Set 5:
Classical E&M and Plasma Processes
Maxwell Equations

- Classical E&M defined by the Maxwell Equations (fields sourced by matter) and the Lorentz force (matter moved by fields)

- In cgs (gaussian) units

\[
\nabla \cdot \mathbf{D} = 4\pi \rho, \quad \nabla \cdot \mathbf{B} = 0, \\
\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \times \mathbf{H} = \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t} + \frac{4\pi}{c} \mathbf{j},
\]

where \( \rho \) = charge density, \( \mathbf{j} \) = current density, \( \mathbf{D} = \varepsilon \mathbf{E} \) and \( \mathbf{B} = \mu \mathbf{H} \) with \( \varepsilon \) = dielectric constant \( \mu \) = magnetic permeability

- In vacuum, \( \varepsilon = \mu = 1 \) and the Maxwell equations simplify to

\[
\nabla \cdot \mathbf{E} = 4\pi \rho, \quad \nabla \cdot \mathbf{B} = 0, \\
\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \times \mathbf{B} = \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c} \mathbf{j},
\]
Charge Conservation

• Maxwell equations are $E$ and $B$ symmetric aside from the lack of magnetic charges

• Divergence of Ampere’s law $\rightarrow$ charge conservation

\[
\nabla \cdot [\nabla \times \mathbf{B}] = \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c} \mathbf{j}
\]

\[
0 = \frac{1}{c} \frac{\partial \nabla \cdot \mathbf{E}}{\partial t} + \frac{4\pi}{c} \nabla \cdot \mathbf{j}
\]

\[
0 = \frac{4\pi}{c} \frac{\partial \rho}{\partial t} + \frac{4\pi}{c} \nabla \cdot \mathbf{j}
\]

\[
0 = \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j}
\]
Wave Equation

- Source free propagation

\[ \nabla \cdot \mathbf{E} = 0, \quad \nabla \cdot \mathbf{B} = 0, \]

\[ \nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \times \mathbf{B} = \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t}, \]

invariant under \( \mathbf{E} \rightarrow \mathbf{B} \) and \( \mathbf{B} \rightarrow -\mathbf{E} \), so work out equation for \( \mathbf{E} \)

- Curl of Faraday’s law \( \rightarrow \) wave equation

\[ \nabla \times [\nabla \times \mathbf{E}] = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} \]

\[ \nabla (\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\frac{1}{c} \frac{\partial \nabla \times \mathbf{B}}{\partial t} \]

\[ -\nabla^2 \mathbf{E} = -\frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} \]

\[ \nabla^2 \mathbf{E} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \]
Wave Equation

• Similarly for B

\[
\begin{pmatrix}
\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}
\end{pmatrix}
\begin{pmatrix}
E \\
B
\end{pmatrix} = 0
\]

• Wave solutions \( k = \frac{2\pi}{\lambda} = \frac{2\pi\nu}{c} \) [real part or superimposed \( k \) and \( -k \)]

\[
\begin{pmatrix}
E \\
B
\end{pmatrix} = \begin{pmatrix}
E_0 e_1 \\
B_0 e_2
\end{pmatrix} e^{i(k \cdot x - \omega t)}
\]

• Wave solution in wave equation provides the “dispersion” relation

\[
k^2 - \frac{1}{c^2} \omega^2 = 0, \quad \omega = kc = 2\pi\nu\]
Wave Equation

- E and B fields related by Maxwell equations

\[ i\mathbf{k} \cdot \hat{e}_1 E_0 = 0, \quad i\mathbf{k} \cdot \hat{e}_2 B_0 = 0, \]

\[ i\mathbf{k} \times \hat{e}_1 E_0 = i \frac{\omega}{c} \hat{e}_2 B_0, \quad i\mathbf{k} \times \hat{e}_2 B_0 = -i \frac{\omega}{c} \hat{e}_1 E_0, \]

so \( \hat{e}_1 \perp \mathbf{k}, \ \hat{e}_2 \perp \mathbf{k}, \ \mathbf{k} \times \hat{e}_1 \parallel \hat{e}_2. \) So \( (\hat{e}_1, \hat{e}_2, \mathbf{k}) \) form a orthonormal right handed basis

\[ i\mathbf{k} \times \hat{e}_1 E_0 = i \frac{\omega}{c} \hat{e}_2 B_0, \quad kE_0 = \frac{\omega}{c} B_0, \quad \rightarrow E_0 = B_0 \]
Lorentz Force

