## Set 5:

Classical E\&M and Plasma Processes

## Maxwell Equations

- Classical E\&M defined by the Maxwell Equations (fields sourced by matter) and the Lorentz force (matter moved by fields)
- In cgs (gaussian) units

$$
\begin{aligned}
\nabla \cdot \mathbf{D}=4 \pi \rho, & \nabla \cdot \mathbf{B}=0 \\
\nabla \times \mathbf{E}=-\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}, & \nabla \times \mathbf{H}=\frac{1}{c} \frac{\partial \mathbf{D}}{\partial t}+\frac{4 \pi}{c} \mathbf{j}
\end{aligned}
$$

where $\rho=$ charge density, $\mathbf{j}=$ current density, $\mathbf{D}=\epsilon \mathbf{E}$ and $\mathbf{B}=\mu \mathbf{H}$ with $\epsilon=$ dielectric constant $\mu=$ magnetic permeability

- In vacuum, $\epsilon=\mu=1$ and the Maxwell equations simplify to

$$
\begin{array}{cl}
\nabla \cdot \mathbf{E}=4 \pi \rho, & \nabla \cdot \mathbf{B}=0 \\
\nabla \times \mathbf{E}=-\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}, & \nabla \times \mathbf{B}=\frac{1}{c} \frac{\partial \mathbf{E}}{\partial t}+\frac{4 \pi}{c} \mathbf{j}
\end{array}
$$

## Charge Conservation

- Maxwell equations are $E$ and $B$ symmetric aside from the lack of magnetic charges
- Divergence of Ampere's law $\rightarrow$ charge conservation

$$
\begin{aligned}
\nabla \cdot[\nabla \times \mathbf{B} & \left.=\frac{1}{c} \frac{\partial \mathbf{E}}{\partial t}+\frac{4 \pi}{c} \mathbf{j}\right] \\
0 & =\frac{1}{c} \frac{\partial \nabla \cdot \mathbf{E}}{\partial t}+\frac{4 \pi}{c} \nabla \cdot \mathbf{j} \\
0 & =\frac{4 \pi}{c} \frac{\partial \rho}{\partial t}+\frac{4 \pi}{c} \nabla \cdot \mathbf{j} \\
0 & =\frac{\partial \rho}{\partial t}+\nabla \cdot \mathbf{j}
\end{aligned}
$$

## Wave Equation

- Source free propagation

$$
\begin{aligned}
\nabla \cdot \mathbf{E}=0, & \nabla \cdot \mathbf{B}=0 \\
\nabla \times \mathbf{E}=-\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}, & \nabla \times \mathbf{B}=\frac{1}{c} \frac{\partial \mathbf{E}}{\partial t}
\end{aligned}
$$

invariant under $\mathbf{E} \rightarrow \mathbf{B}$ and $\mathbf{B} \rightarrow-\mathbf{E}$, so work out equation for $\mathbf{E}$

- Curl of Faraday's law $\rightarrow$ wave equation

$$
\begin{aligned}
\nabla \times[\nabla \times \mathbf{E} & \left.=-\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}\right] \\
\nabla(\nabla \cdot \mathbf{E})-\nabla^{2} \mathbf{E} & =-\frac{1}{c} \frac{\partial \nabla \times \mathbf{B}}{\partial t} \\
-\nabla^{2} \mathbf{E} & =-\frac{1}{c^{2}} \frac{\partial^{2} \mathbf{E}}{\partial t^{2}} \\
\nabla^{2} \mathbf{E}-\frac{1}{c^{2}} \frac{\partial^{2} \mathbf{E}}{\partial t^{2}} & =0
\end{aligned}
$$

## Wave Equation

- Similarly for B

$$
\left[\nabla^{2}-\frac{1}{c^{2}} \frac{\partial^{2}}{\partial t^{2}}\right]\binom{\mathbf{E}}{\mathbf{B}}=0
$$

- Wave solutions $k=2 \pi / \lambda=2 \pi \nu / c$ [real part or superimposed $\mathbf{k}$ and -k ]

$$
\binom{\mathbf{E}}{\mathbf{B}}=\binom{E_{0} \mathbf{e}_{1}}{B_{0} \mathbf{e}_{2}} e^{i(\mathbf{k} \cdot \mathbf{x}-\omega t)}
$$

- Wave solution in wave equation provides the "dispersion" relation

$$
k^{2}-\frac{1}{c^{2}} \omega^{2}=0, \quad \omega=k c=2 \pi \nu
$$

## Wave Equation

- E and B fields related by Maxwell equations

$$
\begin{aligned}
i \mathbf{k} \cdot \hat{\mathbf{e}}_{1} E_{0}=0, & i \mathbf{k} \cdot \hat{\mathbf{e}}_{2} B_{0}=0 \\
i \mathbf{k} \times \hat{\mathbf{e}}_{1} E_{0}=i \frac{\omega}{c} \hat{\mathbf{e}}_{2} B_{0}, & i \mathbf{k} \times \hat{\mathbf{e}}_{2} B_{0}=-i \frac{\omega}{c} \hat{\mathbf{e}}_{1} E_{0}
\end{aligned}
$$

so $\hat{\mathbf{e}}_{1} \perp \mathbf{k}, \hat{\mathbf{e}}_{2} \perp \mathbf{k}, \mathbf{k} \times \hat{\mathbf{e}}_{1} \| \hat{\mathbf{e}}_{2}$. So ( $\left.\hat{\mathbf{e}}_{1}, \hat{\mathbf{e}}_{2}, \hat{\mathbf{k}}\right)$ form a orthonormal right handed basis

$$
i \mathbf{k} \times \hat{\mathbf{e}}_{1} E_{0}=i \frac{\omega}{c} \hat{\mathbf{e}}_{2} B_{0}, \quad k E_{0}=\frac{\omega}{c} B_{0}, \quad \rightarrow E_{0}=B_{0}
$$

