Set 5:

Classical E&M and Plasma Processes

Maxwell Equations

- Classical E&M defined by the Maxwell Equations (fields sourced by matter) and the Lorentz force (matter moved by fields)
- In cgs (gaussian) units

$$\nabla \cdot \mathbf{D} = 4\pi\rho, \qquad \nabla \cdot \mathbf{B} = 0,$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \qquad \nabla \times \mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} + \frac{4\pi}{c}\mathbf{j}.$$

where ρ = charge density, $\mathbf{j}$ = current density, $\mathbf{D} = \epsilon\mathbf{E}$ and $\mathbf{B} = \mu\mathbf{H}$ with ϵ = dielectric constant μ = magnetic permeability

- In vacuum, $\epsilon = \mu = 1$ and the Maxwell equations simplify to

$$\nabla \cdot \mathbf{E} = 4\pi\rho, \qquad \nabla \cdot \mathbf{B} = 0,$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \qquad \nabla \times \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c}\mathbf{j},$$

Charge Conservation

- Maxwell equations are E and B symmetric aside from the lack of magnetic charges
- Divergence of Ampere's law $\rightarrow$ charge conservation

$$\nabla \cdot [\nabla \times \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c}\mathbf{j}]$$

$$0 = \frac{1}{c}\frac{\partial \nabla \cdot \mathbf{E}}{\partial t} + \frac{4\pi}{c}\nabla \cdot \mathbf{j}$$

$$0 = \frac{4\pi}{c}\frac{\partial \rho}{\partial t} + \frac{4\pi}{c}\nabla \cdot \mathbf{j}$$

$$0 = \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j}$$

Wave Equation

- Source free propagation

$$\nabla \cdot \mathbf{E} = 0, \qquad \nabla \cdot \mathbf{B} = 0,$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \qquad \nabla \times \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t},$$

invariant under $\mathbf{E} \to \mathbf{B}$ and $\mathbf{B} \to -\mathbf{E}$, so work out equation for $\mathbf{E}$

- Curl of Faraday's law $\to$ wave equation

$$\nabla \times [\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}]$$

$$\nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\frac{1}{c}\frac{\partial \nabla \times \mathbf{B}}{\partial t}$$

$$-\nabla^2 \mathbf{E} = -\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}$$

$$\nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

Wave Equation

- Similarly for **B**

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\begin{pmatrix}\mathbf{E}\\ \mathbf{B}\end{pmatrix} = 0$$

- Wave solutions $k = 2\pi/\lambda = 2\pi\nu/c$ [real part or superimposed $\mathbf{k}$ and $-\mathbf{k}$]

$$\begin{pmatrix}\mathbf{E}\\ \mathbf{B}\end{pmatrix} = \begin{pmatrix}E_0\mathbf{e}_1\\ B_0\mathbf{e}_2\end{pmatrix} e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$$

- Wave solution in wave equation provides the “dispersion” relation

$$k^2 - \frac{1}{c^2}\omega^2 = 0, \qquad \omega = kc = 2\pi\nu$$

Wave Equation

- E and B fields related by Maxwell equations

$$i\mathbf{k}\cdot\hat{\mathbf{e}}_1 E_0 = 0, \qquad i\mathbf{k}\cdot\hat{\mathbf{e}}_2 B_0 = 0,$$

$$i\mathbf{k}\times\hat{\mathbf{e}}_1 E_0 = i\frac{\omega}{c}\hat{\mathbf{e}}_2 B_0, \qquad i\mathbf{k}\times\hat{\mathbf{e}}_2 B_0 = -i\frac{\omega}{c}\hat{\mathbf{e}}_1 E_0, ,$$

so $\hat{\mathbf{e}}_1 \perp \mathbf{k},\ \hat{\mathbf{e}}_2 \perp \mathbf{k},\ \mathbf{k}\times\hat{\mathbf{e}}_1 \parallel \hat{\mathbf{e}}_2$. So $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{k}})$ form a orthonormal right handed basis

$$i\mathbf{k}\times\hat{\mathbf{e}}_1 E_0 = i\frac{\omega}{c}\hat{\mathbf{e}}_2 B_0, \quad kE_0 = \frac{\omega}{c}B_0, \quad \to E_0 = B_0$$

Lorentz Force

- Lorentz force

$$\mathbf{F} = q\left[\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right]$$

  where q is the charge

- For a distribution of charges in a volume, charge and current density $\mathbf{j} = \rho\mathbf{v}$ provide a force density

$$\mathbf{f} = \rho\mathbf{E} + \frac{1}{c}\mathbf{j} \times \mathbf{B}$$

- Work and change in energy density

$$W = \int \mathbf{F} \cdot d\mathbf{x}, \qquad \frac{dW}{dt} = \mathbf{F} \cdot \mathbf{v} = q\mathbf{E} \cdot \mathbf{v}$$

$$\frac{du_{\rm mech}}{dt} = \rho\mathbf{E} \cdot \mathbf{v} = \mathbf{E} \cdot \mathbf{j}$$