- Lorentz force

\[ \mathbf{F} = q \left[ \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right] \]

where \( q \) is the charge

- For a distribution of charges in a volume, charge and current density \( \mathbf{j} = \rho \mathbf{v} \) provide a force density

\[ \mathbf{f} = \rho \mathbf{E} + \frac{1}{c} \mathbf{j} \times \mathbf{B} \]

- Work and change in energy density

\[ W = \int \mathbf{F} \cdot d\mathbf{x}, \quad \frac{dW}{dt} = \mathbf{F} \cdot \mathbf{v} = q \mathbf{E} \cdot \mathbf{v} \]

\[ \frac{du_{\text{mech}}}{dt} = \rho \mathbf{E} \cdot \mathbf{v} = \mathbf{E} \cdot \mathbf{j} \]
Field Energy

- Use energy conservation to define field and radiation energy

\[
\frac{4\pi}{c} j = \nabla \times H - \frac{1}{c} \frac{\partial D}{\partial t}
\]

\[
j = \frac{c}{4\pi} \left( \nabla \times H - \frac{1}{c} \frac{\partial D}{\partial t} \right)
\]

\[
\frac{du_{\text{mech}}}{dt} = E \cdot j = \frac{c}{4\pi} \left( \nabla \times H - \frac{1}{c} \frac{\partial D}{\partial t} \right) \cdot E
\]

\[
\left[ E \cdot (\nabla \times H) = H \cdot (\nabla \times E) - \nabla \cdot (E \times H) \right]
\]

\[
\frac{du_{\text{mech}}}{dt} = \frac{c}{4\pi} \left[ H \cdot (\nabla \times E) - \nabla \cdot (E \times H) - \frac{1}{c} \frac{\partial D}{\partial t} \cdot E \right]
\]

\[
\left[ \nabla \times E = -\frac{1}{c} \frac{\partial B}{\partial t}, \quad H = B/\mu, \quad D = \epsilon E \right]
\]

\[
= \frac{1}{4\pi} \left[ -\frac{1}{\mu} \frac{\partial B}{\partial t} \cdot B - \epsilon \frac{\partial E}{\partial t} \cdot E - c\nabla \cdot (E \times H) \right]
\]
Field Energy

• Rewrite field terms as a total derivative

\[
\frac{\partial}{\partial t} (B \cdot B) = 2B \cdot \frac{\partial B}{\partial t}, \quad \frac{\partial}{\partial t} (E \cdot E) = 2E \cdot \frac{\partial E}{\partial t}
\]

• Equation forms a conservation law

\[
\frac{\partial}{\partial t} (u_{\text{mech}} + u_{\text{field}}) + \nabla \cdot S = 0
\]

with

\[
u_{\text{field}} = \frac{1}{8\pi} \left( \epsilon E^2 + \frac{1}{\mu} B^2 \right)
\]

and the energy flux carried by the radiation, or Poynting vector

\[
S = \frac{c}{4\pi} E \times H
\]
Field Energy

• For vacuum $\mu = \epsilon = 1$ and so field energy in a monocromatic wave, time averaged over the oscillation

\[
\langle u_{\text{field}} \rangle = \frac{1}{8\pi} \frac{1}{2} (E_0^2 + B_0^2) = \frac{1}{8\pi} E_0^2
\]

\[
\langle S \rangle = \frac{c}{4\pi} \frac{1}{2} E_0 B_0 = \frac{c}{8\pi} E_0^2
\]

which says that $\langle S \rangle / \langle u_{\text{field}} \rangle = c$, (recall $I_\nu / u_\nu = c$)

• Energy Flux

\[
\langle S \rangle = \frac{c}{8\pi} E_0^2 = \int d\nu \int d\Omega \cos \theta I_\nu
\]

so the specific intensity of a monocromatic plane wave is a delta function in frequency and angle
Specific Intensity

- Actual processes will not be monocromatic but have some waveform associated with the acceleration of the emitter: superposition of plane waves

\[ E(t) = \int_{-\infty}^{\infty} E(\omega)e^{-i\omega t} d\omega \]

\[ E(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E(t)e^{i\omega t} dt \]

