \delta(t'-t+\frac{1}{c}|\mathbf{r}-\mathbf{r}'|) \\ &= \int d^3r' \int dt' \frac{q\delta(\mathbf{r}'-\mathbf{r}_0(t'))}{|\mathbf{r}-\mathbf{r}'|} \delta(t'-t+\frac{1}{c}|\mathbf{r}-\mathbf{r}'|) \\ &= q \int dt' \delta(t'-t+\frac{R(t')}{c}) \frac{1}{R(t')} \qquad \mathbf{R}(t) = |\mathbf{r}-\mathbf{r}_0(t)| \end{split}$$

Lienard-Wiechart Potential

Change variables so that explicit integration possible

$$t'' = t' - t + \frac{R(t')}{c}$$

$$dt'' = dt' + \frac{1}{c}\dot{R}(t')dt'$$

$$= [1 + \frac{1}{c}\dot{R}(t')]dt'$$

$$\dot{\mathbf{R}} = -\dot{\mathbf{r}}_0 = -\mathbf{u}, \qquad \hat{\mathbf{n}} = \frac{\mathbf{R}}{R}, \qquad \dot{\mathbf{R}} \cdot \hat{\mathbf{n}} = -\mathbf{u} \cdot \hat{\mathbf{n}}$$

$$2R\dot{R} = 2\dot{\mathbf{R}} \cdot \mathbf{R} \rightarrow \dot{R} = \dot{\mathbf{R}} \cdot \hat{\mathbf{n}} = -\mathbf{u} \cdot \hat{\mathbf{n}}$$

$$\kappa(t') \equiv 1 + \frac{1}{c}\dot{R}(t') = 1 - \frac{1}{c}\mathbf{u} \cdot \hat{\mathbf{n}}$$

which is the origin of relativistic beaming effects

Lienard-Wiechart Potential

• Thus since $t'' = 0 \rightarrow t' = t - R(t')/c$

$$\phi(\mathbf{r},t) = q \int dt'' \delta(t'') \frac{1}{\kappa(t')R(t')} = \frac{q}{\kappa R}\Big|_{t''=0} = \left[\frac{q}{\kappa R}\right]$$

• Similarly

$$\mathbf{A}(\mathbf{r},t) = \left[\frac{q\mathbf{u}}{c\kappa R}\right]$$

- For non-relativistic velocities $\kappa = 1 \mathbf{u}/c \cdot \hat{\mathbf{n}} \approx 1$ and potential are just retarded versions of electrostatic potentials
- For u → c then enhanced radiation along u || n from relativistic beaming

E&B Field

• Plug and chug with

$$\begin{split} \mathbf{E} &= -\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} - \nabla \phi, \qquad \mathbf{B} = \nabla \times \mathbf{A} \\ \mathbf{E} &= \mathbf{E}_{\text{vel}} + \mathbf{E}_{\text{rad}} \qquad \mathbf{B} = \hat{\mathbf{n}} \times \mathbf{E} \end{split}$$

• Velocity field falls off as $1/R^2$, $\beta = \mathbf{u}/c$ as a generalization of Coulomb's law

$$\mathbf{E}_{\rm vel} = q \left[\frac{(\hat{\mathbf{n}} - \beta)(1 - \beta^2)}{\kappa^3 R^2} \right]$$

• The radiation field depends on the acceleration and falls off as 1/R so that there is a flux $E^2 \propto 1/R^2$ that propagates to infinity

$$\mathbf{E}_{\rm rad} = \frac{q}{c} \left[\frac{\hat{\mathbf{n}}}{\kappa^3 R} \times \left((\hat{\mathbf{n}} - \beta) \times \dot{\beta} \right) \right]$$

Larmor Formula

• Larmor

formula: non relativistic case $\beta \ll 1$

$$\mathbf{E}_{\text{rad}} = \frac{q}{c} \frac{\hat{\mathbf{n}}}{R} \times (\hat{\mathbf{n}} \times \dot{\beta})$$
$$= \frac{q}{c^2 R} \hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \dot{\mathbf{u}})$$

• Let $\dot{\mathbf{u}}/\dot{u}\cdot\hat{\mathbf{n}}=\cos\Theta$

$$|\mathbf{E}_{\rm rad}| = \frac{q\dot{u}}{Rc^2}\sin\Theta$$

• So flux

$$S = \frac{c}{4\pi} E_{\rm rad}^2 \hat{\mathbf{n}} = \frac{c}{4\pi} \frac{q^2 \dot{u}^2}{R^2 c^4} \sin^2 \Theta = \frac{1}{4\pi} \frac{q^2 \dot{u}^2}{R^2 c^3} \sin^2 \Theta$$



Larmor Formula

• Power per unit angle $dA = R^2 d\Omega$

$$\frac{dW}{dtd\Omega} = \frac{q^2 \dot{u}^2}{4\pi c^3} \sin^2 \Theta$$
$$P = \frac{dW}{dt} = \frac{q^2 \dot{u}^2}{4\pi c^3} \int d\Omega \sin^2 \Theta = \frac{2q^2 \dot{u}^2}{3c^3}$$

- So dipole pattern of radiation is perpendicular to acceleration and polarization is in plane spanned by \dot{u} and \hat{n}
- Larmor formula can be used more generally in that one can transform to a frame where the particles are non-relativistic via Lorentz transformation