Field Energy

- Use energy conservation to define field and radiation energy

$$\frac{4\pi}{c}\mathbf{j} = \nabla \times \mathbf{H} - \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t}$$

$$\mathbf{j} = \frac{c}{4\pi}\left(\nabla \times \mathbf{H} - \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t}\right)$$

$$\frac{du_{\rm mech}}{dt} = \mathbf{E}\cdot\mathbf{j} = \frac{c}{4\pi}\left(\nabla \times \mathbf{H} - \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t}\right)\cdot\mathbf{E}$$

$$[\mathbf{E}\cdot(\nabla\times\mathbf{H}) = \mathbf{H}\cdot(\nabla\times\mathbf{E}) - \nabla\cdot(\mathbf{E}\times\mathbf{H})]$$

$$\frac{du_{\rm mech}}{dt} = \frac{c}{4\pi}\left[\mathbf{H}\cdot(\nabla\times\mathbf{E}) - \nabla\cdot(\mathbf{E}\times\mathbf{H}) - \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t}\cdot\mathbf{E}\right]$$

$$\left[\nabla\times\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \quad \mathbf{H} = \mathbf{B}/\mu, \quad \mathbf{D} = \epsilon\mathbf{E}\right]$$

$$= \frac{1}{4\pi}\left[-\frac{1}{\mu}\frac{\partial \mathbf{B}}{\partial t}\cdot\mathbf{B} - \epsilon\frac{\partial \mathbf{E}}{\partial t}\cdot\mathbf{E} - c\nabla\cdot(\mathbf{E}\times\mathbf{H})\right]$$

Field Energy

- Rewrite field terms as a total derivative

$$\frac{\partial}{\partial t}(\mathbf{B}\cdot\mathbf{B}) = 2\mathbf{B}\cdot\frac{\partial \mathbf{B}}{\partial t}, \qquad \frac{\partial}{\partial t}(\mathbf{E}\cdot\mathbf{E}) = 2\mathbf{E}\cdot\frac{\partial \mathbf{E}}{\partial t}$$

- Equation forms a conservation law

$$\frac{\partial}{\partial t}(u_{\mathrm{mech}} + u_{\mathrm{field}}) + \nabla\cdot\mathbf{S} = 0$$

with

$$u_{\mathrm{field}} = \frac{1}{8\pi}\left(\epsilon E^2 + \frac{1}{\mu}B^2\right)$$

and the energy flux carried by the radiation, or Poynting vector

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{H}$$

Field Energy

- For vacuum $\mu = \epsilon = 1$ and so field energy in a monocromatic wave, time averaged over the oscillation

$$\langle u_{\rm field} \rangle = \frac{1}{8\pi}\frac{1}{2}(E_0^2 + B_0^2) = \frac{1}{8\pi}E_0^2$$

$$\langle S \rangle = \frac{c}{4\pi}\frac{1}{2}E_0 B_0 = \frac{c}{8\pi}E_0^2$$

which says that $\langle S \rangle / \langle u_{\rm field} \rangle = c$, (recall $I_\nu / u_\nu = c$)

- Energy Flux

$$\langle S \rangle = \frac{c}{8\pi}E_0^2 = \int d\nu \int d\Omega \cos\theta I_\nu$$

so the specific intensity of a monocromatic plane wave is a delta function in frequency and angle

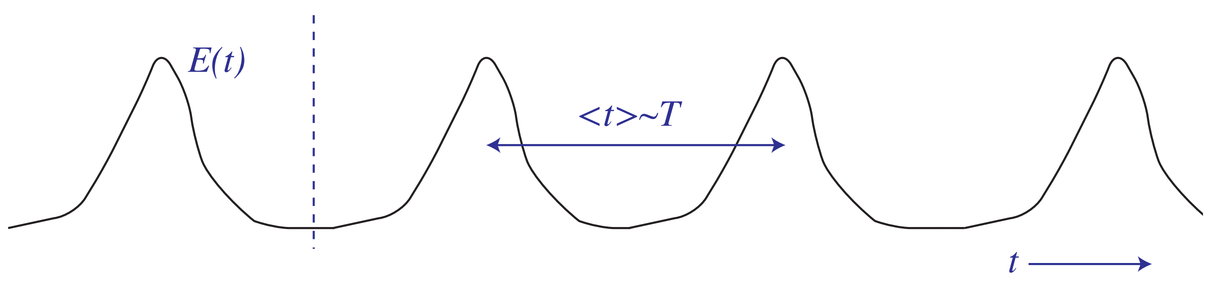
Specific Intensity

- Actual processes will not be monocromatic but have some waveform associated with the acceleration of the emitter: superposition of plane waves

$$E(t) = \int_{-\infty}^{\infty} E(\omega) e^{-i\omega t} d\omega$$

$$E(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E(t) e^{i\omega t} dt$$

Total emission will be an incoherent superposition of these single particle sources



Specific Intensity

- Energy flux normal to propagation direction

$$\frac{dW}{dtdA} = S = \frac{c}{4\pi} E^2(t)$$

- Total energy passing through dA

$$\frac{dW}{dA} = \frac{c}{4\pi} \int_{-\infty}^{\infty} E^2(t) dt$$

$$\left[\int_{-\infty}^{\infty} E^2(t) dt = 2\pi \int_{-\infty}^{\infty} |E(\omega)|^2 d\omega \right]$$