Total emission will be an incoherent superposition of these single particle sources
Specific Intensity

- Energy flux normal to propagation direction

\[ \frac{dW}{dt dA} = S = \frac{c}{4\pi} E^2(t) \]

- Total energy passing through \( dA \)

\[ \frac{dW}{dA} = \frac{c}{4\pi} \int_{-\infty}^{\infty} E^2(t) dt \]

\[ = \frac{c}{4\pi} \left[ \int_{-\infty}^{\infty} E^2(t) dt = 2\pi \int_{-\infty}^{\infty} |E(\omega)|^2 d\omega \right] \]

\[ = \frac{c}{4\pi} \left[ 4\pi \int_{0}^{\infty} |E(\omega)|^2 d\omega \right] \]

\[ \frac{dW}{dAd\omega} = c |E(\omega)|^2 \]
Specific Intensity

- Now given a timescale $T$ for single particle processes

$$\frac{dW}{dA d\omega dt} = \frac{1}{T} \frac{dW}{dA d\omega} = \frac{c}{T} |E(\omega)|^2$$

which given $\omega = 2\pi \nu$ is the specific flux. Given a resolved source divide by the source solid angle to get $I_\nu$
Stokes Parameters

- Specific intensity is related to quadratic combinations of the field. Define the intensity matrix (time averaged over oscillations) \( \langle \mathbf{E} \mathbf{E}^\dagger \rangle \)

- Hermitian matrix can be decomposed into Pauli matrices

\[
P = \langle \mathbf{E} \mathbf{E}^\dagger \rangle = \frac{1}{2} \left( \mathbf{I} \sigma_0 + Q \sigma_3 + U \sigma_1 - V \sigma_2 \right),
\]

where

\[
\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

- Stokes parameters recovered as \( \text{Tr}(\sigma_i P) \)
Stokes Parameters

- Consider a general plane wave solution

\[ \mathbf{E}(t, z) = E_1(t, z) \hat{e}_1 + E_2(t, z) \hat{e}_2 \]

\[ E_1(t, z) = A_1 e^{i\phi_1} e^{i(kz-\omega t)} \]

\[ E_2(t, z) = A_2 e^{i\phi_2} e^{i(kz-\omega t)} \]

- Explicitly:

\[ I = \langle E_1 E_1^* + E_2 E_2^* \rangle = A_1^2 + A_2^2 \]

\[ Q = \langle E_1 E_1^* - E_2 E_2^* \rangle = A_1^2 - A_2^2 \]

\[ U = \langle E_1 E_2^* + E_2 E_1^* \rangle = 2A_1 A_2 \cos(\phi_2 - \phi_1) \]

\[ V = -i \langle E_1 E_2^* - E_2 E_1^* \rangle = 2A_1 A_2 \sin(\phi_2 - \phi_1) \]

so that the Stokes parameters define the state up to an unobservable overall phase of the wave.
Polarization

- Radiation field involves a directed quantity, the electric field vector, which defines the polarization
- Consider a general plane wave solution

\[ E(t, z) = E_1(t, z)\hat{e}_1 + E_2(t, z)\hat{e}_2 \]

\[ E_1(t, z) = \text{Re}A_1 e^{i\phi_1} e^{i(kz - \omega t)} \]

\[ E_2(t, z) = \text{Re}A_2 e^{i\phi_2} e^{i(kz - \omega t)} \]

or at \( z = 0 \) the field vector traces out an ellipse

\[ E(t, 0) = A_1 \cos(\omega t - \phi_1)\hat{e}_1 + A_2 \cos(\omega t - \phi_2)\hat{e}_2 \]

with principal axes defined by

\[ E(t, 0) = A'_1 \cos(\omega t)\hat{e}'_1 - A'_2 \sin(\omega t)\hat{e}'_2 \]

so as to trace out a clockwise rotation for \( A'_1, A'_2 > 0 \)
Polarization

- Define polarization angle

\[ \hat{e}'_1 = \cos \chi \hat{e}_1 + \sin \chi \hat{e}_2 \]
\[ \hat{e}'_2 = -\sin \chi \hat{e}_1 + \cos \chi \hat{e}_2 \]

