Set 6:

Relativity

The Metric

- The metric defines a measure on a space, distance between points, length of vectors.
- 3D Cartesian coordinates: separation vector between 2 points dx^i (upper or contravariant indices)

$$ds^2 = \sum_{i=1}^{3} dx^i dx^i$$

- Generalize to curvilinear coordinates, e.g. for spherical coordinates (r, θ, ϕ) the distances along an orthonormal set of vectors

$$\begin{aligned}
\hat{\mathbf{e}}_r &: \quad dr \\
\hat{\mathbf{e}}_\theta &: \quad r d\theta \\
\hat{\mathbf{e}}_\phi &: \quad r \sin\theta d\phi
\end{aligned}$$

The Metric

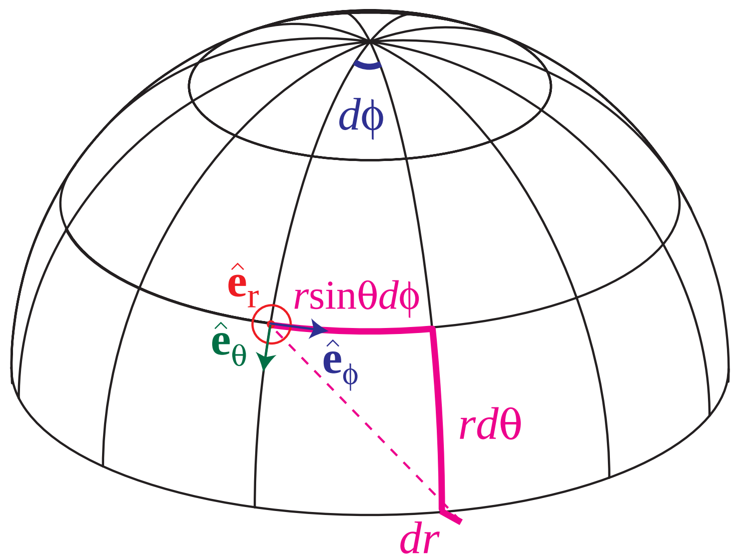
- Length in spherical coordinates

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

$$= \sum_{ij} g_{ij} dx^i dx^j$$

defines the metric

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix}$$



The Metric

- This would look like an ordinary dot product if we introduce the dual or "covariant" components of the vector

$$dx_i \equiv g_{ij} dx^j$$

$$ds^2 = \sum_i dx_i dx^i \equiv dx_i dx^i$$

where the Einstein summation convention is that repeated pairs of upper and lower indices are summed

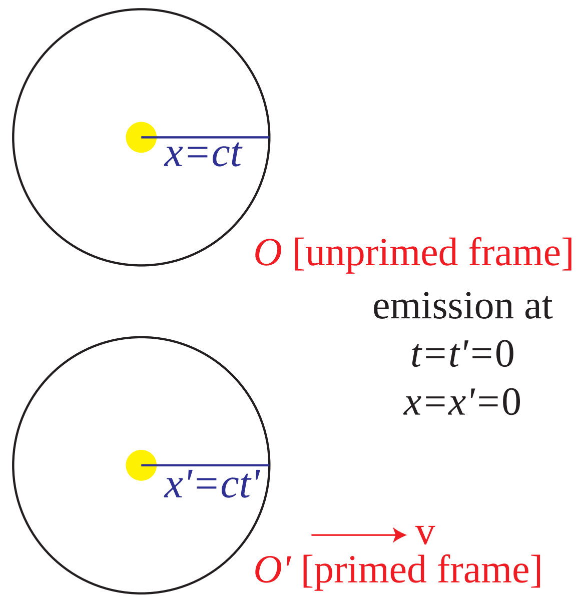
- Similarly the dot product between two vectors is given by

$$V^i X_i = g_{ij} V^i X^j = V_i X^i$$

Special Relativity

- In space time, the coordinates run from $0-3$ with $\mu=0$ as the temporal coordinate, and ds^2 represents the space-time separation

$$dx^{\mu} = (cdt, dx^1, dx^2, dx^3)$$



- Metric is defined by the requirement that two observers will see light propagating at the speed of light.