$$= \frac{c}{4\pi} \left[4\pi \int_{0}^{\infty} |E(\omega)|^2 d\omega \right]$$

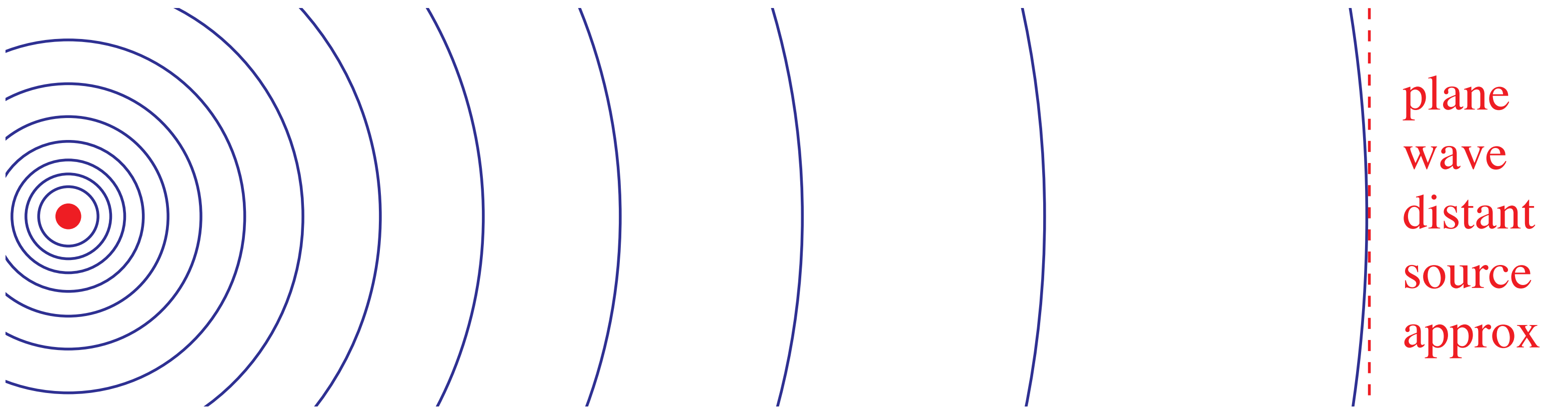
$$\frac{dW}{dAd\omega} = c|E(\omega)|^2$$

Specific Intensity

- Now given a timescale T for single particle processes

$$\frac{dW}{dA d\omega dt} = \frac{1}{T}\frac{dW}{dA d\omega} = \frac{c}{T}|E(\omega)|^2$$

which given $\omega = 2\pi\nu$ is the specific flux. Given a resolved source divide by the source solid angle to get I_ν



Stokes Parameters

- Specific intensity is related to quadratic combinations of the field. Define the intensity matrix (time averaged over oscillations) $\langle \mathbf{E}\,\mathbf{E}^\dagger \rangle$
- Hermitian matrix can be decomposed into Pauli matrices

$$\mathbf{P} = \langle \mathbf{E}\,\mathbf{E}^\dagger \rangle = \frac{1}{2}\left(I\boldsymbol{\sigma}_0 + Q\,\boldsymbol{\sigma}_3 + U\,\boldsymbol{\sigma}_1 - V\,\boldsymbol{\sigma}_2\right)\,,$$

where

$$\boldsymbol{\sigma}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \boldsymbol{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \boldsymbol{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \boldsymbol{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- Stokes parameters recovered as $\mathrm{Tr}(\sigma_i \mathbf{P})$

Stokes Parameters

- Consider a general plane wave solution

$$\mathbf{E}(t,z) = E_1(t,z)\hat{\mathbf{e}}_1 + E_2(t,z)\hat{\mathbf{e}}_2$$

$$E_1(t,z) = A_1 e^{i\phi_1} e^{i(kz-\omega t)}$$

$$E_2(t,z) = A_2 e^{i\phi_2} e^{i(kz-\omega t)}$$

- Explicitly:

$$I = \langle E_1 E_1^* + E_2 E_2^* \rangle = A_1^2 + A_2^2$$

$$Q = \langle E_1 E_1^* - E_2 E_2^* \rangle = A_1^2 - A_2^2$$

$$U = \langle E_1 E_2^* + E_2 E_1^* \rangle = 2A_1 A_2 \cos(\phi_2 - \phi_1)$$

$$V = -i \langle E_1 E_2^* - E_2 E_1^* \rangle = 2A_1 A_2 \sin(\phi_2 - \phi_1)$$

so that the Stokes parameters define the state up to an unobservable overall phase of the wave

Polarization

- Radiation field involves a directed quantity, the electric field vector, which defines the polarization
- Consider a general plane wave solution

$$\mathbf{E}(t,z) = E_1(t,z)\hat{\mathbf{e}}_1 + E_2(t,z)\hat{\mathbf{e}}_2$$

$$E_1(t,z) = \mathrm{Re} A_1 e^{i\phi_1} e^{i(kz-\omega t)}$$

$$E_2(t,z) = \mathrm{Re} A_2 e^{i\phi_2} e^{i(kz-\omega t)}$$

or at $z = 0$ the field vector traces out an ellipse

$$\mathbf{E}(t,0) = A_1 \cos(\omega t - \phi_1)\hat{\mathbf{e}}_1 + A_2 \cos(\omega t - \phi_2)\hat{\mathbf{e}}_2$$

with principal axes defined by

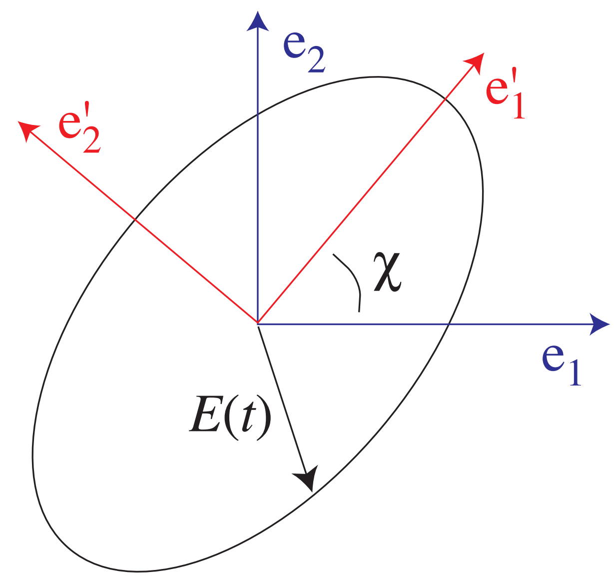
$$\mathbf{E}(t,0) = A_1' \cos(\omega t)\hat{\mathbf{e}}_1' - A_2' \sin(\omega t)\hat{\mathbf{e}}_2'$$

so as to trace out a clockwise rotation for $A_1', A_2' > 0$

Polarization

- Define polarization angle

$$\hat{\mathbf{e}}'_1 = \cos\chi\hat{\mathbf{e}}_1 + \sin\chi\hat{\mathbf{e}}_2$$
$$\hat{\mathbf{e}}'_2 = -\sin\chi\hat{\mathbf{e}}_1 + \cos\chi\hat{\mathbf{e}}_2$$



