Set 8: Compton Scattering
Energy-Momentum Conservation

- Thomson scattering $e_i + \gamma_i \rightarrow e_f + \gamma_f$ where the frequencies $\omega_i = \omega_f$ (elastic scattering) cannot strictly be true.

- Photons carry off $E/c$ momentum and so to conserve momentum the electron must recoil.

- The electron then carries away some of the available energy and so the Thomson limit requires $\hbar \omega_i/mc^2 \ll 1$ in the electron rest frame.

- General case (arbitrary electron velocity)
Energy-Momentum Conservation

- 4 momenta

\[ P_i = \frac{E_i}{c} (1, \hat{n}_i) , \quad P_f = \frac{E_f}{c} (1, \hat{n}_f) , \quad \text{photon} \]

\[ Q_i = \gamma_i m (c, \hat{v}_i) , \quad Q_f = \gamma_f m (c, \hat{v}_f) , \quad \text{electron} \]

- To work out change in photon energy consider Lorentz invariants:

\[ Q_\mu Q^\mu = \gamma^2 m^2 (v^2 - c^2) = m^2 \frac{v^2 - c^2}{1 - v^2/c^2} = -m^2 c^2 \]

\[ P_\mu P^\mu = 0 \]
Energy-Momentum Conservation

- Conservation

\[ P_i + Q_i = P_f + Q_f \]

also \[ P_f \cdot [P_i + Q_i = P_f + Q_f] \]

\[ P_f \cdot P_i + P_f \cdot Q_i = P_f \cdot Q_f \]

- Some identities to express final state in terms of initial state

\[ (P_i + Q_i)_\mu(P_i + Q_i)^\mu = (P_f + Q_f)_\mu(P_f + Q_f)^\mu \]

\[ 2P_{i\mu}Q_i^\mu - m^2 c^2 = 2P_{f\mu}Q_f^\mu - m^2 c^2 \]

\[ P_i \cdot Q_i = P_f \cdot Q_f \]

- So

\[ P_f \cdot P_i + P_f \cdot Q_i = P_i \cdot Q_i \]
Energy-Momentum Conservation

- In three vector notation

\[
\frac{E_i}{c} \frac{E_f}{c} (\hat{n}_i \cdot \hat{n}_f - 1) + \frac{E_f}{c} \gamma_i m (\hat{n}_f \cdot \mathbf{v}_i - c) = \frac{E_i}{c} \gamma_i m (\hat{n}_i \cdot \mathbf{v}_i - c)
\]

- Introducing the scattering angle as \( \hat{n}_i \cdot \hat{n}_f = \cos \theta \) and auxiliary angles \( \hat{n}_f \cdot \mathbf{v}_i = v_i \cos \alpha_f \) and \( \hat{n}_i \cdot \mathbf{v}_i = v_i \cos \alpha_i \)

\[
E_i E_f (\cos \theta - 1) + E_f \gamma_i m c^2 (\beta_i \cos \alpha_f - 1) = E_i \gamma_i m c^2 (\beta_i \cos \alpha_i - 1)
\]

\[
E_f = \frac{E_i \gamma_i m c^2 (\beta_i \cos \alpha_i - 1)}{E_i (\cos \theta - 1) + \gamma_i m c^2 (\beta_i \cos \alpha_i - 1)}
\]

- So the change in photon energy is given by

\[
\frac{E_f}{E_i} = \frac{1 - \beta_i \cos \alpha_i}{1 - \beta_i \cos \alpha_f + \frac{E_i}{\gamma m c^2} (1 - \cos \theta)}
\]
Recoil Effect

- Two ways of changing the energy: Doppler boost $\beta_i$ from incoming electron velocity and $E_i$ comparable to $\gamma mc^2$
- Take the incoming electron rest frame $\beta_i = 0$ and $\gamma = 1$

$$\left| \frac{E_f}{E_i} \right|_{\text{rest}} = \frac{1}{1 + \frac{E_i}{mc^2}(1 - \cos \theta)}$$