## Lorentz Force

- Lorentz force

$$
\mathbf{F}=q\left[\mathbf{E}+\frac{\mathbf{v}}{c} \times \mathbf{B}\right]
$$

where $q$ is the charge

- For a distribution of charges in a volume, charge and current density $\mathbf{j}=\rho \mathbf{v}$ provide a force density

$$
\mathbf{f}=\rho \mathbf{E}+\frac{1}{c} \mathbf{j} \times \mathbf{B}
$$

- Work and change in energy density

$$
\begin{aligned}
W & =\int \mathbf{F} \cdot d \mathbf{x}, \quad \frac{d W}{d t}=\mathbf{F} \cdot \mathbf{v}=q \mathbf{E} \cdot \mathbf{v} \\
\frac{d u_{\mathrm{mech}}}{d t} & =\rho \mathbf{E} \cdot \mathbf{v}=\mathbf{E} \cdot \mathbf{j}
\end{aligned}
$$

## Field Energy

- Use energy conservation to define field and radiation energy

$$
\begin{aligned}
\frac{4 \pi}{c} \mathbf{j}= & \nabla \times \mathbf{H}-\frac{1}{c} \frac{\partial \mathbf{D}}{\partial t} \\
\mathbf{j}= & \frac{c}{4 \pi}\left(\nabla \times \mathbf{H}-\frac{1}{c} \frac{\partial \mathbf{D}}{\partial t}\right) \\
\frac{d u_{\text {mech }}}{d t}= & \mathbf{E} \cdot \mathbf{j}=\frac{c}{4 \pi}\left(\nabla \times \mathbf{H}-\frac{1}{c} \frac{\partial \mathbf{D}}{\partial t}\right) \cdot \mathbf{E} \\
& {[\mathbf{E} \cdot(\nabla \times \mathbf{H})=\mathbf{H} \cdot(\nabla \times \mathbf{E})-\nabla \cdot(\mathbf{E} \times \mathbf{H})] } \\
\frac{d u_{\text {mech }}}{d t}= & \frac{c}{4 \pi}\left[\mathbf{H} \cdot(\nabla \times \mathbf{E})-\nabla \cdot(\mathbf{E} \times \mathbf{H})-\frac{1}{c} \frac{\partial \mathbf{D}}{\partial t} \cdot \mathbf{E}\right] \\
& {\left[\nabla \times \mathbf{E}=-\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}, \quad \mathbf{H}=\mathbf{B} / \mu, \quad \mathbf{D}=\epsilon \mathbf{E}\right] } \\
= & \frac{1}{4 \pi}\left[-\frac{1}{\mu} \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{B}-\epsilon \frac{\partial \mathbf{E}}{\partial t} \cdot \mathbf{E}-c \nabla \cdot(\mathbf{E} \times \mathbf{H})\right]
\end{aligned}
$$

## Field Energy

- Rewrite field terms as a total derivative

$$
\frac{\partial}{\partial t}(\mathbf{B} \cdot \mathbf{B})=2 \mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t}, \quad \frac{\partial}{\partial t}(\mathbf{E} \cdot \mathbf{E})=2 \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t}
$$

- Equation forms a conservation law

$$
\frac{\partial}{\partial t}\left(u_{\text {mech }}+u_{\text {field }}\right)+\nabla \cdot \mathbf{S}=0
$$

with

$$
u_{\text {field }}=\frac{1}{8 \pi}\left(\epsilon E^{2}+\frac{1}{\mu} B^{2}\right)
$$

and the energy flux carried by the radiation, or Poynting vector

$$
\mathbf{S}=\frac{c}{4 \pi} \mathbf{E} \times \mathbf{H}
$$

## Field Energy

- For vacuum $\mu=\epsilon=1$ and so field energy in a monocromatic wave, time averaged over the oscillation

$$
\begin{aligned}
\left\langle u_{\text {field }}\right\rangle & =\frac{1}{8 \pi} \frac{1}{2}\left(E_{0}^{2}+B_{0}^{2}\right)=\frac{1}{8 \pi} E_{0}^{2} \\
\langle S\rangle & =\frac{c}{4 \pi} \frac{1}{2} E_{0} B_{0}=\frac{c}{8 \pi} E_{0}^{2}
\end{aligned}
$$

which says that $\langle S\rangle /\left\langle u_{\text {field }}\right\rangle=c$, (recall $\left.I_{\nu} / u_{\nu}=c\right)$

- Energy Flux

$$
\langle S\rangle=\frac{c}{8 \pi} E_{0}^{2}=\int d \nu \int d \Omega \cos \theta I_{\nu}
$$

so the specific intensity of a monocromatic plane wave is a delta function in frequency and angle

## Specific Intensity

- Actual processes will not be monocromatic but have some waveform associated with the acceleration of the emitter: superposition of plane waves

$$
\begin{aligned}
E(t) & =\int_{-\infty}^{\infty} E(\omega) e^{-i \omega t} d \omega \\
E(\omega) & =\frac{1}{2 \pi} \int_{-\infty}^{\infty} E(t) e^{i \omega t} d t
\end{aligned}
$$

Total emission will be an incoherent superposition of these single particle sources