- Match

$$\begin{aligned}\mathbf{E}(t,0) &= A'_1\cos\omega t[\cos\chi\hat{\mathbf{e}}_1 + \sin\chi\hat{\mathbf{e}}_2] \\ &\quad - A'_2\cos\omega t[-\sin\chi\hat{\mathbf{e}}_1 + \cos\chi\hat{\mathbf{e}}_2] \\ &= A_1[\cos\phi_1\cos\omega t + \sin\phi_1\sin\omega t]\hat{\mathbf{e}}_1 \\ &\quad + A_2[\cos\phi_2\cos\omega t + \sin\phi_2\sin\omega t]\hat{\mathbf{e}}_2\end{aligned}$$

Polarization

- Define relative strength of two principal states

$$A_1' = E_0 \cos\beta \quad A_2' = E_0 \sin\beta$$

- Characterize the polarization by two angles

$$A_1 \cos\phi_1 = E_0 \cos\beta \cos\chi, \qquad A_1 \sin\phi_1 = E_0 \sin\beta \sin\chi,$$
$$A_2 \cos\phi_2 = E_0 \cos\beta \sin\chi, \qquad A_2 \sin\phi_2 = -E_0 \sin\beta \cos\chi$$

Or Stokes parameters by

$$I = E_0^2\,, \quad Q = E_0^2 \cos 2\beta \cos 2\chi$$
$$U = E_0^2 \cos 2\beta \sin 2\chi\,, \quad V = E_0^2 \sin 2\beta$$

- So $I^2 = Q^2 + U^2 + V^2$, double angles reflect the spin 2 field or headless vector nature of polarization

Polarization

Special cases

- If $\beta = 0, \pi/2, \pi$ then only one principal axis, ellipse collapses to a line and $V = 0 \rightarrow$ linear polarization oriented at angle χ

  If $\chi = 0, \pi/2, \pi$ then $I = \pm Q$ and $U = 0$

  If $\chi = \pi/4, 3\pi/4...$ then $I = \pm U$ and $Q = 0$ - so U is Q in a frame rotated by 45 degrees

- If $\beta = \pi/4, 3\pi/4$, then principal components have equal strength and E field rotates on a circle: $I = \pm V$ and $Q = U = 0 \rightarrow$ circular polarization

- $U/Q = \tan 2\chi$ defines angle of linear polarization and $V/I = \sin 2\beta$ defines degree of circular polarization

Natural Light

- A monochromatic plane wave is completely polarized $I^2 = Q^2 + U^2 + V^2$
- Polarization matrix is like a density matrix in quantum mechanics and allows for pure (coherent) states and mixed states
- Suppose the total $\mathbf{E}_{\rm tot}$ field is composed of different (frequency) components

$$\mathbf{E}_{\rm tot} = \sum_i \mathbf{E}_i$$

- Then components decorrelate in time average

$$\left\langle \mathbf{E}_{\rm tot} \mathbf{E}_{\rm tot}^\dagger \right\rangle = \sum_{ij} \left\langle \mathbf{E}_i \mathbf{E}_j^\dagger \right\rangle = \sum_i \left\langle \mathbf{E}_i \mathbf{E}_i^\dagger \right\rangle$$

Natural Light

- So Stokes parameters of incoherent contributions add

$$I = \sum_i I_i \quad Q = \sum_i Q_i \quad U = \sum_i U_i \quad V = \sum_i V_i$$

and since individual Q, U and V can have either sign:
$I^2 \geq Q^2 + U^2 + V^2$, all 4 Stokes parameters needed

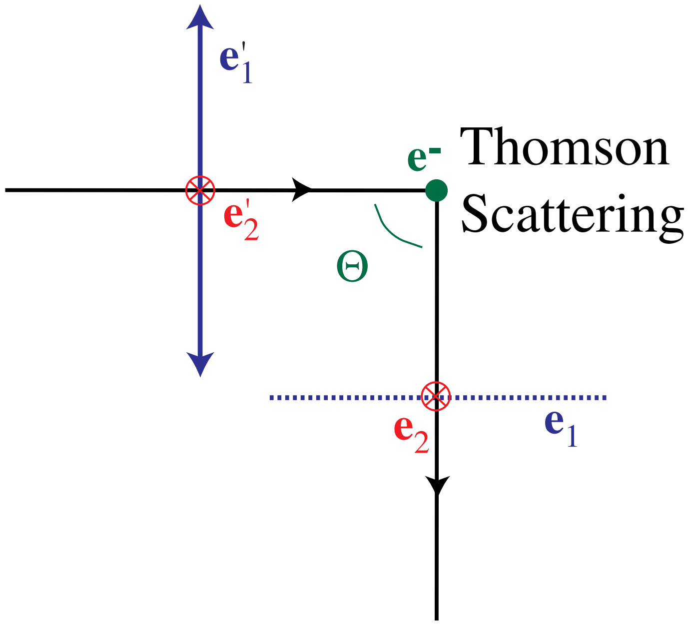
Polarized Radiative Transfer

- Define a specific intensity "vector": $\mathbf{I}_\nu = (I_{\nu 1}, I_{\nu 2}, U, V)$ where $I = I_{\nu 1} + I_{\nu 2}, Q = I_{\nu 1} - I_{\nu 2}$

$$\frac{d\mathbf{I}_\nu}{ds} = \alpha_\nu(\mathbf{S}_\nu - \mathbf{I}_\nu)$$

- Source vector in practice can be complicated
- Thomson collision based on differential cross section

$$\frac{d\sigma_T}{d\Omega} = \frac{3}{8\pi}|\hat{\mathbf{E}}' \cdot \hat{\mathbf{E}}|^2 \sigma_T \ ,$$