- Since $-1 \leq \cos \theta \leq 1$, $E_f \leq E_i$, energy is lost from the recoil except for purely forward scattering via comparing the incoming photon energy $E_i$ and the electron rest mass $mc^2$
- The backwards scattering limit is easy to derive

$$|q_f| = m|v_f| = 2\frac{E_i}{c}$$

$$\Delta E = \frac{1}{2}mv_f^2 = \frac{1}{2}m \left( \frac{2E_i}{mc} \right)^2 = \frac{2E_i}{mc^2} E_i$$

$$E_f = E_i - \Delta E = \left(1 - \frac{2E_i}{mc^2}\right)E_i \approx \frac{E_i}{1 + \frac{2E_i}{mc^2}}$$
Compton Wavelength

- Alternate phrasing in terms of length scales $E = h\nu = hc/\lambda$ so

$$\frac{\lambda_i}{\lambda_f} \bigg|_{\text{rest}} = \frac{1}{1 + \frac{hc}{\lambda_imc^2}(1 - \cos \theta)}$$

$$\frac{\lambda_f - 1}{\lambda_i} \bigg|_{\text{rest}} = \frac{h}{mc\lambda_i}(1 - \cos \theta) \equiv \frac{\lambda_c}{\lambda_i}(1 - \cos \theta)$$

where the Compton wavelength is $\lambda_c = h/mc \approx 2.4 \times 10^{-10}$ cm.

- Doppler effect: consider the limit of $\beta_i \ll 1$ then expand to first order

$$\frac{E_f}{E_i} = 1 - \beta_i \cos \alpha_i + \beta_i \cos \alpha_f - \frac{E_i}{mc^2}(1 - \cos \theta)$$

however averaging over angles the Doppler shifts don’t change the energies
Second Order Doppler Shift

- To second order in the velocities, the Doppler shift transfers energy from the electron to the photon in opposition to the recoil

\[
\frac{E_f}{E_i} = 1 - \beta_i \cos \alpha_i + \beta_i \cos \alpha_f + \beta_i^2 \cos^2 \alpha_f - \frac{E_i}{mc^2}
\]

\[
\langle \frac{E_f}{E_i} \rangle \approx 1 + \frac{1}{3} \beta_i^2 - \frac{E_i}{mc^2}
\]

- For a thermal distribution of velocities

\[
\frac{1}{2} m \langle v^2 \rangle = \frac{3kT}{2} \quad \beta_i^2 \approx \frac{3kT}{mc^2} \rightarrow \langle \frac{E_f}{E_i} - 1 \rangle \sim \frac{kT - E_i}{mc^2}
\]

so that if \( E_i \ll kT \) the photon gains energy and \( E_i \gg kT \) it loses energy \( \rightarrow \) this is a thermalization process
Energy Transfer

- A more proper treatment of the radiative transfer equation (below) for the distribution gives the Kompaneets equation and

\[
\frac{E_f}{E_i} - 1 = \frac{4kT - E_i}{mc^2}
\]

where the coefficient reflects the fact that averaged over the Wien tail \( \langle E_i \rangle = 3kT \) and \( \langle E_i^2 \rangle = 12(kT)^2 \) so that

\[
E_f - E_i \propto 4kTE_i - E_i^2 = 0
\]

in thermodynamic equilibrium. For multiple scattering events one finds that the energy transfer goes as

\[
4 \frac{k(T_e - T_\gamma)}{mc^2} \tau
\]
Inverse Compton Scattering

- Relativistic electrons boost energy of low energy photons by a potentially enormous amount - a way of getting $\gamma$ rays in astrophysics

$$\frac{E_f}{E_i} = \frac{1 - \beta_i \cos \alpha_i}{1 - \beta_i \cos \alpha_f + \frac{E_i}{\gamma_i mc^2}(1 - \cos \theta)}$$

- Take $\beta_i \to 1$ and $\gamma \gg 1$ but $E_i < \gamma_i mc^2$ so that the scattering is Thomson in the rest frame

- Qualitative: $\alpha_i$ incoming angle wrt electron velocity can be anything; BUT outgoing $\alpha_f$ is strongly beamed in direction of velocity with $\alpha_f \sim 1/\gamma \ll 1$ or $\cos \alpha_f \approx 1$ so

$$\frac{E_f}{E_i} \approx \frac{1 - \beta_i \cos \alpha_i}{1 - \beta_i} \left[ \frac{1}{\gamma_i^2} = (1 - \beta)(1 + \beta) \approx 2(1 - \beta) \right]$$

$$\approx (1 - \beta_i \cos \alpha_i)2\gamma_i^2$$
Inverse Compton Scattering

- Since the incoming angle is random, a typical photon gains energy by a factor of $\gamma_i^2$!