- Spherical pulse travels for time dt at the speed of light c

$$c^2dt^2 = dx_i dx^i \rightarrow -c^2dt^2 + dx_i dx^i = 0 = -c^2dt'^2 + dx'_i dx'^i$$

Special Relativity

- The space time separation for light is null and invariant
- So as an invariant measure on the space time, the temporal coordinate has the opposite sign: in Cartesian coordinates

$$[g_{\mu\nu} =]\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$ds^2 = dx_\mu dx^\mu = \eta_{\mu\nu} dx^\mu dx^\nu = -c^2 dt^2 + dx_i dx^i$$

Lorentz Transformation

- Set of all linear coordinate transformations that leave ds^2, and hence the speed of light, invariant
- 3D example: rotations leave the length of vectors invariant, generalization of a 4D rotation is a Lorentz transformation
- Begin with a general linear transformation (excluding a constant term)

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \rightarrow \Lambda^{\mu}{}_{\nu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}}$$

$$\begin{aligned} ds'^2 &= \eta_{\alpha\beta} dx'^{\alpha} dx'^{\beta} \\ &= \eta_{\alpha\beta} \Lambda^{\alpha}{}_{\mu} dx^{\mu} \Lambda^{\beta}{}_{\nu} dx^{\nu} \\ &= ds^2 = \eta_{\mu\nu} dx^{\mu} dx^{\nu} \end{aligned}$$

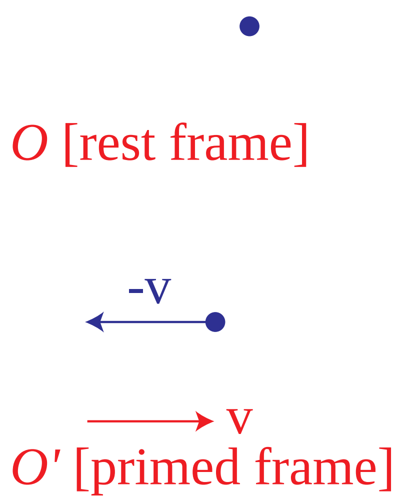
$$\eta_{\mu\nu} = \eta_{\alpha\beta} \Lambda^{\alpha}{}_{\mu} \Lambda^{\beta}{}_{\nu}$$

Lorentz Transformation

- Suppose a particle at rest in frame O is viewed in frame O' moving with a velocity v

$$dx'^{\mu} = \Lambda^{\mu}{}_{\nu} dx^{\nu}$$

$$dx^{\nu} = (cdt, 0, 0, 0)$$



- Take $\mu = 0$, $\mu = i$

$$cdt' = \Lambda^{0}{}_{0} dx^{0} + \Lambda^{0}{}_{i} dx^{i} = \Lambda^{0}{}_{0} cdt$$

$$dx'^{i} = \Lambda^{i}{}_{0} cdt$$

Lorentz Transformation

- Combine

$$\frac{dx'^i}{cdt'} \equiv -\frac{v^i}{c} = \frac{\Lambda^i{}_0}{\Lambda^0{}_0}$$

$$\Lambda^i{}_0 = -\frac{v^i}{c}\Lambda^0{}_0$$

- Recall invariance of ds^2 implies

$$\eta_{\mu\nu} = \eta_{\alpha\beta}\Lambda^\alpha{}_\mu\Lambda^\beta{}_\nu$$

- Evaluate 00 component

$$\eta_{00} = -1 = \eta_{\alpha\beta}\Lambda^\alpha{}_0\Lambda^\beta{}_0$$

$$-1 = -(\Lambda^0{}_0)^2 + (\Lambda^i{}_0)^2$$

Lorentz Transformation

- Plug in relationship between Λ's

$$-1 = -(\Lambda^0{}_0)^2 + \frac{v^2}{c^2}(\Lambda^0{}_0)^2$$
$$1 = (\Lambda^0{}_0)^2(1 - \frac{v^2}{c^2})$$

- Solve for $\Lambda^0{}_0$

$$\Lambda^0{}_0 = \frac{1}{\sqrt{1 - v^2/c^2}} \equiv \gamma$$
$$\Lambda^i{}_0 = -\gamma v^i/c = -\gamma\beta^i$$