Polarized Radiative Transfer

- $\hat{\mathbf{E}}'$ and $\hat{\mathbf{E}}$ denote the incoming and outgoing directions of the electric field or polarization vector.
- Thomson scattering into $\mathbf{e}_1$: $I_{\nu 2} \to I_{\nu 2}$ but $I_{\nu 1}$ does not scatter
- More generally if Θ is the scattering angle then referenced to the plane of the scattering $\alpha_\nu = n_e \sigma_T$ and

$$\mathbf{S}_\nu = \frac{3}{8\pi} \int d\Omega' \begin{pmatrix} \cos^2\Theta & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\Theta & 0 \\ 0 & 0 & 0 & \cos\Theta \end{pmatrix} \mathbf{I}'_\nu$$

- But to calculate Stokes parameters in a fixed coordinate system must rotate into the scattering basis, scatter and rotate back out to the fixed coordinate system

Plasma Effects

- Astrophysical media are typically ionized so that radiation does not propagate in a vacuum but through an ionized plasma.
- However the plasma is typically so rarified that only the very lowest frequency radiation is affected
- Maxwell equations for plane wave radiation $\exp[i(\mathbf{k}\cdot\mathbf{r}-\omega t)]$ with sources

$$i\mathbf{k}\cdot\mathbf{E} = 4\pi\rho, \qquad i\mathbf{k}\cdot\mathbf{B} = 0,$$

$$i\mathbf{k}\times\mathbf{E} = i\frac{\omega}{c}\mathbf{B}, \qquad i\mathbf{k}\times\mathbf{B} = -i\frac{\omega}{c}\mathbf{E} + \frac{4\pi}{c}\mathbf{j},$$

- Medium is globally neutral but electric field of the radiation cause a high frequency electron drift $\rightarrow$ current $\rightarrow$ charge via continuity

Plasma Sources

- Lorentz force

$$m\dot{\mathbf{v}} = e\mathbf{E} \quad \to \mathbf{v} = -\frac{e\mathbf{E}}{i\omega m}$$

- Current density carried by electrons of number density n

$$\mathbf{j} = ne\mathbf{v} = -\frac{ne^2\mathbf{E}}{i\omega m} \equiv \sigma\mathbf{E}$$

$$\sigma = \frac{ine^2}{\omega m} \quad \text{conductivity}$$

- Charge conservation

$$-i\omega\rho + i\mathbf{k}\cdot\mathbf{j} = 0$$

$$\rho = \frac{\mathbf{k}\cdot\mathbf{j}}{\omega} = \frac{\sigma}{\omega}\mathbf{k}\cdot\mathbf{E}$$

Plasma Frequency

- Maxwell equation with ρ

$$i\mathbf{k}\cdot\mathbf{E} = 4\pi\frac{\sigma}{\omega}\mathbf{k}\cdot\mathbf{E}$$

$$i\left(1-\frac{4\pi\sigma}{\omega i}\right)\mathbf{k}\cdot\mathbf{E} = 0$$

$$i\mathbf{k}\cdot\epsilon\mathbf{E} = 0$$

with the dielectric constant

$$\epsilon = 1-\frac{4\pi\sigma}{\omega i} = 1-\frac{4\pi n e^2}{m\omega^2}$$

$$= 1-\frac{\omega_p^2}{\omega^2} \qquad \left[\omega_p^2 = \frac{4\pi n e^2}{m}\right]$$

Plasma Frequency

- Likewise the Maxwell equation with $\mathbf{j}$

$$
\begin{aligned}
i\mathbf{k} \times \mathbf{B} &= \frac{4\pi}{c}\mathbf{j} - i\frac{\omega}{c}\mathbf{E} \\
&= \left(\frac{4\pi}{c}\sigma - i\frac{\omega}{c}\right)\mathbf{E} \\
i\mathbf{k} \times \mathbf{B} &= -i\frac{\omega}{c}\epsilon\mathbf{E}
\end{aligned}
$$

- So that the Maxwell equations become source free equations

$$
i\mathbf{k} \cdot \epsilon\mathbf{E} = 0, \qquad i\mathbf{k} \cdot \mathbf{B} = 0,
$$

$$
i\mathbf{k} \times \mathbf{E} = i\frac{\omega}{c}\mathbf{B}, \qquad i\mathbf{k} \times \mathbf{B} = -i\frac{\omega}{c}\epsilon\mathbf{E},
$$

Wave Equation

- Wave equation becomes (similarly for $\mathbf{B} \perp \mathbf{E}$)

$$i[\mathbf{k} \times (\mathbf{k} \times \mathbf{E})] = -ik^2\mathbf{E} = i\frac{\omega}{c}\mathbf{k} \times \mathbf{B} = -i\frac{\omega^2}{c^2}\epsilon\mathbf{E}$$

- Modified dispersion relation

$$k^2 = \frac{\omega^2}{c^2}\epsilon = \frac{\omega^2}{c^2}\left(1 - \frac{\omega_p^2}{\omega^2}\right)$$

$$k = \frac{1}{c}\sqrt{\omega^2 - \omega_p^2}$$

- If $\omega < \omega_p$ then k is imaginary and the wave function has an exponential suppression - waves don't propagate below the plasma frequency

$$\nu_p = \frac{\omega_p}{2\pi} = 0.01\,\mathrm{MHz}\left(\frac{n}{1\mathrm{cm}^{-3}}\right)^{1/2}$$