## Specific Intensity

- Energy flux normal to propagation direction

$$
\frac{d W}{d t d A}=S=\frac{c}{4 \pi} E^{2}(t)
$$

- Total energy passing through $d A$

$$
\begin{aligned}
\frac{d W}{d A}= & \frac{c}{4 \pi} \int_{-\infty}^{\infty} E^{2}(t) d t \\
& {\left[\int_{-\infty}^{\infty} E^{2}(t) d t=2 \pi \int_{-\infty}^{\infty}|E(\omega)|^{2} d \omega\right] } \\
= & \frac{c}{4 \pi}\left[4 \pi \int_{0}^{\infty}|E(\omega)|^{2} d \omega\right] \\
\frac{d W}{d A d \omega}= & c|E(\omega)|^{2}
\end{aligned}
$$

## Specific Intensity

- Now given a timescale $T$ for single particle processes

$$
\frac{d W}{d A d \omega d t}=\frac{1}{T} \frac{d W}{d A d \omega}=\frac{c}{T}|E(\omega)|^{2}
$$

which given $\omega=2 \pi \nu$ is the specific flux. Given a resolved source divide by the source solid angle to get $I_{\nu}$


## Stokes Parameters

- Specific intensity is related to quadratic combinations of the field. Define the intensity matrix (time averaged over oscillations) $\left\langle\mathbf{E} \mathbf{E}^{\dagger}\right\rangle$
- Hermitian matrix can be decomposed into Pauli matrices

$$
\mathbf{P}=\left\langle\mathbf{E} \mathbf{E}^{\dagger}\right\rangle=\frac{1}{2}\left(I \boldsymbol{\sigma}_{0}+Q \boldsymbol{\sigma}_{3}+U \boldsymbol{\sigma}_{1}-V \boldsymbol{\sigma}_{2}\right)
$$

where

$$
\boldsymbol{\sigma}_{0}=\left(\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right), \boldsymbol{\sigma}_{1}=\left(\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right), \boldsymbol{\sigma}_{2}=\left(\begin{array}{cc}
0 & -i \\
i & 0
\end{array}\right), \boldsymbol{\sigma}_{3}=\left(\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right)
$$

- Stokes parameters recovered as $\operatorname{Tr}\left(\sigma_{i} \mathbf{P}\right)$


## Stokes Parameters

- Consider a general plane wave solution

$$
\begin{aligned}
\mathbf{E}(t, z) & =E_{1}(t, z) \hat{\mathbf{e}}_{1}+E_{2}(t, z) \hat{\mathbf{e}}_{2} \\
E_{1}(t, z) & =A_{1} e^{i \phi_{1}} e^{i(k z-\omega t)} \\
E_{2}(t, z) & =A_{2} e^{i \phi_{2}} e^{i(k z-\omega t)}
\end{aligned}
$$

- Explicitly:

$$
\begin{aligned}
I & =\left\langle E_{1} E_{1}^{*}+E_{2} E_{2}^{*}\right\rangle=A_{1}^{2}+A_{2}^{2} \\
Q & =\left\langle E_{1} E_{1}^{*}-E_{2} E_{2}^{*}\right\rangle=A_{1}^{2}-A_{2}^{2} \\
U & =\left\langle E_{1} E_{2}^{*}+E_{2} E_{1}^{*}\right\rangle=2 A_{1} A_{2} \cos \left(\phi_{2}-\phi_{1}\right) \\
V & =-i\left\langle E_{1} E_{2}^{*}-E_{2} E_{1}^{*}\right\rangle=2 A_{1} A_{2} \sin \left(\phi_{2}-\phi_{1}\right)
\end{aligned}
$$

so that the Stokes parameters define the state up to an unobservable overall phase of the wave

## Polarization

- Radiation field involves a directed quantity, the electric field vector, which defines the polarization
- Consider a general plane wave solution

$$
\begin{aligned}
\mathbf{E}(t, z) & =E_{1}(t, z) \hat{\mathbf{e}}_{1}+E_{2}(t, z) \hat{\mathbf{e}}_{2} \\
E_{1}(t, z) & =\operatorname{Re} A_{1} e^{i \phi_{1}} e^{i(k z-\omega t)} \\
E_{2}(t, z) & =\operatorname{Re} A_{2} e^{i \phi_{2}} e^{i(k z-\omega t)}
\end{aligned}
$$

or at $z=0$ the field vector traces out an ellipse

$$
\mathbf{E}(t, 0)=A_{1} \cos \left(\omega t-\phi_{1}\right) \hat{\mathbf{e}}_{1}+A_{2} \cos \left(\omega t-\phi_{2}\right) \hat{\mathbf{e}}_{2}
$$

with principal axes defined by

$$
\mathbf{E}(t, 0)=A_{1}^{\prime} \cos (\omega t) \hat{\mathbf{e}}_{1}^{\prime}-A_{2}^{\prime} \sin (\omega t) \hat{\mathbf{e}}_{2}^{\prime}
$$

so as to trace out a clockwise rotation for $A_{1}^{\prime}, A_{2}^{\prime}>0$

## Polarization

- Define polarization angle

$$
\begin{aligned}
& \hat{\mathbf{e}}_{1}^{\prime}=\cos \chi \hat{\mathbf{e}}_{1}+\sin \chi \hat{\mathbf{e}}_{2} \\
& \hat{\mathbf{e}}_{2}^{\prime}=-\sin \chi \hat{\mathbf{e}}_{1}+\cos \chi \hat{\mathbf{e}}_{2}
\end{aligned}
$$