- Spatial components determined from $\eta_{\mu\nu} = \eta_{\alpha\beta}\Lambda^\alpha{}_\mu\Lambda^\beta{}_\nu$, excluding rotation before boost in $\hat{\mathrm{e}}_1$ direction: $\Lambda^0{}_1 = -\beta\Lambda^1{}_1$, $1 = -(\Lambda^0{}_1)^2 + (\Lambda^1{}_1)^2$

Lorentz Transformation

- Lorentz transformation for boost in $\hat{\mathrm{e}}_1$ direction

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

or with $dx'^{\mu} = \Lambda^{\mu}{}_{\nu} dx^{\nu}$

$$ct' = \gamma ct - \beta\gamma x = \gamma c(t - \frac{\beta}{c}x)$$

$$t' = \gamma(t - \frac{v}{c^2}x)$$

$$x' = -\beta\gamma ct + \gamma x$$

$$x' = \gamma(x - vt)$$

Lorentz Transformation

- Relativity paradoxes by holding various things fixed: simultaneity in one frame not same as another
- Lorentz contraction: (primed: rest frame) length measured at fixed t

$$\Delta x'|_t = \gamma \Delta x \to \Delta x = \Delta x'/\gamma|_t$$

- Time dilation measured at $x' = 0$

$$\begin{aligned}\Delta t' &= \gamma(\Delta t - \frac{v}{c^2} x|_{x=vt}) \\ &= \gamma(\Delta t - \frac{v^2}{c^2}\Delta t) = \frac{1}{\gamma}\Delta t \\ \Delta t' &= \frac{1}{\gamma}\Delta t\end{aligned}$$

Lorentz Transformation

- Boost of a covariant vector

$$x'_\mu = \tilde{\Lambda}_\mu{}^\nu x_\nu \rightarrow \tilde{\Lambda}_\mu{}^\nu \equiv \frac{\partial x'_\mu}{\partial x_\nu}$$

$$x_\alpha x^\alpha = x'_\mu x'^\mu = \tilde{\Lambda}_\mu{}^\alpha x_\alpha \Lambda^\mu{}_\beta x^\beta$$

$$\tilde{\Lambda}_\mu{}^\alpha \Lambda^\mu{}_\beta = \delta^\alpha{}_\beta \rightarrow \tilde{\mathbf{\Lambda}}\mathbf{\Lambda} = \mathbf{I} \rightarrow \tilde{\mathbf{\Lambda}} = \mathbf{\Lambda}^{-1}$$

$$\tilde{\Lambda}_\mu{}^\nu \equiv \frac{\partial x^\nu}{\partial x^{\mu\prime}}$$

$$\tilde{\Lambda}_\mu{}^\nu = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Lorentz Transformation

- Tensors: multi-index objects that transform under a Lorentz transformation as, e.g.

$$T'^{\mu}{}_{\nu} = \Lambda^{\mu}{}_{\sigma}\tilde{\Lambda}_{\nu}{}^{\tau}T^{\sigma}{}_{\tau}$$

- Special relativity: laws of physics invariant under Lorentz transformation = laws of physics can be written as relationships between scalars, 4 vectors and tensors
- Like ds^2 the contraction of a set of 4 vectors or tensors is a Lorentz invariant
- What about laws involving derivatives?

Derivative Operator

- Derivative operator on a scalar transforms as a covariant vector

$$\frac{\partial}{\partial x'^{\alpha}} = \frac{\partial x^{\beta}}{\partial x'^{\alpha}} \frac{\partial}{\partial x^{\beta}} = \tilde{\Lambda}_{\alpha}{}^{\beta} \frac{\partial}{\partial x^{\beta}}$$

$$\nabla'_{\alpha} = \tilde{\Lambda}_{\alpha}{}^{\beta} \nabla_{\beta}$$

- Derivative operator on a vector transforms as a tensor

$$T_{\alpha}{}^{\beta} = \nabla_{\alpha} V^{\beta} = \frac{\partial}{\partial x^{\alpha}} V^{\beta}$$

$$T'_{\alpha}{}^{\beta} = \nabla'_{\alpha} V'^{\beta} = \frac{\partial}{\partial x'^{\alpha}} V'^{\beta} = \frac{\partial}{\partial x'^{\alpha}} \left(\frac{\partial x'^{\beta}}{\partial x^{\mu}} V^{\mu} \right)$$

$$= \frac{\partial x'^{\beta}}{\partial x^{\mu}} \frac{\partial}{\partial x'^{\alpha}} V^{\mu} + V^{\mu} \frac{\partial}{\partial x'^{\alpha}} \frac{\partial x'^{\beta}}{\partial x^{\mu}}$$