Plasma Cutoff & Refraction

- For the ionosphere $n \sim 10^4$ cm^{-3} and radio waves at < 1MHz cannot propagate
- For ISM $n < 1$ cm^{-3} and the cut off is a much smaller < 0.01 MHz
- The phase velocity defines the index of refaction

$$v_p = \frac{\omega}{k} \equiv \frac{c}{n_r} \quad \rightarrow n_r \equiv \sqrt{\epsilon} = \sqrt{1 - \frac{\omega_p^2}{\omega^2}}$$

- Radio waves can be refracted according to Snell's law and change their direction of propagation along the path s

$$\frac{dn_r\hat{\mathbf{k}}}{ds} = \nabla n_r$$

Dispersion Measure

- For wave packet propagation the relevant quantity is the group velocity defined by demanding that the phase remain stationary for constructive interference

$$\phi(k) = kz - \omega(k)t$$

$$\frac{\partial \phi}{\partial k} = 0 = z - \frac{\partial \omega}{\partial k} t = z - v_g t$$

$$v_g = \frac{\partial \omega}{\partial k} = c\sqrt{1 - \frac{\omega_p^2}{\omega^2}} \leq c$$

$$\approx c\left(1 - \frac{1}{2}\frac{\omega_p^2}{\omega^2}\right) \qquad [\omega \gg \omega_p]$$

- Photons effectively gain a mass leading to a delay in arrival times

Dispersion Measure

- For a pulse of radiation from a pulsar

$$t_p = \int_0^d \frac{ds}{v_g} \approx \frac{d}{c} + \frac{1}{2c\omega^2} \int_0^d \omega_p^2 ds$$

$$t_p = \frac{d}{c} + \frac{2\pi e^2}{mc\omega^2} \left[\int_0^d n ds \equiv D \right]$$

$$\frac{\partial t_p}{\partial \omega} = -\frac{4\pi e^2}{mc\omega^3} D$$

Change in arrival time with frequency $\rightarrow$ dispersion measure $\rightarrow$ distance given a mean n

Faraday Rotation

- In an external magnetic field $\mathbf{B}_0 = B_0\mathbf{e}_3$ the electron responds to the magnetic field as well as the electric field of the radiation

$$m\frac{d\mathbf{v}}{dt} = e\mathbf{E} + \frac{e}{c}\mathbf{v} \times \mathbf{B}_0$$

- Examine the propagation of circularly polarized states

$$\mathbf{E}_\pm(t) = E_\pm e^{-i\omega t}[\mathbf{e}_1 \pm i\mathbf{e}_2]$$

- Take a trial solution

$$\mathbf{v}_\pm(t) = v_\pm e^{-i\omega t}[\mathbf{e}_1 \pm i\mathbf{e}_2]$$

$$\begin{aligned}-im\omega v_\pm[\mathbf{e}_1 \pm i\mathbf{e}_2] &= eE_\pm[\mathbf{e}_1 \pm i\mathbf{e}_2] + \frac{e}{c}v_\pm B_0[-\mathbf{e}_2 \pm i\mathbf{e}_1] \\ &= [eE_\pm \pm i\frac{e}{c}B_0 v_\pm][\mathbf{e}_1 \pm i\mathbf{e}_2]\end{aligned}$$

Faraday Rotation

$$-i(\omega m \pm \frac{e}{c}B_0)v_\pm = eE_\pm$$

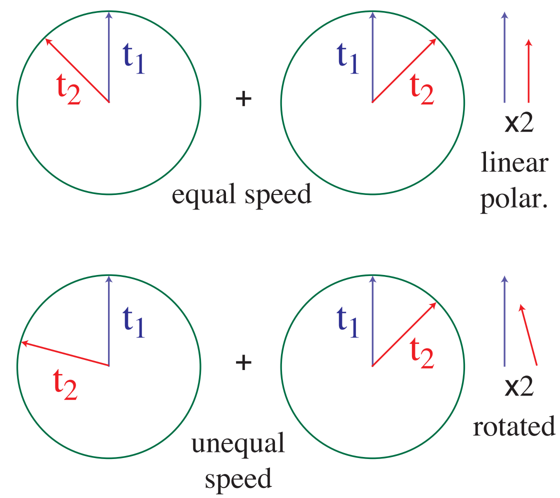
$$v_\pm = \frac{ieE_\pm}{m(\omega \pm \omega_B)} \qquad \left[\omega_B = \frac{eB_0}{mc}\right]$$

- Conductivity

$$\sigma = \frac{\mathbf{j}_\pm}{\mathbf{E}_\pm} = \frac{env_\pm}{E_\pm} = \frac{ie^2 n}{m(\omega \pm \omega_B)}$$

$$\begin{aligned}\epsilon_\pm &= 1 - \frac{4\pi\sigma}{\omega i} \\ &= 1 - \frac{4\pi n e^2}{m(\omega \pm \omega_B)\omega} \\ &= 1 - \frac{\omega_p^2}{\omega(\omega \pm \omega_B)}\end{aligned}$$

Faraday Rotation



- Right and left polarizations
  travel at different velocities: disperson
  relation for $\omega \gg \omega_B$ and $\omega \gg \omega_p$

$$k_{\pm} = \frac{\omega}{c}\sqrt{\epsilon_{\pm}} \approx \frac{\omega}{c}\left[1 - \frac{1}{2}\frac{\omega_p^2}{\omega^2}\left(1 \pm \frac{\omega_p}{\omega}\right)\right]$$

- Considering linear polarization
  as a superposition of right and left circular
  polarization, the difference in propagation
  speeds will lead to a Faraday rotation of the linear polarization

Faraday Rotation

- Phase

$$\phi_\pm = \int_0^d k_\pm ds$$

$$\frac{\Delta\phi}{2} = \frac{1}{2}\int_0^d (k_+ - k_-)ds = \frac{1}{2c}\int_0^d \frac{\omega_p^2}{\omega^2}\omega_B ds$$