- Match

$$
\begin{aligned}
\mathbf{E}(t, 0)= & A_{1}^{\prime} \cos \omega t\left[\cos \chi \hat{\mathbf{e}}_{1}+\sin \chi \hat{\mathbf{e}}_{2}\right] \\
& -A_{2}^{\prime} \cos \omega t\left[-\sin \chi \hat{\mathbf{e}}_{1}+\cos \chi \hat{\mathbf{e}}_{2}\right] \\
= & A_{1}\left[\cos \phi_{1} \cos \omega t+\sin \phi_{1} \sin \omega t\right] \hat{\mathbf{e}}_{1} \\
& +A_{2}\left[\cos \phi_{2} \cos \omega t+\sin \phi_{2} \sin \omega t\right] \hat{\mathbf{e}}_{2}
\end{aligned}
$$

## Polarization

- Define relative strength of two principal states

$$
A_{1}^{\prime}=E_{0} \cos \beta \quad A_{2}^{\prime}=E_{0} \sin \beta
$$

- Characterize the polarization by two angles

$$
\begin{array}{ll}
A_{1} \cos \phi_{1}=E_{0} \cos \beta \cos \chi, & A_{1} \sin \phi_{1}=E_{0} \sin \beta \sin \chi \\
A_{2} \cos \phi_{2}=E_{0} \cos \beta \sin \chi, & A_{2} \sin \phi_{2}=-E_{0} \sin \beta \cos \chi
\end{array}
$$

Or Stokes parameters by

$$
\begin{aligned}
I & =E_{0}^{2}, \quad Q=E_{0}^{2} \cos 2 \beta \cos 2 \chi \\
U & =E_{0}^{2} \cos 2 \beta \sin 2 \chi, \quad V=E_{0}^{2} \sin 2 \beta
\end{aligned}
$$

- So $I^{2}=Q^{2}+U^{2}+V^{2}$, double angles reflect the spin 2 field or headless vector nature of polarization


## Polarization

Special cases

- If $\beta=0, \pi / 2, \pi$ then only one principal axis, ellipse collapses to a line and $V=0 \rightarrow$ linear polarization oriented at angle $\chi$

$$
\begin{aligned}
& \text { If } \chi=0, \pi / 2, \pi \text { then } I= \pm Q \text { and } U=0 \\
& \text { If } \chi=\pi / 4,3 \pi / 4 \ldots \text { then } I= \pm U \text { and } Q=0-\text { so } U \text { is } Q \text { in a } \\
& \text { frame rotated by } 45 \text { degrees }
\end{aligned}
$$

- If $\beta=\pi / 4,3 \pi / 4$, then principal components have equal strength and $E$ field rotates on a circle: $I= \pm V$ and $Q=U=0 \rightarrow$ circular polarization
- $U / Q=\tan 2 \chi$ defines angle of linear polarization and $V / I=\sin 2 \beta$ defines degree of circular polarization


## Natural Light

- A monochromatic plane wave is completely polarized $I^{2}=Q^{2}+U^{2}+V^{2}$
- Polarization matrix is like a density matrix in quantum mechanics and allows for pure (coherent) states and mixed states
- Suppose the total $\mathbf{E}_{\text {tot }}$ field is composed of different (frequency) components

$$
\mathbf{E}_{\mathrm{tot}}=\sum_{i} \mathbf{E}_{i}
$$

- Then components decorrelate in time average

$$
\left\langle\mathbf{E}_{\mathrm{tot}} \mathbf{E}_{\mathrm{tot}}^{\dagger}\right\rangle=\sum_{i j}\left\langle\mathbf{E}_{i} \mathbf{E}_{j}^{\dagger}\right\rangle=\sum_{i}\left\langle\mathbf{E}_{i} \mathbf{E}_{i}^{\dagger}\right\rangle
$$

## Natural Light

- So Stokes parameters of incoherent contributions add

$$
I=\sum_{i} I_{i} \quad Q=\sum_{i} Q_{i} \quad U=\sum_{i} U_{i} \quad V=\sum_{i} V_{i}
$$

and since individual $Q, U$ and $V$ can have either sign:
$I^{2} \geq Q^{2}+U^{2}+V^{2}$, all 4 Stokes parameters needed

## Polarized Radiative Transfer

- Define a specific intensity "vector": $\mathbf{I}_{\nu}=\left(I_{\nu 1}, I_{\nu 2}, U, V\right)$ where $I=I_{\nu 1}+I_{\nu 2}, Q=I_{\nu 1}-I_{\nu 2}$

$$
\frac{d \mathbf{I}_{\nu}}{d s}=\alpha_{\nu}\left(\mathbf{S}_{\nu}-\mathbf{I}_{\nu}\right)
$$

- Source vector in practice can be complicated
- Thomson collision
based on differential cross section

$$
\frac{d \sigma_{T}}{d \Omega}=\frac{3}{8 \pi}\left|\hat{\mathbf{E}}^{\prime} \cdot \hat{\mathbf{E}}\right|^{2} \sigma_{T}
$$



## Polarized Radiative Transfer

- $\hat{\mathbf{E}}^{\prime}$ and $\hat{\mathbf{E}}$ denote the incoming and outgoing directions of the electric field or polarization vector.
- Thomson scattering into $\mathbf{e}_{1}: I_{\nu 2} \rightarrow I_{\nu 2}$ but $I_{\nu 1}$ does not scatter
- More generally if $\Theta$ is the scattering angle then referenced to the plane of the scattering $\alpha_{\nu}=n_{e} \sigma_{T}$ and

$$
\mathbf{S}_{\nu}=\frac{3}{8 \pi} \int d \Omega^{\prime}\left(\begin{array}{cccc}
\cos ^{2} \Theta & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \cos \Theta & 0 \\
0 & 0 & 0 & \cos \Theta
\end{array}\right) \mathbf{I}_{\nu}^{\prime}
$$

- But to calculate Stokes parameters in a fixed coordinate system must rotate into the scattering basis, scatter and rotate back out to the fixed coordinate system