Derivative Operator

- For a Lorentz transformation, the coordinates are linearly related

$$\frac{\partial}{\partial x'^{\alpha}} \frac{\partial x'^{\beta}}{\partial x^{\mu}} = \frac{\partial}{\partial x'^{\alpha}} \Lambda^{\beta}{}_{\mu} = 0$$

$$T_{\alpha}{}^{\beta} = \frac{\partial x'^{\beta}}{\partial x^{\mu}} \frac{\partial x^{\nu}}{\partial x'^{\alpha}} \frac{\partial}{\partial x^{\nu}} V^{\mu} = \Lambda^{\beta}{}_{\mu} \tilde{\Lambda}_{\alpha}{}^{\nu} T^{\alpha}_{\nu}$$

  so that the derivative of a vector transforms as a tensor as long as the coordinate transformation is "special", i.e. linear. In general relativity this condition is lifted by promoting the ordinary derivative to a covariant derivative through the connection coefficients

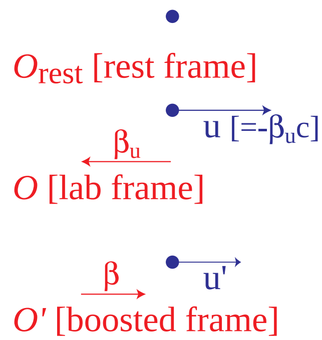
4 Velocity

- Four velocity:

  dx^μ is a vector $d\tau^2 = -ds^2/c^2$ is a scalar

$$\frac{dx^\mu}{d\tau} \equiv U^\mu$$

$$(c, 0, 0, 0)_{\text{restframe}}$$



in a boosted frame (particle velocity is opposite to boost $u = -\beta_u c$)

$$U^\mu = \Lambda^\mu{}_\nu U^\nu_{\text{rest}} = (\gamma_u c, -\beta_u \gamma_u c, 0, 0) = \gamma_u (c, \mathbf{u})$$

4 Velocity

- Boost again by β to show how velocity transforms

$$U'^0 = \gamma(U^0 - \beta U^1) = \gamma\gamma_u(c - \beta u^1) \equiv \gamma_{u'} c$$

$$U'^1 = \gamma(-\beta U^0 + U^1) = \gamma\gamma_u(-\beta c + u^1) \equiv \gamma_{u'} u'^1$$

$$U'^2 = U^2 = \gamma_u u^2 = \gamma_{u'} u'^2$$

$$U'^3 = U^3 = \gamma_u u^3 = \gamma_{u'} u'^3$$

which imply

$$\gamma_{u'} = \gamma\gamma_u(1 - vu^1/c^2)$$

$$\gamma_{u'} u'^1 = \gamma\gamma_u(u^1 - v)$$

$$\gamma_{u'} u'^{2,3} = \gamma\gamma_u(1 - vu^1/c^2)u'^{2,3} = \gamma_u u^{2,3}$$

4 Velocity

- Divide two relations

$$u'^1 = \frac{u^1 - v}{1 - vu^1/c^2}$$

$$u'^2 = \frac{\gamma_u}{\gamma_{u'}} u^2 = \frac{1}{\gamma(1 - vu^1/c^2)} u^2$$

$$u'^3 = \frac{1}{\gamma(1 - vu^1/c^2)} u^3$$

- Note that $U^\mu U_\mu = -(\gamma c)^2 + (\gamma u)^2 = -c^2$ also dot product is a way of evaluating rest frame time component of a vector A^μ: $U^\mu A_\mu = -U^0 A^0 + U^i A^i = -cA^0$.