- Match

\[ \mathbf{E}(t, 0) = A'_1 \cos \omega t [\cos \chi \hat{e}_1 + \sin \chi \hat{e}_2] \]
\[ - A'_2 \cos \omega t [-\sin \chi \hat{e}_1 + \cos \chi \hat{e}_2] \]
\[ = A_1 [\cos \phi_1 \cos \omega t + \sin \phi_1 \sin \omega t] \hat{e}_1 \]
\[ + A_2 [\cos \phi_2 \cos \omega t + \sin \phi_2 \sin \omega t] \hat{e}_2 \]
Polarization

- Define relative strength of two principal states

\[ A'_1 = E_0 \cos \beta \quad A'_2 = E_0 \sin \beta \]

- Characterize the polarization by two angles

\[ A_1 \cos \phi_1 = E_0 \cos \beta \cos \chi, \quad A_1 \sin \phi_1 = E_0 \sin \beta \sin \chi, \]
\[ A_2 \cos \phi_2 = E_0 \cos \beta \sin \chi, \quad A_2 \sin \phi_2 = -E_0 \sin \beta \cos \chi \]

Or Stokes parameters by

\[ I = E_0^2, \quad Q = E_0^2 \cos 2\beta \cos 2\chi \]
\[ U = E_0^2 \cos 2\beta \sin 2\chi, \quad V = E_0^2 \sin 2\beta \]

- So \( I^2 = Q^2 + U^2 + V^2 \), double angles reflect the spin 2 field or headless vector nature of polarization
Polarization

Special cases

- If $\beta = 0, \pi/2, \pi$ then only one principal axis, ellipse collapses to a line and $V = 0 \rightarrow$ linear polarization oriented at angle $\chi$
  
  If $\chi = 0, \pi/2, \pi$ then $I = \pm Q$ and $U = 0$
  
  If $\chi = \pi/4, 3\pi/4...$ then $I = \pm U$ and $Q = 0$ - so $U$ is $Q$ in a frame rotated by 45 degrees

- If $\beta = \pi/4, 3\pi/4$, then principal components have equal strength and $E$ field rotates on a circle: $I = \pm V$ and $Q = U = 0 \rightarrow$ circular polarization

- $U/Q = \tan 2\chi$ defines angle of linear polarization and $V/I = \sin 2\beta$ defines degree of circular polarization
Natural Light

- A monochromatic plane wave is completely polarized
  \[ I^2 = Q^2 + U^2 + V^2 \]

- Polarization matrix is like a density matrix in quantum mechanics and allows for pure (coherent) states and mixed states

- Suppose the total \( E_{\text{tot}} \) field is composed of different (frequency) components

  \[ E_{\text{tot}} = \sum_i E_i \]

- Then components decorrelate in time average

  \[ \langle E_{\text{tot}} E_{\text{tot}}^\dagger \rangle = \sum_{ij} \langle E_i E_j^\dagger \rangle = \sum_i \langle E_i E_i^\dagger \rangle \]
Natural Light

- So Stokes parameters of incoherent contributions add

\[ I = \sum_i I_i \quad Q = \sum_i Q_i \quad U = \sum_i U_i \quad V = \sum_i V_i \]

and since individual \( Q, U \) and \( V \) can have either sign:

\[ I^2 \geq Q^2 + U^2 + V^2, \] all 4 Stokes parameters needed
Polarized Radiative Transfer

- Define a specific intensity “vector”: \( \mathbf{I}_\nu = (I_{\nu 1}, I_{\nu 2}, U, V) \) where \( I = I_{\nu 1} + I_{\nu 2}, Q = I_{\nu 1} - I_{\nu 2} \)

\[
\frac{d\mathbf{I}_\nu}{ds} = \alpha_\nu (\mathbf{S}_\nu - \mathbf{I}_\nu)
\]

- Source vector in practice can be complicated
- Thomson collision based on differential cross section

\[
\frac{d\sigma_T}{d\Omega} = \frac{3}{8\pi} |\hat{\mathbf{E}}' \cdot \hat{\mathbf{E}}|^2 \sigma_T,
\]
Polarized Radiative Transfer