## Plasma Effects

- Astrophysical media are typically ionized so that radiation does not propagate in a vacuum but through an ionized plasma.
- However the plasma is typically so rarified that only the very lowest frequency radiation is affected
- Maxwell equations for plane wave radiation $\exp [i(\mathbf{k} \cdot \mathbf{r}-\omega t)]$ with sources

$$
\begin{aligned}
i \mathbf{k} \cdot \mathbf{E}=4 \pi \rho, & i \mathbf{k} \cdot \mathbf{B}=0 \\
i \mathbf{k} \times \mathbf{E}=i \frac{\omega}{c} \mathbf{B}, & i \mathbf{k} \times \mathbf{B}=-i \frac{\omega}{c} \mathbf{E}+\frac{4 \pi}{c} \mathbf{j}
\end{aligned}
$$

- Medium is globally neutral but electric field of the radiation cause a high frequency electron drift $\rightarrow$ current $\rightarrow$ charge via continuity


## Plasma Sources

- Lorentz force

$$
m \dot{\mathbf{v}}=e \mathbf{E} \quad \rightarrow \mathbf{v}=-\frac{e \mathbf{E}}{i \omega m}
$$

- Current density carried by electrons of number density $n$

$$
\begin{aligned}
& \mathbf{j}=n e \mathbf{v}=-\frac{n e^{2} \mathbf{E}}{i \omega m} \equiv \sigma \mathbf{E} \\
& \sigma=\frac{i n e^{2}}{\omega m} \quad \text { conductivity }
\end{aligned}
$$

- Charge conservation

$$
\begin{aligned}
-i \omega \rho+i \mathbf{k} \cdot \mathbf{j} & =0 \\
\rho=\frac{\mathbf{k} \cdot \mathbf{j}}{\omega} & =\frac{\sigma}{\omega} \mathbf{k} \cdot \mathbf{E}
\end{aligned}
$$

## Plasma Frequency

- Maxwell equation with $\rho$

$$
\begin{aligned}
i \mathbf{k} \cdot \mathbf{E} & =4 \pi \frac{\sigma}{\omega} \mathbf{k} \cdot \mathbf{E} \\
i\left(1-\frac{4 \pi \sigma}{\omega i}\right) \mathbf{k} \cdot \mathbf{E} & =0 \\
i \mathbf{k} \cdot \epsilon \mathbf{E} & =0
\end{aligned}
$$

with the dielectric constant

$$
\begin{aligned}
\epsilon & =1-\frac{4 \pi \sigma}{\omega i}=1-\frac{4 \pi n e^{2}}{m \omega^{2}} \\
& =1-\frac{\omega_{p}^{2}}{\omega^{2}} \quad\left[\omega_{p}^{2}=\frac{4 \pi n e^{2}}{m}\right]
\end{aligned}
$$

## Plasma Frequency

- Likewise the Maxwell equation with $\mathbf{j}$

$$
\begin{aligned}
i \mathbf{k} \times \mathbf{B} & =\frac{4 \pi}{c} \mathbf{j}-i \frac{\omega}{c} \mathbf{E} \\
& =\left(\frac{4 \pi}{c} \sigma-i \frac{\omega}{c}\right) \mathbf{E} \\
i \mathbf{k} \times \mathbf{B} & =-i \frac{\omega}{c} \epsilon \mathbf{E}
\end{aligned}
$$

- So that the Maxwell equations become source free equations

$$
\begin{aligned}
i \mathbf{k} \cdot \epsilon \mathbf{E}=0, & i \mathbf{k} \cdot \mathbf{B}=0 \\
i \mathbf{k} \times \mathbf{E}=i \frac{\omega}{c} \mathbf{B}, & i \mathbf{k} \times \mathbf{B}=-i \frac{\omega}{c} \epsilon \mathbf{E}
\end{aligned}
$$

## Wave Equation

- Wave equation becomes (similarly for $\mathbf{B} \perp \mathbf{E}$ )

$$
i[\mathbf{k} \times(\mathbf{k} \times \mathbf{E})]=-i k^{2} \mathbf{E}=i \frac{\omega}{c} \mathbf{k} \times \mathbf{B}=-i \frac{\omega^{2}}{c^{2}} \epsilon \mathbf{E}
$$

- Modified dispersion relation

$$
\begin{array}{r}
k^{2}=\frac{\omega^{2}}{c^{2}} \epsilon=\frac{\omega^{2}}{c^{2}}\left(1-\frac{\omega_{p}^{2}}{\omega^{2}}\right) \\
k=\frac{1}{c} \sqrt{\omega^{2}-\omega_{p}^{2}}
\end{array}
$$

- If $\omega<\omega_{p}$ then $k$ is imaginary and the wave function has an exponential suppression - waves don't propagate below the plasma frequency

$$
\nu_{p}=\frac{\omega_{p}}{2 \pi}=0.01 \mathrm{MHz}\left(\frac{n}{1 \mathrm{~cm}^{-3}}\right)^{1 / 2}
$$

## Plasma Cutoff \& Refraction

- For the ionosphere $n \sim 10^{4} \mathrm{~cm}^{-3}$ and radio waves at $<1 \mathrm{MHz}$ cannot propagate
- For ISM $n<1 \mathrm{~cm}^{-3}$ and the cut off is a much smaller $<0.01$ MHz
- The phase velocity defines the index of refaction

$$
v_{p}=\frac{\omega}{k} \equiv \frac{c}{n_{r}} \quad \rightarrow n_{r} \equiv \sqrt{\epsilon}=\sqrt{1-\frac{\omega_{p}^{2}}{\omega^{2}}}
$$