- From the electron’s perspective: see a bunch of (beamed) photons coming head on boosted in energy by a factor of $\gamma_i$ and scatter out roughly isotropic *preserving energy* in the rest frame. Reverse the boost and pick up another factor of $\gamma_i$ in the lab frame beamed into the direction of motion.

- Limit to energy transfer - total energy of the electron in lab frame

\[
E_f - E_i \approx E_f < \gamma_i m c^2
\]
\[
\gamma_i^2 E_i < \gamma_i m c^2
\]
\[
\gamma_i E_i < m c^2
\]
Inverse Compton Scattering

- Maximum energy boost

\[ E_f = \gamma_i (\gamma_i E_i) < \gamma_i mc^2 = \gamma_i (511\text{keV}) \]

so that \( E_f \) can be an enormous energy and the scattering is still Thomson in the rest frame

- Is it consistent to neglect recoil compared with \( 1/\gamma_i^2 \) in the lab frame? The condition becomes

\[
\text{Recoil} = \frac{E_i}{\gamma_i mc^2} \ll \frac{1}{\gamma_i^2} \\
E_i \gamma_i \ll mc^2
\]

Yes until the maximum energy is reached.
Inverse Compton Scattering

- With $\gamma_i = 10^3$

  - radio \hspace{1cm} 1GHz = $10^9$Hz $\rightarrow$ $10^{15}$Hz $\approx$ 300nm  
  - optical \hspace{1cm} $4 \times 10^{14}$Hz $\rightarrow$ $4 \times 10^{20}$Hz $\approx$ 1.6MeV  

- These energies are less than the maximal $\gamma_i (512\text{keV}) = 512\text{MeV}$ so still Thomson in rest frame.
Single Scattering

- Consider a distribution of photons (and electrons) in the lab frame. Characterize the radiative transfer in the optically thin single scattering regime.

- Consider the energy density ($A$ is a constant)

$$u = 2 \int \frac{d^3p}{(2\pi\hbar)^2} Ef = A \int f p^3 dp d\Omega$$

- In the electron rest frame the energy density transforms with the aid of the Lorentz transformations

$$p' = p\gamma(1 - \beta\mu)$$

$$d\Omega' = \frac{d\Omega}{\gamma^2(1 - \beta\mu)^2} \quad \rightarrow \quad p'^2 d\Omega' = p^2 d\Omega$$

$$f' = f$$
Single Scattering

- Therefore energy density transforms as

\[ u' = A \int f' p'^3 dp' d\Omega' = A \int f p'^2 dp' d\Omega \]

\[ = A \int \gamma^2 (1 - \beta \mu)^2 f p^3 dp d\Omega \]

- Assume that \( f \) is isotropic (in lab frame) then the energy densities are related as

\[ u' = \int d\Omega \gamma^2 (1 - \beta \mu)^2 A \int f p^3 dp \]

\[ = \int d\Omega \gamma^2 (1 - \beta \mu)^2 \frac{u}{4\pi} \]

\[ = \gamma^2 (1 + \frac{1}{3} \beta^2) u \]
Single Scattering

- In the rest frame the emitted power is given by the Thomson scattering strength $\sigma_T = P/S$ and power is a Lorentz invariant

\[
\frac{dE_{\text{rad}}}{dt} = \frac{dE'}{dt'} = c\sigma_T u'
\]

\[
= c\sigma_T \gamma^2 [1 + \frac{1}{3} \beta^2] u
\]

- The total power accounts for the “absorbed” energy of the incoming photons

\[
\frac{dE}{dt} = c\sigma_T [\gamma^2 (1 + \frac{1}{3} \beta^2) - 1] u, \quad \gamma^2 - 1 = \gamma^2 \beta^2
\]

\[
= \frac{4}{3} \sigma_T c \gamma^2 \beta^2 u
\]
Single Scattering

- If we assume a thermal distribution of electrons $\langle \beta^2 \rangle = \frac{3kT}{mc^2}$ we get

  $$\frac{du}{dt} = \frac{4kT}{mc^2}c\sigma_T n_e u \quad \rightarrow \quad \frac{du}{d\tau} = \frac{4kT}{mc^2}u$$