4 Acceleration, Momentum, Force

- Acceleration

$$\frac{dU^\mu}{d\tau} = a^\mu$$

- Momentum (finite rest mass)

$$P^\mu = mU^\mu = \gamma m(c, \mathbf{u})$$

- Force

$$F^\mu = \frac{dP^\mu}{d\tau}$$

E & M

- Wavevector: phase of wave is a Lorentz scalar

$$\phi = k_i x^i - \omega t$$

$$\phi = k_\mu x^\mu$$

$$K^\mu = (\omega/c, \mathbf{k})$$

- Momentum $\mathbf{q} = \hbar \mathbf{k}$

$$P^\mu = (E/c, \mathbf{q}) = \hbar K^\mu$$

- Doppler shift $\cos\theta = \hat{\mathbf{u}} \cdot \hat{\mathbf{k}}$

$$K'^\mu = \Lambda^\mu{}_\nu K^\nu$$

$$\omega' = \gamma(\omega - k_i u^i) = \gamma\omega(1 - \frac{u}{c}\cos\theta)$$

γ factor purely relativistic

E & M

- Charge conservation and 4 current

$$\frac{\partial \rho}{\partial t} + \nabla_i j^i = 0$$

$$\nabla_\mu J^\mu = 0 \qquad J^\mu = (\rho c, \mathbf{j})$$

- 4 potential: $A^\mu = (\phi, \mathbf{A})$ obeys the wave equation

$$\nabla_\alpha \nabla^\alpha A^\mu = [\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}]A^\mu = -\frac{4\pi}{c} J^\mu$$

- Lorentz gauge condition

$$\nabla_i A^i + \frac{1}{c}\frac{\partial \phi}{\partial t} = \nabla_\mu A^\mu = 0$$

- Fields are related to derivatives of potential: field strength tensor

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$$

E & M

- E and B field

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}$$

- Maxwell Equations

$$\nabla^\mu F_{\mu\nu} = \frac{4\pi}{c} J_\nu \,, \quad \nabla_\sigma F_{\mu\nu} + \nabla_\nu F_{\sigma\mu} + \nabla_\mu F_{\nu\sigma} = 0$$

- Lorentz scalars (an electromagnetic field cannot be transformed away)

$$F_{\mu\nu} F^{\mu\nu} = 2(B^2 - E^2) \,, \quad \det F = (\mathbf{E} \cdot \mathbf{B})^2$$

E & M

- Lorentz force

$$F^{\mu} = \frac{e}{c} F^{\mu}{}_{\nu} U^{\nu}$$

- Phase space occupation

$$d^3x = \gamma^{-1} d^3x' \,, \qquad d^3q = \gamma d^3q'$$

so $d^3x d^3q$ is a Lorentz scalar and photon number is conserved so f is a Lorentz scalar (note that the energy spread is negligible in the rest frame)

- Field transformation

$$F'_{\mu\nu} = \tilde{\Lambda}_{\mu}{}^{\alpha} \tilde{\Lambda}_{\nu}{}^{\beta} F_{\alpha\beta}$$

$$E'_{\parallel} = E_{\parallel} \,, \qquad B'_{\parallel} = B_{\parallel}$$

$$\mathbf{E}'_{\perp} = \gamma(\mathbf{E}_{\perp} + \beta \times \mathbf{B}) \,, \qquad \mathbf{B}'_{\perp} = \gamma(\mathbf{B}_{\perp} - \beta \times \mathbf{E})$$

Coulomb Field Transformation

- Take Coulomb field in the particle rest frame $\mathbf{E}' = (q/{r'}^2)\hat{\mathbf{r}}'$ and boost in the x direction

$$E_x = \frac{qx'}{r'^3} = \frac{q\gamma(x-vt)}{r'^3}, \qquad B_x = 0$$

$$E_y = \frac{q\gamma y'}{r'^3} = \frac{q\gamma y'}{r'^3}, \qquad B_y = -\frac{q\gamma\beta z'}{r'^3} = -\frac{q\gamma\beta z'}{r'^3}$$

$$E_z = \frac{q\gamma z'}{r'^3} = \frac{q\gamma z}{r'^3}, \qquad B_z = \frac{q\gamma\beta y'}{r'^3} = \frac{q\gamma\beta y}{r'^3}$$

$${r'}^2 = \gamma^2(x-vt)^2 + y^2 + z^2$$

This may be rewritten as the velocity term in the Lienard-Wiechart fields

Power

- Power is a Lorentz scalar (4 momentum transformation with zero spatial momentum from symmetry)

$$dW = \gamma dW' \,, \qquad dt = \gamma dt' \,, \quad P = \frac{dW}{dt} = \frac{dW'}{dt'} = P'$$