- \( \hat{E}' \) and \( \hat{E} \) denote the incoming and outgoing directions of the electric field or polarization vector.
- Thomson scattering into \( e_1 \): \( I_{\nu 2} \to I_{\nu 2} \) but \( I_{\nu 1} \) does not scatter.
- More generally if \( \Theta \) is the scattering angle then referenced to the plane of the scattering \( \alpha_\nu = n_e \sigma_T \) and

\[
S_\nu = \frac{3}{8\pi} \int d\Omega' \begin{pmatrix}
\cos^2 \Theta & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \cos \Theta & 0 \\
0 & 0 & 0 & \cos \Theta
\end{pmatrix} I'_\nu
\]

- But to calculate Stokes parameters in a fixed coordinate system must rotate into the scattering basis, scatter and rotate back out to the fixed coordinate system.
Astrophysical media are typically ionized so that radiation does not propagate in a vacuum but through an ionized plasma.

However the plasma is typically so rarified that only the very lowest frequency radiation is affected.

Maxwell equations for plane wave radiation $\exp[i(k \cdot r - \omega t)]$ with sources

\[
\begin{align*}
    ik \cdot E &= 4\pi \rho, & ik \cdot B &= 0, \\
    ik \times E &= i \frac{\omega}{c} B, & ik \times B &= -i \frac{\omega}{c} E + \frac{4\pi}{c} j,
\end{align*}
\]

Medium is globally neutral but electric field of the radiation cause a high frequency electron drift $\rightarrow$ current $\rightarrow$ charge via continuity.
Plasma Sources

- Lorentz force

\[ m \dot{v} = eE \quad \rightarrow v = - \frac{eE}{i\omega m} \]

- Current density carried by electrons of number density \( n \)

\[ j = nev = - \frac{ne^2E}{i\omega m} \equiv \sigma E \]

\[ \sigma = \frac{ine^2}{\omega m} \quad \text{conductivity} \]

- Charge conservation

\[ -i\omega \rho + ik \cdot j = 0 \]

\[ \rho = \frac{k \cdot j}{\omega} = \frac{\sigma}{\omega} k \cdot E \]
Plasma Frequency

- Maxwell equation with $\rho$

\[
i k \cdot E = 4\pi \frac{\sigma}{\omega} k \cdot E
\]

\[
i \left(1 - \frac{4\pi \sigma}{\omega i}\right) k \cdot E = 0
\]

\[
i k \cdot \epsilon E = 0
\]

with the dielectric constant

\[
\epsilon = 1 - \frac{4\pi \sigma}{\omega i} = 1 - \frac{4\pi ne^2}{m\omega^2}
\]

\[
= 1 - \frac{\omega_p^2}{\omega^2} \quad \left[\omega_p^2 = \frac{4\pi ne^2}{m}\right]
\]
Likewise the Maxwell equation with $j$

\[
  ik \times B = \frac{4\pi}{c} j - i \frac{\omega}{c} E
\]

\[
  = \left( \frac{4\pi}{c} \sigma - i \frac{\omega}{c} \right) E
\]

\[
  ik \times B = -i \frac{\omega}{c} \epsilon E
\]

So that the Maxwell equations become source free equations

\[
  ik \cdot \epsilon E = 0, \quad ik \cdot B = 0,
\]

\[
  ik \times E = i \frac{\omega}{c} B, \quad ik \times B = -i \frac{\omega}{c} \epsilon E,
\]
Wave Equation

- Wave equation becomes (similarly for $\mathbf{B} \perp \mathbf{E}$)

$$i[k \times (k \times \mathbf{E})] = -ik^2 \mathbf{E} = i\frac{\omega}{c} k \times \mathbf{B} = -i\frac{\omega^2}{c^2} \epsilon \mathbf{E}$$

- Modified dispersion relation

$$k^2 = \frac{\omega^2}{c^2} \epsilon = \frac{\omega^2}{c^2} \left(1 - \frac{\omega_p^2}{\omega^2}\right)$$

$$k = \frac{1}{c} \sqrt{\omega^2 - \omega_p^2}$$

- If $\omega < \omega_p$ then $k$ is imaginary and the wave function has an exponential suppression - waves don’t propagate below the plasma frequency

$$\nu_p = \frac{\omega_p}{2\pi} = 0.01 \text{ MHz} \left(\frac{n}{1\text{cm}^{-3}}\right)^{1/2}$$
Plasma Cutoff & Refraction

- For the ionosphere $n \sim 10^4 \text{ cm}^{-3}$ and radio waves at $< 1 \text{MHz}$ cannot propagate