- Assume a power law distribution of electron energies $dE = mc^2 d\gamma$ confined to a range $\gamma_{\text{min}} \leq \gamma \leq \gamma_{\text{max}}$

  $$n_e = \int n_{e,\gamma} d\gamma \quad n_{e,\gamma} = K\gamma^{-p}$$

- Change in the radiation energy density given $\gamma \gg 1 (\beta \approx 1)$

  $$\frac{du}{dt} = \frac{d^2 E}{dtdV} = \int \frac{dE}{dt} n_{e,\gamma} d\gamma$$
  
  $$= \frac{4}{3} \sigma_T cu \frac{K}{3-p} (\gamma_{\text{max}}^{3-p} - \gamma_{\text{min}}^{3-p})$$
Single Scattering

- Energy spectrum in the power law case

\[ \frac{d^2 E}{dtdV} \propto \int \gamma^{2-p} d\gamma \]

and the scattered photon energy \( E_f \propto \gamma^2 E_i \)

\[ \frac{d^2 E}{dtdV dE_f} = \frac{d\gamma}{dE_f} \frac{d^2 E}{dtdV d\gamma} \propto \frac{1}{\gamma} \gamma^{2-p} = \gamma^{1-p} \propto E_f^{(1-p)/2} \]

so the scattered spectrum is also a power law \( E_f^{-s} \) but with a power law index \( s = (p - 1)/2 \).
Radiative Transfer Equation

- For full radiative transfer, we must go beyond the single scattering, optically thin limit. Generally (set $\hbar = c = k = 1$ and neglect Pauli blocking and polarization)

$$\frac{\partial f}{\partial t} = \frac{1}{2E(p_f)} \int \frac{d^3p_i}{(2\pi)^3} \frac{1}{2E(p_i)} \int \frac{d^3q_f}{(2\pi)^3} \frac{1}{2E(q_f)} \int \frac{d^3q_i}{(2\pi)^3} \frac{1}{2E(q_i)}$$

$$\times (2\pi)^4 \delta(p_f + q_f - p_i - q_i) |M|^2$$

$$\times \left\{ f_e(q_i) f(p_i) [1 + f(p_f)] - f_e(q_f) f(p_f) [1 + f(p_i)] \right\}$$

where the matrix element is calculated in field theory and is Lorentz invariant. In terms of the rest frame $\alpha = e^2/\hbar c$ (c.f. Klein Nishina Cross Section)

$$|M|^2 = 2(4\pi)^2 \alpha^2 \left[ \frac{E(p_i)}{E(p_f)} + \frac{E(p_f)}{E(p_i)} - \sin^2 \beta \right]$$

with $\beta$ as the rest frame scattering angle.
Kompaneets Equation

- The Kompaneets equation is the radiative transfer equation in the limit that electrons are thermal

\[ f_e = e^{-(m-\mu)/T_e} e^{-q^2/2mT_e} \]

\[ n_e = e^{-(m-\mu)/T_e} \left( \frac{mT_e}{2\pi} \right)^{3/2} \]

\[ = \left( \frac{2\pi}{mT_e} \right)^{3/2} n_e e^{-q^2/2mT_e} \]

and assume that the energy transfer is small (non-relativistic electrons, \( E_i \ll m \))

\[ \frac{E_f - E_i}{E_i} \ll 1 \quad [O(T_e/m, E_i/m)] \]
Kompaneets Equation

- Kompaneets equation (restoring $\hbar, c k$)

$$\frac{\partial f}{\partial t} = n_e \sigma_T c \left( \frac{kT_e}{mc^2} \right) \frac{1}{x^2} \frac{\partial}{\partial x} \left[ x^4 \left( \frac{\partial f}{\partial x} + f(1 + f) \right) \right] \quad x = \hbar \omega / kT_e$$

- Equilibrium solution must be a Bose-Einstein distribution

$$\frac{\partial f}{\partial t} = 0$$

$$\left[ x^4 \left( \frac{\partial f}{\partial x} + f(1 + f) \right) \right] = K$$

$$\frac{\partial f}{\partial x} + f(1 + f) = \frac{K}{x^4}$$
Kompaneets Equation

Assume that as $x \to 0$, $f \to 0$ then $K = 0$ and

$$\frac{df}{dx} = -f(1 + f) \quad \rightarrow \quad \frac{df}{f(1 + f)} = dx$$

$$\ln \frac{f}{1 + f} = -x + c \quad \rightarrow \quad \frac{f}{1 + f} = e^{-x + c}$$

$$f = \frac{e^{-x+c}}{1 - e^{-x+c}} = \frac{1}{e^{x-c} - 1}$$
Kompaneets Equation