- $\Delta\phi/2$ gives the rotation of linear polarization

$$\Delta\theta = \frac{\Delta\phi}{2} = \frac{1}{2c\omega^2}\frac{4\pi e^2}{m}\frac{e}{mc}\int_0^d B_0 n ds$$
$$= \frac{2\pi e^3}{m^2c^2\omega^2}\int B_0 n ds$$

- More generally $B_0 \rightarrow B_\parallel$ the line of sight component
- Given an average n measure B – e.g. magnetic field of ISM, cluster

E & M Potentials

- Introduce the vector and scalar potential to simplify source calculation
- $\nabla \cdot \mathbf{B} = 0$ implies that $\mathbf{B} = \nabla \times \mathbf{A}$, where $\mathbf{A}$ is the vector potential
- So Faraday's law becomes

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial}{\partial t}(\nabla \times \mathbf{A})$$

$$\nabla \times [\mathbf{E} + \frac{1}{c}\frac{\partial}{\partial t}\mathbf{A}] = 0$$

implying a scalar potential ϕ

$$\mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} = -\nabla\phi$$

Gauge

- Potentials $(\phi, \mathbf{A})$ allow for gauge freedom. Given a change in the potentials through an arbitrary field ψ

$$\mathbf{A}' = \mathbf{A} + \nabla\psi$$

$$\phi' = \phi - \frac{1}{c}\frac{\partial\psi}{\partial t}$$

the observable $\mathbf{E}$ and $\mathbf{B}$ fields invariant

$$\begin{aligned}
\mathbf{E} &= -\frac{1}{c}\frac{\partial\mathbf{A}'}{\partial t} - \nabla\phi' \\
&= -\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t} - \frac{1}{c}\frac{\partial}{\partial t}\nabla\psi - \nabla\phi + \frac{1}{c}\frac{\partial}{\partial t}\nabla\psi \\
&= -\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t} - \nabla\phi \\
\mathbf{B} &= \nabla\times\mathbf{A}' = \nabla\times\mathbf{A}
\end{aligned}$$

Lorentz Gauge

- Gauge freedom allows one to choose a convenient gauge to simplify equations
- Choose a gauge where the relationship between the potentials is

$$\nabla \cdot \mathbf{A} + \frac{1}{c}\frac{\partial \phi}{\partial t} = 0$$

- Maxwell equations simplify to

$$\nabla^2 \phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = -4\pi\rho$$

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\frac{4\pi}{c}\mathbf{j}$$

Retarded Potentials

- Green function solutions (propagate a δ function disturbance; superimpose to get arbitrary source. [See Jackson]
- Looks like electrostatics but accounts for the finite propagation time of light

$$\phi(\mathbf{r},t) = \int \frac{[\rho] d^3 r'}{|\mathbf{r}-\mathbf{r}'|}$$

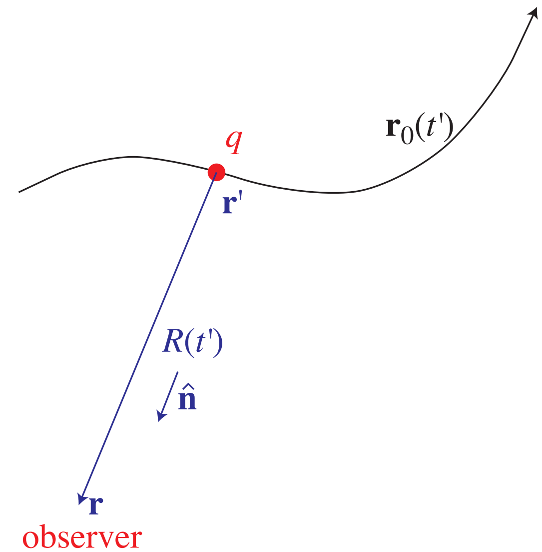
$$\mathbf{A}(\mathbf{r},t) = \frac{1}{c}\int \frac{[\mathbf{j}] d^3 r'}{|\mathbf{r}-\mathbf{r}'|}$$

where the [] denotes evaluation at the retarded time

$$[f](\mathbf{r}',t) = f(\mathbf{r}', t - \frac{1}{c}|\mathbf{r}-\mathbf{r}'|)$$

Lienard-Wiechart Potential

- Consider a single charge on a trajectory $\mathbf{r}_0(t)$ with velocity $\mathbf{u} = \dot{\mathbf{r}}_0(t)$



$$\rho(\mathbf{r}, t) = q\delta(\mathbf{r} - \mathbf{r}_0(t))$$

$$\mathbf{j}(\mathbf{r}, t) = q\mathbf{u}\delta(\mathbf{r} - \mathbf{r}_0(t))$$

- Scalar potential

$$\phi(\mathbf{r}, t) = \int d^3r' \int dt' \frac{\rho(\mathbf{r}', t')}{|\mathbf{r} - \mathbf{r}'|}\delta(t' - t + \frac{1}{c}|\mathbf{r} - \mathbf{r}'|)$$

$$= \int d^3r' \int dt' \frac{q\delta(\mathbf{r}' - \mathbf{r}_0(t'))}{|\mathbf{r} - \mathbf{r}'|}\delta(t' - t + \frac{1}{c}|\mathbf{r} - \mathbf{r}'|)$$

$$= q \int dt' \delta(t' - t + \frac{R(t')}{c})\frac{1}{R(t')} \qquad \mathbf{R}(t) = |\mathbf{r} - \mathbf{r}_0(t)|$$