- Radio waves can be refracted according to Snell's law and change their direction of propagation along the path $s$

$$
\frac{d n_{r} \hat{\mathbf{k}}}{d s}=\nabla n_{r}
$$

## Dispersion Measure

- For wave packet propagation the relevant quantity is the group velocity defined by demanding that the phase remain stationary for constructive interference

$$
\begin{aligned}
\phi(k) & =k z-\omega(k) t \\
\frac{\partial \phi}{\partial k} & =0=z-\frac{\partial \omega}{\partial k} t=z-v_{g} t \\
v_{g} & =\frac{\partial \omega}{\partial k}=c \sqrt{1-\frac{\omega_{p}^{2}}{\omega^{2}}} \leq c \\
& \approx c\left(1-\frac{1}{2} \frac{\omega_{p}^{2}}{\omega^{2}}\right) \quad\left[\omega \gg \omega_{p}\right]
\end{aligned}
$$

- Photons effectively gain a mass leading to a delay in arrival times


## Dispersion Measure

- For a pulse of radiation from a pulsar

$$
\begin{array}{r}
t_{p}=\int_{0}^{d} \frac{d s}{v_{g}} \approx \frac{d}{c}+\frac{1}{2 c \omega^{2}} \int_{0}^{d} \omega_{p}^{2} d s \\
t_{p}=\frac{d}{c}+\frac{2 \pi e^{2}}{m c \omega^{2}}\left[\int_{0}^{d} n d s \equiv D\right] \\
\frac{\partial t_{p}}{\partial \omega}=-\frac{4 \pi e^{2}}{m c \omega^{3}} D
\end{array}
$$

Change in arrival time with frequency $\rightarrow$ dispersion measure $\rightarrow$ distance given a mean $n$

## Faraday Rotation

- In an external magnetic field $\mathbf{B}_{0}=B_{0} \mathbf{e}_{3}$ the electron responds to the magnetic field as well as the electric field of the radiation

$$
m \frac{d \mathbf{v}}{d t}=e \mathbf{E}+\frac{e}{c} \mathbf{v} \times \mathbf{B}_{0}
$$

- Examine the propagation of circularly polarized states

$$
\mathbf{E}_{ \pm}(t)=E_{ \pm} e^{-i \omega t}\left[\mathbf{e}_{1} \pm i \mathbf{e}_{2}\right]
$$

- Take a trial solution

$$
\begin{aligned}
\mathbf{v}_{ \pm}(t) & =v_{ \pm} e^{-i \omega t}\left[\mathbf{e}_{1} \pm i \mathbf{e}_{2}\right] \\
-i m \omega v_{ \pm}\left[\mathbf{e}_{1} \pm i \mathbf{e}_{2}\right] & =e E_{ \pm}\left[\mathbf{e}_{1} \pm i \mathbf{e}_{2}\right]+\frac{e}{c} v_{ \pm} B_{0}\left[-\mathbf{e}_{2} \pm i \mathbf{e}_{1}\right] \\
& =\left[e E_{ \pm} \pm i \frac{e}{c} B_{0} v_{ \pm}\right]\left[\mathbf{e}_{1} \pm i \mathbf{e}_{2}\right]
\end{aligned}
$$

## Faraday Rotation

$$
\begin{gathered}
-i\left(\omega m \pm \frac{e}{c} B_{0}\right) v_{ \pm}=e E_{ \pm} \\
v_{ \pm}=\frac{i e E_{ \pm}}{m\left(\omega \pm \omega_{B}\right)}\left[\omega_{B}=\frac{e B_{0}}{m c}\right]
\end{gathered}
$$

- Conductivity

$$
\begin{aligned}
\sigma & =\frac{\mathbf{j}_{ \pm}}{\mathbf{E}_{ \pm}}=\frac{e n v_{ \pm}}{E_{ \pm}}=\frac{i e^{2} n}{m\left(\omega \pm \omega_{B}\right)} \\
\epsilon_{ \pm} & =1-\frac{4 \pi \sigma}{\omega i} \\
& =1-\frac{4 \pi n e^{2}}{m\left(\omega \pm \omega_{B}\right) \omega} \\
& =1-\frac{\omega_{p}^{2}}{\omega\left(\omega \pm \omega_{B}\right)}
\end{aligned}
$$

## Faraday Rotation

- Right and left polarizations travel at different velocities: disperson relation for $\omega \gg \omega_{B}$ and $\omega \gg \omega_{p}$


$$
k_{ \pm}=\frac{\omega}{c} \sqrt{\epsilon_{ \pm}} \approx \frac{\omega}{c}\left[1-\frac{1}{2} \frac{\omega_{p}^{2}}{\omega^{2}}\left(1 \pm \frac{\omega_{B}}{\omega}\right)\right]
$$



- Considering linear polarization as a superposition of right and left circular polarization, the difference in propagation speeds will lead to a Faraday rotation of the linear polarization


## Faraday Rotation

- Phase

$$
\begin{aligned}
\phi_{ \pm} & =\int_{0}^{d} k_{ \pm} d s \\
\frac{\Delta \phi}{2} & =\frac{1}{2} \int_{0}^{d}\left(k_{+}-k_{-}\right) d s=\frac{1}{2 c} \int_{0}^{d} \frac{\omega_{p}^{2}}{\omega^{2}} \omega_{B} d s
\end{aligned}
$$

- $\Delta \phi / 2$ gives the rotation of linear polarization

$$
\begin{aligned}
\Delta \theta & =\frac{\Delta \phi}{2}=\frac{1}{2 c \omega^{2}} \frac{4 \pi e^{2}}{m} \frac{e}{m c} \int_{0}^{d} B_{0} n d s \\
& =\frac{2 \pi e^{3}}{m^{2} c^{2} \omega^{2}} \int B_{0} n d s
\end{aligned}
$$