- For ISM $n < 1 \text{ cm}^{-3}$ and the cut off is a much smaller $< 0.01 \text{ MHz}$

- The phase velocity defines the index of refraction

$$v_p = \frac{\omega}{k} \equiv \frac{c}{n_r} \rightarrow n_r \equiv \sqrt{\epsilon} = \sqrt{1 - \frac{\omega_p^2}{\omega^2}}$$

- Radio waves can be refracted according to Snell’s law and change their direction of propagation along the path $s$

$$\frac{dn_r \hat{k}}{ds} = \nabla n_r$$
Displacement Measure

- For wave packet propagation the relevant quantity is the group velocity defined by demanding that the phase remain stationary for constructive interference

\[ \phi(k) = kz - \omega(k)t \]

\[ \frac{\partial \phi}{\partial k} = 0 = z - \frac{\partial \omega}{\partial k}t = z - v_g t \]

\[ v_g = \frac{\partial \omega}{\partial k} = c\sqrt{1 - \frac{\omega_p^2}{\omega^2}} \leq c \]

\[ \approx c(1 - \frac{1}{2} \frac{\omega_p^2}{\omega^2}) \quad [\omega \gg \omega_p] \]

- Photons effectively gain a mass leading to a delay in arrival times
Dispersion Measure

- For a pulse of radiation from a pulsar

\[ t_p = \int_0^d \frac{ds}{v_g} \approx \frac{d}{c} + \frac{1}{2c\omega^2} \int_0^d \omega^2 ds \]

\[ t_p = \frac{d}{c} + \frac{2\pi e^2}{mc\omega^2} \left[ \int_0^d nds \equiv D \right] \]

\[ \frac{\partial t_p}{\partial \omega} = -\frac{4\pi e^2}{mc\omega^3} D \]

Change in arrival time with frequency → dispersion measure → distance given a mean \( n \)
Faraday Rotation

- In an external magnetic field $B_0 = B_0 e_3$ the electron responds to the magnetic field as well as the electric field of the radiation

$$m \frac{d \mathbf{v}}{d t} = e \mathbf{E} + \frac{e}{c} \mathbf{v} \times \mathbf{B}_0$$

- Examine the propagation of circularly polarized states

$$\mathbf{E}_\pm(t) = E_\pm e^{-i\omega t} [e_1 \pm i e_2]$$

- Take a trial solution

$$\mathbf{v}_\pm(t) = v_\pm e^{-i\omega t} [e_1 \pm i e_2]$$

$$-i m \omega v_\pm [e_1 \pm i e_2] = e E_\pm [e_1 \pm i e_2] + \frac{e}{c} v_\pm B_0 [-e_2 \pm i e_1]$$

$$= [e E_\pm \pm i \frac{e}{c} B_0 v_\pm] [e_1 \pm i e_2]$$
Faraday Rotation

\[-i(\omega m \pm \frac{e}{c}B_0)v_\pm = eE_\pm\]

\[v_\pm = \frac{ieE_\pm}{m(\omega \pm \omega_B)}\]

\[\omega_B = \frac{eB_0}{mc}\]

- Conductivity

\[\sigma = \frac{\mathbf{j}_\pm}{E_\pm} = \frac{env_\pm}{E_\pm} = \frac{ie^2n}{m(\omega \pm \omega_B)}\]

\[\epsilon_\pm = 1 - \frac{4\pi\sigma}{\omega i}\]

\[= 1 - \frac{4\pi ne^2}{m(\omega \pm \omega_B)\omega}\]

\[= 1 - \frac{\omega_p^2}{\omega(\omega \pm \omega_B)}\]
Faraday Rotation

- Right and left polarizations travel at different velocities: dispersion relation for $\omega \gg \omega_B$ and $\omega \gg \omega_p$

$$k_{\pm} = \frac{\omega}{c} \sqrt{\epsilon_{\pm}} \approx \frac{\omega}{c} \left[ 1 - \frac{1}{2} \frac{\omega_p^2}{\omega^2} \left( 1 \pm \frac{\omega_p}{\omega} \right) \right]$$

- Considering linear polarization as a superposition of right and left circular polarization, the difference in propagation speeds will lead to a Faraday rotation of the linear polarization
Faraday Rotation