- In particular in the instananeous rest frame the power can be calculated using the Larmor formula

$$P' = \frac{2q^2}{3c^2}|a'|^2 = \frac{2q^2}{3c^3}a^\mu a_\mu$$

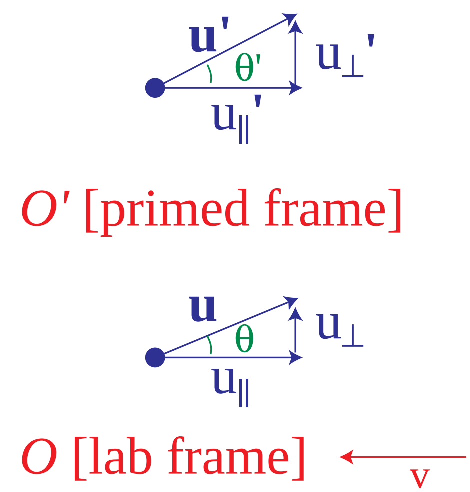
- given $a^\mu = d^2x^\mu/d\tau^2$ one can show $a'_\parallel = \gamma^3 a_\parallel$ and $a'_\perp = \gamma^2 a_\perp$

$$P = \frac{2q^2}{3c^3}\gamma^4(a_\perp^2 + \gamma^2 a_\parallel^2)$$

so that a parallel acceleration causes a much larger power radiation

Relativistic Beaming

- Isotropic emission also becomes beamed: by the addition of velocities, the angle changes with a boost



$$u_{\parallel} = \frac{u'_{\parallel} + v}{1 + vu'_{\parallel}/c^2} \qquad u_{\perp} = \frac{u'_{\perp}}{\gamma(1 + vu'_{\parallel}/c^2)}$$

$$\tan\theta = \frac{u_{\perp}}{u_{\parallel}} = \frac{u'_{\perp}}{u'_{\parallel} + v}\frac{1}{\gamma} = \frac{u'\sin\theta'}{\gamma(u'\cos\theta' + v)}$$

for light $u' = c$ and the aberration formula is

$$\tan\theta = \frac{c}{\gamma}\frac{\sin\theta'}{c\cos\theta' + v}$$

Relativistic Beaming

- For $\theta' = \pi/2$ then $\tan\theta = c/\gamma v$ and for $\gamma \gg 1$, $v \to c$ and so $\tan\theta \approx \theta \approx 1/\gamma$ - beamed a tight half angle
- Explains the differential power transformation: Larmor in [primed] rest frame
- Solid angle transformation: again apply addition for light to get

$$\cos\theta = \frac{u_\parallel}{\sqrt{u_\perp^2 + u_\parallel^2}} = \frac{u_\parallel}{c}, \qquad \cos\theta' = \frac{u'_\parallel}{c}$$

$$\cos\theta = \frac{\cos\theta' + v/c}{1 + v/c\,\cos\theta'}$$

- So $d\Omega = d\cos\theta d\phi$ and

$$d\Omega = d\Omega' \frac{1}{\gamma^2(1 + \beta\cos\theta')^2}$$

Differential power

- Energy and arrival time as ($\mu = \cos\theta$)

$$dW = \gamma(dW' + v dP'_x) = \gamma(1 + \beta\mu')dW'$$
$$dt_A = \gamma(1 - \beta\mu)dt'$$

- Identity

$$\gamma(1 - \beta\mu) = \frac{1}{\gamma(1 + \beta\mu')}$$

- Transformation of differential power

$$\frac{dP}{d\Omega} = \frac{dW}{d\Omega dt_A} = \frac{1}{\gamma^4(1 - \beta\mu)^4}\frac{dP'}{d\Omega'}$$
$$= \frac{1}{\gamma^4(1 - \beta\mu)^4}\frac{q^2 a'^2}{4\pi c^3}\sin^2\Theta'$$

Differential power

- Acceleration parallel to velocity: dipole pattern gets perpendicular lobes bent toward the velocity direction
- Acceleration perpendicular to velocity: forward dipole enhanced, second lobe distorted