Lienard-Wiechart Potential

Change variables so that explicit integration possible

$$t'' = t' - t + \frac{R(t')}{c}$$

$$dt'' = dt' + \frac{1}{c}\dot{R}(t')dt'$$

$$= [1 + \frac{1}{c}\dot{R}(t')]dt'$$

$$\dot{\mathbf{R}} = -\dot{\mathbf{r}}_0 = -\mathbf{u}, \qquad \hat{\mathbf{n}} = \frac{\mathbf{R}}{R}, \qquad \dot{\mathbf{R}} \cdot \hat{\mathbf{n}} = -\mathbf{u} \cdot \hat{\mathbf{n}}$$

$$2R\dot{R} = 2\dot{\mathbf{R}} \cdot \mathbf{R} \rightarrow \dot{R} = \dot{\mathbf{R}} \cdot \hat{\mathbf{n}} = -\mathbf{u} \cdot \hat{\mathbf{n}}$$

$$\kappa(t') \equiv 1 + \frac{1}{c}\dot{R}(t') = 1 - \frac{1}{c}\mathbf{u} \cdot \hat{\mathbf{n}}$$

which is the origin of relativistic beaming effects

Lienard-Wiechart Potential

- Thus since $t'' = 0 \rightarrow t' = t - R(t')/c$

$$\phi(\mathbf{r}, t) = q \int dt'' \delta(t'') \frac{1}{\kappa(t')R(t')} = \left. \frac{q}{\kappa R} \right|_{t''=0} = \left[\frac{q}{\kappa R} \right]$$

- Similarly

$$\mathbf{A}(\mathbf{r}, t) = \left[\frac{q\mathbf{u}}{c\kappa R} \right]$$

- For non-relativistic velocities $\kappa = 1 - \mathbf{u}/c \cdot \hat{\mathbf{n}} \approx 1$ and potential are just retarded versions of electrostatic potentials
- For $u \rightarrow c$ then enhanced radiation along $\mathbf{u} \parallel \mathbf{n}$ from relativistic beaming

E&B Field

- Plug and chug with

$$\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \nabla\phi, \qquad \mathbf{B} = \nabla \times \mathbf{A}$$

$$\mathbf{E} = \mathbf{E}_{\rm vel} + \mathbf{E}_{\rm rad} \qquad \mathbf{B} = \hat{\mathbf{n}} \times \mathbf{E}$$

- Velocity field falls off as $1/R^2$, $\beta = \mathbf{u}/c$ as a generalization of Coulomb's law

$$\mathbf{E}_{\rm vel} = q\left[\frac{(\hat{\mathbf{n}} - \beta)(1-\beta^2)}{\kappa^3 R^2}\right]$$

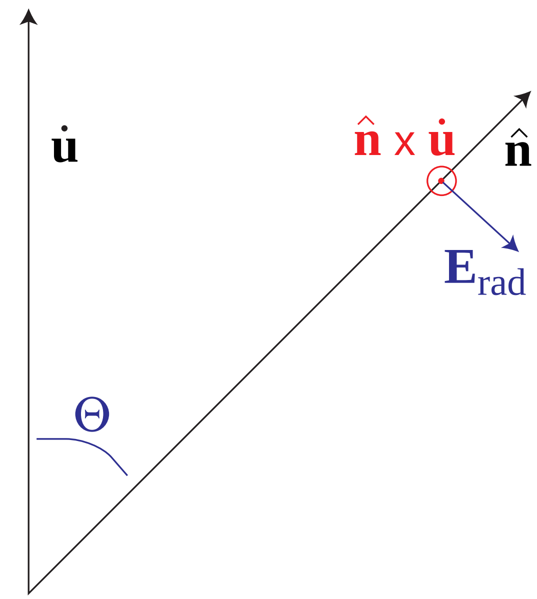
- The radiation field depends on the acceleration and falls off as $1/R$ so that there is a flux $E^2 \propto 1/R^2$ that propagates to infinity

$$\mathbf{E}_{\rm rad} = \frac{q}{c}\left[\frac{\hat{\mathbf{n}}}{\kappa^3 R} \times ((\hat{\mathbf{n}} - \beta) \times \dot{\beta})\right]$$

Larmor Formula

- Larmor formula: non relativistic case $\beta \ll 1$

$$\mathbf{E}_{\rm rad} = \frac{q}{c}\frac{\hat{\mathbf{n}}}{R} \times (\hat{\mathbf{n}} \times \dot{\beta})$$
$$= \frac{q}{c^2 R}\hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \dot{\mathbf{u}})$$



- Let $\dot{\mathbf{u}}/\dot{u} \cdot \hat{\mathbf{n}} = \cos\Theta$

$$|\mathbf{E}_{\rm rad}| = \frac{q\dot{u}}{Rc^2}\sin\Theta$$

- So flux

$$S = \frac{c}{4\pi}E^2_{\rm rad}\hat{\mathbf{n}} = \frac{c}{4\pi}\frac{q^2\dot{u}^2}{R^2c^4}\sin^2\Theta = \frac{1}{4\pi}\frac{q^2\dot{u}^2}{R^2c^3}\sin^2\Theta$$

Larmor Formula

- Power per unit angle $dA = R^2 d\Omega$

$$\frac{dW}{dt d\Omega} = \frac{q^2 \dot{u}^2}{4\pi c^3} \sin^2 \Theta$$

$$P = \frac{dW}{dt} = \frac{q^2 \dot{u}^2}{4\pi c^3} \int d\Omega \sin^2 \Theta = \frac{2q^2 \dot{u}^2}{3c^3}$$

- So dipole pattern of radiation is perpendicular to acceleration and polarization is in plane spanned by $\dot{\mathbf{u}}$ and $\hat{\mathbf{n}}$
- Larmor formula can be used more generally in that one can transform to a frame where the particles are non-relativistic via Lorentz transformation