- Phase

\[ \phi_\pm = \int_0^d k_\pm ds \]

\[ \frac{\Delta \phi}{2} = \frac{1}{2} \int_0^d (k_+ - k_-) ds = \frac{1}{2c} \int_0^d \frac{\omega_p^2}{\omega^2} \omega_B ds \]

- \( \Delta \phi / 2 \) gives the rotation of linear polarization

\[ \Delta \theta = \frac{\Delta \phi}{2} = \frac{1}{2c\omega^2} \frac{4\pi e^2}{m} \frac{e}{mc} \int_0^d B_0 n ds \]

\[ = \frac{2\pi e^3}{m^2 c^2 \omega^2} \int B_0 n ds \]

- More generally \( B_0 \to B_\parallel \) the line of sight component

- Given an average \( n \) measure \( B \) – e.g. magnetic field of ISM, cluster
E & M Potentials

• Introduce the vector and scalar potential to simplify source calculation

• $\nabla \cdot \mathbf{B} = 0$ implies that $\mathbf{B} = \nabla \times \mathbf{A}$, where $\mathbf{A}$ is the vector potential

• So Faraday’s law becomes

$$ \nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial}{\partial t} (\nabla \times \mathbf{A}) $$

$$ \nabla \times [\mathbf{E} + \frac{1}{c} \frac{\partial}{\partial t} \mathbf{A}] = 0 $$

implying a scalar potential $\phi$

$$ \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} = -\nabla \phi $$
Gauge

- Potentials \((\phi, A)\) allow for gauge freedom. Given a change in the potentials through an arbitrary field \(\psi\)

\[
A' = A + \nabla \psi
\]

\[
\phi' = \phi - \frac{1}{c} \frac{\partial \psi}{\partial t}
\]

the observable \(E\) and \(B\) fields invariant

\[
E = -\frac{1}{c} \frac{\partial A'}{\partial t} - \nabla \phi'
\]

\[
= -\frac{1}{c} \frac{\partial A}{\partial t} - \frac{1}{c} \frac{\partial}{\partial t} \nabla \psi - \nabla \phi + \frac{1}{c} \frac{\partial}{\partial t} \nabla \psi
\]

\[
= -\frac{1}{c} \frac{\partial A}{\partial t} - \nabla \phi
\]

\[
B = \nabla \times A' = \nabla \times A
\]
Lorentz Gauge

• Gauge freedom allows one to choose a convenient gauge to simplify equations

• Choose a gauge where the relationship between the potentials is

\[ \nabla \cdot \mathbf{A} + \frac{1}{c} \frac{\partial \phi}{\partial t} = 0 \]

• Maxwell equations simplify to

\[ \nabla^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = -4\pi\rho \]
\[ \nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\frac{4\pi}{c} \mathbf{j} \]
Retarded Potentials

- Green function solutions (propagate a $\delta$ function disturbance; superimpose to get arbitrary source. [See Jackson]

- Looks like electrostatics but accounts for the finite propagation time of light

\[
\phi(\mathbf{r}, t) = \int \frac{[\rho] d^3 r'}{|\mathbf{r} - \mathbf{r}'|}
\]

\[
\mathbf{A}(\mathbf{r}, t) = \frac{1}{c} \int \frac{[\mathbf{j}] d^3 r'}{|\mathbf{r} - \mathbf{r}'|}
\]

where the $[]$ denotes evaluation at the retarded time

\[
[f](\mathbf{r}', t) = f(\mathbf{r}', t - \frac{1}{c} |\mathbf{r} - \mathbf{r}'|)
\]
Consider a single charge on a trajectory $r_0(t)$ with velocity $u = \dot{r}_0(t)$

$$\rho(r,t) = q\delta(r - r_0(t))$$

$$j(r,t) = qu\delta(r - r_0(t))$$

Scalar potential

$$\phi(r,t) = \int d^3r' \int dt' \frac{\rho(r',t')}{|r - r'|} \delta(t' - t + \frac{1}{c}|r - r'|)$$

$$= \int d^3r' \int dt' \frac{q\delta(r' - r_0(t'))}{|r - r'|} \delta(t' - t + \frac{1}{c}|r - r'|)$$

$$= q \int dt' \delta (t' - t + \frac{R(t')}{c}) \frac{1}{R(t')} \mathbf{R}(t) = |r - r_0(t)|$$
Lienard-Wiechart Potential