- More generally, no evolution in the number density

\[
\begin{align*}
n_\gamma & \propto \int d^3 p \, f \propto \int d x \, x^2 \, f \\
\frac{\partial n_\gamma}{\partial t} & \propto \int d x \, x^2 \left( \frac{1}{x^2} \frac{\partial}{\partial x} \right) \left[ x^4 \left( \frac{\partial f}{\partial x} + f (1 + f) \right) \right] \\
& \propto x^4 \left[ \frac{\partial f}{\partial x} + f (1 + f) \right]_0^\infty = 0
\end{align*}
\]

- Energy evolution \( R \equiv n_e \sigma_T c (kT_e / mc^2) \)

\[
\begin{align*}
u & = 2 \int \frac{d^3 p}{(2\pi \hbar)^3} E f = 2 \int \frac{p^3 dp c}{2\pi^2 \hbar^3} f = \left[ \frac{(kT_e)^4}{c^4 \hbar^3} \frac{1}{\pi^2} \equiv A \right] \int x^3 dx f \\
\frac{\partial u}{\partial t} & = AR \int d x \, x \frac{\partial}{\partial x} \left[ x^4 \left( \frac{\partial f}{\partial x} + f (1 + f) \right) \right]
\end{align*}
\]
Kompaneets Equation

\[
\frac{\partial u}{\partial t} = -AR \int dx x^4 \left( \frac{\partial f}{\partial x} + f (1 + f) \right)
\]

\[
= AR \int dx 4x^3 f - AR \int dx x^4 f (1 + f)
\]

\[
= 4n_e \sigma_T c \frac{kT_e}{mc^2} u - AR \int dx x^4 f (1 + f)
\]

Change in energy is difference between Doppler and recoil

• If \( f \) is a Bose-Einstein distribution at temperature \( T_\gamma \)

\[
\frac{\partial f}{\partial x_\gamma} = -f (1 + f) \quad x_\gamma = \frac{pc}{kT_\gamma}
\]

\[
AR \int dx x^4 f (1 + f) = -AR \int dx x^4 \frac{\partial f}{\partial x_\gamma} = AR \int dx 4x^3 \frac{dx}{dx_\gamma} f
\]
Kompaneets Equation

- Radiative transfer equation for energy density

\[
\frac{\partial u}{\partial t} = 4n_e \sigma_T c \frac{k T_e}{m c^2} \left[ 1 - \frac{T_\gamma}{T_e} \right] u
\]

\[
\frac{1}{u} \frac{\partial u}{\partial t} = 4n_e \sigma_T c \frac{k (T_e - T_\gamma)}{m c^2}
\]

which is our original form from the energy-momentum conservation argument

- The analogue to the optical depth for energy transfer is the Compton \( y \) parameter

\[
d\tau = n_e \sigma_T ds = n_e \sigma_t c dt
\]

\[
dy = \frac{k (T_e - T_\gamma)}{mc^2} d\tau
\]
Kompaneets Equation

- Example: hot X-ray cluster with $kT \sim$ keV and the CMB: $T_e \gg T_\gamma$

- Inverse Compton scattering transfers energy to the photons while conserving the photon number

- Optically thin conditions: low energy photons boosted to high energy leaving a deficit in the number density in the RJ tail and an enhancement in the Wien tail called a Compton-y distortion — see problem set

- Compton scattering off high energy electrons can give low energy photons a large boost in energy but cannot create the photons in the first place
Kompaneets Equation

- Numerical solution of the Kompaneets equation going from a Compton-$y$ distortion to a chemical potential distortion of a blackbody
Compton Scattering Map

- WMAP measured the Compton “emission” of photons, i.e. the Compton scattering of the CMB at $z \sim 1000$. At 61 GHz:

- Red band at equator is galactic contamination from the next two processes.
• Gamma ray bubble from center of galaxy thought to be inverse Compton from synchrotron radiation seen at radio frequencies.

Compton Scattering Map

- Fermi 1-5 GeV
- Haslam 408 MHz
- Rosat Band 6 and 7
- WMAP 23 GHz haze