- More generally $B_{0} \rightarrow B_{\|}$the line of sight component
- Given an average $n$ measure $B$ - e.g. magnetic field of ISM, cluster


## E \& M Potentials

- Introduce the vector and scalar potential to simplify source calculation
- $\nabla \cdot \mathbf{B}=0$ implies that $\mathbf{B}=\nabla \times \mathbf{A}$, where $\mathbf{A}$ is the vector potential
- So Faraday's law becomes

$$
\begin{aligned}
\nabla & \times \mathbf{E}=-\frac{1}{c} \frac{\partial}{\partial t}(\nabla \times \mathbf{A}) \\
\nabla \times\left[\mathbf{E}+\frac{1}{c} \frac{\partial}{\partial t} \mathbf{A}\right] & =0
\end{aligned}
$$

implying a scalar potential $\phi$

$$
\mathbf{E}+\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}=-\nabla \phi
$$

## Gauge

- Potentials $(\phi, \mathbf{A})$ allow for gauge freedom. Given a change in the potentials through an arbitrary field $\psi$

$$
\begin{aligned}
\mathbf{A}^{\prime} & =\mathbf{A}+\nabla \psi \\
\phi^{\prime} & =\phi-\frac{1}{c} \frac{\partial \psi}{\partial t}
\end{aligned}
$$

the observable $\mathbf{E}$ and $\mathbf{B}$ fields invariant

$$
\begin{aligned}
\mathbf{E} & =-\frac{1}{c} \frac{\partial \mathbf{A}^{\prime}}{\partial t}-\nabla \phi^{\prime} \\
& =-\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}-\frac{1}{c} \frac{\partial}{\partial t} \nabla \psi-\nabla \phi+\frac{1}{c} \frac{\partial}{\partial t} \nabla \psi \\
& =-\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}-\nabla \phi \\
\mathbf{B} & =\nabla \times \mathbf{A}^{\prime}=\nabla \times \mathbf{A}
\end{aligned}
$$

## Lorentz Gauge

- Gauge freedom allows one to choose a convenient gauge to simplify equations
- Choose a gauge where the relationship between the potentials is

$$
\nabla \cdot \mathbf{A}+\frac{1}{c} \frac{\partial \phi}{\partial t}=0
$$

- Maxwell equations simplify to

$$
\begin{aligned}
\nabla^{2} \phi-\frac{1}{c^{2}} \frac{\partial^{2} \phi}{\partial t^{2}} & =-4 \pi \rho \\
\nabla^{2} \mathbf{A}-\frac{1}{c^{2}} \frac{\partial^{2} \mathbf{A}}{\partial t^{2}} & =-\frac{4 \pi}{c} \mathbf{j}
\end{aligned}
$$

## Retarded Potentials

- Green function solutions (propagate a $\delta$ function disturbance; superimpose to get arbitrary source. [See Jackson]
- Looks like electrostatics but accounts for the finite propagation time of light

$$
\begin{aligned}
\phi(\mathbf{r}, t) & =\int \frac{[\rho] d^{3} r^{\prime}}{\left|\mathbf{r}-\mathbf{r}^{\prime}\right|} \\
\mathbf{A}(\mathbf{r}, t) & =\frac{1}{c} \int \frac{[\mathbf{j}] d^{3} r^{\prime}}{\left|\mathbf{r}-\mathbf{r}^{\prime}\right|}
\end{aligned}
$$

where the [] denotes evaluation at the retarded time

$$
[f]\left(\mathbf{r}^{\prime}, t\right)=f\left(\mathbf{r}^{\prime}, t-\frac{1}{c}\left|\mathbf{r}-\mathbf{r}^{\prime}\right|\right)
$$

## Lienard-Wiechart Potential

- Consider a single charge on a trajectory $\mathbf{r}_{0}(t)$ with velocity $\mathbf{u}=\dot{\mathbf{r}}_{0}(t)$

$$
\begin{aligned}
\rho(\mathbf{r}, t) & =q \delta\left(\mathbf{r}-\mathbf{r}_{0}(t)\right) \\
\mathbf{j}(\mathbf{r}, t) & =q \mathbf{u} \delta\left(\mathbf{r}-\mathbf{r}_{0}(t)\right)
\end{aligned}
$$

- Scalar potential


$$
\begin{aligned}
\phi(\mathbf{r}, t) & =\int d^{3} r^{\prime} \int d t^{\prime} \frac{\rho\left(\mathbf{r}^{\prime}, t^{\prime}\right)}{\left|\mathbf{r}-\mathbf{r}^{\prime}\right|} \delta\left(t^{\prime}-t+\frac{1}{c}\left|\mathbf{r}-\mathbf{r}^{\prime}\right|\right) \\
& =\int d^{3} r^{\prime} \int d t^{\prime} \frac{q \delta\left(\mathbf{r}^{\prime}-\mathbf{r}_{0}\left(t^{\prime}\right)\right)}{\left|\mathbf{r}-\mathbf{r}^{\prime}\right|} \delta\left(t^{\prime}-t+\frac{1}{c}\left|\mathbf{r}-\mathbf{r}^{\prime}\right|\right) \\
& =q \int d t^{\prime} \delta\left(t^{\prime}-t+\frac{R\left(t^{\prime}\right)}{c}\right) \frac{1}{R\left(t^{\prime}\right)} \quad \mathbf{R}(t)=\left|\mathbf{r}-\mathbf{r}_{0}(t)\right|
\end{aligned}
$$