Change variables so that explicit integration possible

\[ t'' = t' - t + \frac{R(t')}{c} \]

\[ dt'' = dt' + \frac{1}{c} \dot{R}(t') dt' \]

\[ = [1 + \frac{1}{c} \dot{R}(t')] dt' \]

\[ \dot{R} = -\dot{r}_0 = -u, \quad \hat{n} = \frac{R}{R}, \quad \dot{R} \cdot \hat{n} = -u \cdot \hat{n} \]

\[ 2R\dot{R} = 2\dot{R} \cdot R \rightarrow \dot{R} = \dot{R} \cdot \hat{n} = -u \cdot \hat{n} \]

\[ \kappa(t') \equiv 1 + \frac{1}{c} \dot{R}(t') = 1 - \frac{1}{c} u \cdot \hat{n} \]

which is the origin of relativistic beaming effects
Lienard-Wiechart Potential

- Thus since $t'' = 0 \rightarrow t' = t - \frac{R(t')}{c}$

$$\phi(r, t) = q \int dt'' \delta(t'') \frac{1}{\kappa(t') R(t')} = \frac{q}{\kappa R} \bigg|_{t''=0} = \left[ \frac{q}{\kappa R} \right]$$

- Similarly

$$A(r, t) = \left[ \frac{qu}{c\kappa R} \right]$$

- For non-relativistic velocities $\kappa = 1 - \frac{u}{c} \cdot \hat{n} \approx 1$ and potential are just retarded versions of electrostatic potentials

- For $u \rightarrow c$ then enhanced radiation along $u \parallel n$ from relativistic beaming
E&B Field

- Plug and chug with

\[ \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} - \nabla \phi, \quad \mathbf{B} = \nabla \times \mathbf{A} \]

\[ \mathbf{E} = \mathbf{E}_{\text{vel}} + \mathbf{E}_{\text{rad}} \quad \mathbf{B} = \hat{n} \times \mathbf{E} \]

- Velocity field falls off as \(1/R^2\), \(\beta = u/c\) as a generalization of Coulomb’s law

\[ \mathbf{E}_{\text{vel}} = q \left[ \frac{(\hat{n} - \beta)(1 - \beta^2)\kappa^3 R^2}{\kappa^3 R^2} \right] \]

- The radiation field depends on the acceleration and falls off as \(1/R\) so that there is a flux \(E^2 \propto 1/R^2\) that propagates to infinity

\[ \mathbf{E}_{\text{rad}} = \frac{q}{c} \left[ \frac{\hat{n}}{\kappa^3 R} \times ((\hat{n} - \beta) \times \dot{\beta}) \right] \]
Larmor Formula

- Larmor formula: non relativistic case $\beta \ll 1$

\[ E_{\text{rad}} = \frac{q}{c} \hat{n} \times (\hat{n} \times \dot{\beta}) \]
\[ = \frac{q}{c^2 R} \hat{n} \times (\hat{n} \times \dot{u}) \]

- Let $\dot{u} / \dot{u} \cdot \hat{n} = \cos \Theta$

\[ |E_{\text{rad}}| = \frac{q \dot{u}}{R c^2} \sin \Theta \]

- So flux

\[ S = \frac{c}{4\pi} E_{\text{rad}}^2 \hat{n} = \frac{c}{4\pi} \frac{q^2 \dot{u}^2}{R^2 c^4} \sin^2 \Theta = \frac{1}{4\pi} \frac{q^2 \dot{u}^2}{R^2 c^3} \sin^2 \Theta \]
Larmor Formula

- Power per unit angle \( dA = R^2 d\Omega \)

\[
\frac{dW}{dt d\Omega} = \frac{q^2 \dot{u}^2}{4\pi c^3} \sin^2 \Theta
\]

\[
P = \frac{dW}{dt} = \frac{q^2 \dot{u}^2}{4\pi c^3} \int d\Omega \sin^2 \Theta = \frac{2q^2 \dot{u}^2}{3c^3}
\]

- So dipole pattern of radiation is perpendicular to acceleration and polarization is in plane spanned by \( \dot{u} \) and \( \hat{n} \)

- Larmor formula can be used more generally in that one can transform to a frame where the particles are non-relativistic via Lorentz transformation