## Lienard-Wiechart Potential

Change variables so that explicit integration possible

$$
\begin{aligned}
t^{\prime \prime} & =t^{\prime}-t+\frac{R\left(t^{\prime}\right)}{c} \\
d t^{\prime \prime} & =d t^{\prime}+\frac{1}{c} \dot{R}\left(t^{\prime}\right) d t^{\prime} \\
& =\left[1+\frac{1}{c} \dot{R}\left(t^{\prime}\right)\right] d t^{\prime} \\
\dot{\mathbf{R}} & =-\dot{\mathbf{r}}_{0}=-\mathbf{u}, \quad \hat{\mathbf{n}}=\frac{\mathbf{R}}{R}, \quad \dot{\mathbf{R}} \cdot \hat{\mathbf{n}}=-\mathbf{u} \cdot \hat{\mathbf{n}} \\
2 R \dot{R} & =2 \dot{\mathbf{R}} \cdot \mathbf{R} \rightarrow \dot{R}=\dot{\mathbf{R}} \cdot \hat{\mathbf{n}}=-\mathbf{u} \cdot \hat{\mathbf{n}} \\
\kappa\left(t^{\prime}\right) & \equiv 1+\frac{1}{c} \dot{R}\left(t^{\prime}\right)=1-\frac{1}{c} \mathbf{u} \cdot \hat{\mathbf{n}}
\end{aligned}
$$

which is the origin of relativistic beaming effects

## Lienard-Wiechart Potential

- Thus since $t^{\prime \prime}=0 \rightarrow t^{\prime}=t-R\left(t^{\prime}\right) / c$

$$
\phi(\mathbf{r}, t)=q \int d t^{\prime \prime} \delta\left(t^{\prime \prime}\right) \frac{1}{\kappa\left(t^{\prime}\right) R\left(t^{\prime}\right)}=\left.\frac{q}{\kappa R}\right|_{t^{\prime \prime}=0}=\left[\frac{q}{\kappa R}\right]
$$

- Similarly

$$
\mathbf{A}(\mathbf{r}, t)=\left[\frac{q \mathbf{u}}{c \kappa R}\right]
$$

- For non-relativistic velocities $\kappa=1-\mathbf{u} / c \cdot \hat{\mathbf{n}} \approx 1$ and potential are just retarded versions of electrostatic potentials
- For $u \rightarrow c$ then enhanced radiation along $\mathbf{u} \| \mathbf{n}$ from relativistic beaming


## E\&B Field

- Plug and chug with

$$
\begin{array}{rl}
\mathbf{E}=-\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}-\nabla \phi, & \mathbf{B}=\nabla \times \mathbf{A} \\
\mathbf{E}=\mathbf{E}_{\mathrm{vel}}+\mathbf{E}_{\mathrm{rad}} & \mathbf{B}=\hat{\mathbf{n}} \times \mathbf{E}
\end{array}
$$

- Velocity field falls off as $1 / R^{2}, \beta=\mathbf{u} / c$ as a generalization of Coulomb's law

$$
\mathbf{E}_{\mathrm{vel}}=q\left[\frac{(\hat{\mathbf{n}}-\beta)\left(1-\beta^{2}\right)}{\kappa^{3} R^{2}}\right]
$$

- The radiation field depends on the acceleration and falls off as $1 / R$ so that there is a flux $E^{2} \propto 1 / R^{2}$ that propagates to infinity

$$
\mathbf{E}_{\mathrm{rad}}=\frac{q}{c}\left[\frac{\hat{\mathbf{n}}}{\kappa^{3} R} \times((\hat{\mathbf{n}}-\beta) \times \dot{\beta})\right]
$$

## Larmor Formula

- Larmor
formula: non relativistic case $\beta \ll 1$

$$
\begin{aligned}
\mathbf{E}_{\mathrm{rad}} & =\frac{q}{c} \frac{\hat{\mathbf{n}}}{R} \times(\hat{\mathbf{n}} \times \dot{\beta}) \\
& =\frac{q}{c^{2} R} \hat{\mathbf{n}} \times(\hat{\mathbf{n}} \times \dot{\mathbf{u}})
\end{aligned}
$$

- Let $\dot{\mathbf{u}} / \dot{u} \cdot \hat{\mathbf{n}}=\cos \Theta$

$$
\left|\mathbf{E}_{\mathrm{rad}}\right|=\frac{q \dot{u}}{R c^{2}} \sin \Theta
$$

- So flux

$$
S=\frac{c}{4 \pi} E_{\mathrm{rad}}^{2} \hat{\mathbf{n}}=\frac{c}{4 \pi} \frac{q^{2} \dot{u}^{2}}{R^{2} c^{4}} \sin ^{2} \Theta=\frac{1}{4 \pi} \frac{q^{2} \dot{u}^{2}}{R^{2} c^{3}} \sin ^{2} \Theta
$$

## Larmor Formula

- Power per unit angle $d A=R^{2} d \Omega$

$$
\begin{aligned}
\frac{d W}{d t d \Omega} & =\frac{q^{2} \dot{u}^{2}}{4 \pi c^{3}} \sin ^{2} \Theta \\
P=\frac{d W}{d t} & =\frac{q^{2} \dot{u}^{2}}{4 \pi c^{3}} \int d \Omega \sin ^{2} \Theta=\frac{2 q^{2} \dot{u}^{2}}{3 c^{3}}
\end{aligned}
$$

- So dipole pattern of radiation is perpendicular to acceleration and polarization is in plane spanned by $\dot{\mathbf{u}}$ and $\hat{\mathbf{n}}$
- Larmor formula can be used more generally in that one can transform to a frame where the particles are non-relativistic via Lorentz transformation

