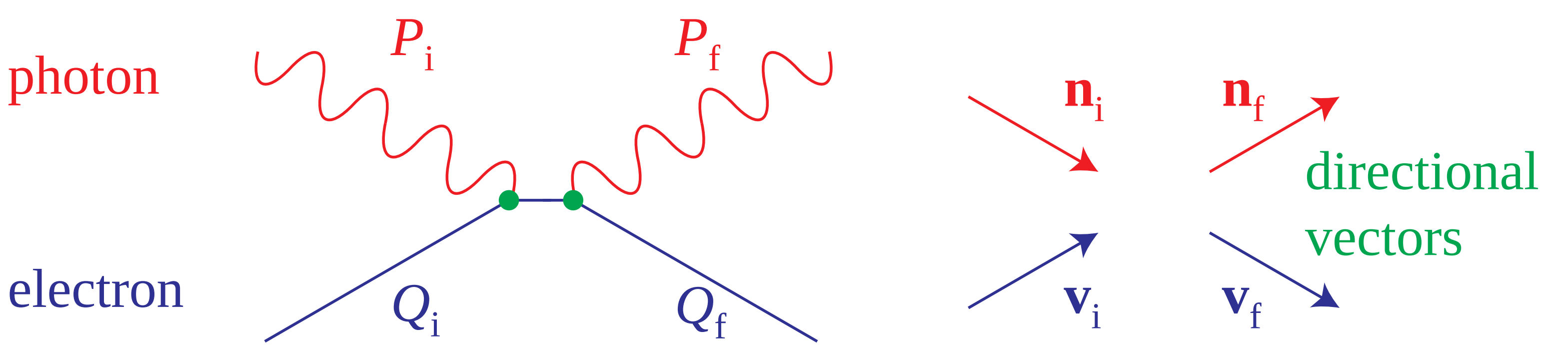
Set 8:

Compton Scattering

Energy-Momentum Conservation

- Thomson scattering $e_i + \gamma_i \to e_f + \gamma_f$ where the frequencies $\omega_i = \omega_f$ (elastic scattering) cannot strictly be true
- Photons carry off E/c momentum and so to conserve momentum the electron must recoil.
- The electron then carries away some of the available energy and so the Thomson limit requires $\hbar\omega_i/mc^2 \ll 1$ in the electron rest frame
- General case (arbitrary electron velocity)



Energy-Momentum Conservation

- 4 momenta

$$P_i = \frac{E_i}{c}(1, \hat{\mathbf{n}}_i)\,, \quad P_f = \frac{E_f}{c}(1, \hat{\mathbf{n}}_f)\,, \qquad \text{photon}$$
$$Q_i = \gamma_i m(c, \hat{\mathbf{v}}_i)\,, \quad Q_f = \gamma_f m(c, \hat{\mathbf{v}}_f)\,, \qquad \text{electron}$$

- To work out change in photon energy consider Lorentz invariants:

$$Q_\mu Q^\mu = \gamma^2 m^2 (v^2 - c^2) = m^2 \frac{v^2 - c^2}{1 - v^2/c^2} = -m^2 c^2$$
$$P_\mu P^\mu = 0$$

Energy-Momentum Conservation

- Conservation

$$P_i + Q_i = P_f + Q_f$$

$$\text{also} \qquad P_f \cdot [P_i + Q_i = P_f + Q_f]$$

$$P_f \cdot P_i + P_f \cdot Q_i = P_f \cdot Q_f$$

- Some identities to express final state in terms of initial state

$$(P_i + Q_i)_\mu (P_i + Q_i)^\mu = (P_f + Q_f)_\mu (P_f + Q_f)^\mu$$

$$2P_{i\mu} Q_i^\mu - m^2 c^2 = 2P_{f\mu} Q_f^\mu - m^2 c^2$$

$$P_i \cdot Q_i = P_f \cdot Q_f$$

- So

$$P_f \cdot P_i + P_f \cdot Q_i = P_i \cdot Q_i$$

Energy-Momentum Conservation

- In three vector notation

$$\frac{E_i}{c}\frac{E_f}{c}(\hat{\mathbf{n}}_i\cdot\hat{\mathbf{n}}_f-1)+\frac{E_f}{c}\gamma_i m(\hat{\mathbf{n}}_f\cdot\mathbf{v}_i-c)=\frac{E_i}{c}\gamma_i m(\hat{\mathbf{n}}_i\cdot\mathbf{v}_i-c)$$

- Introducing the scattering angle as $\hat{\mathbf{n}}_i\cdot\hat{\mathbf{n}}_f=\cos\theta$ and auxiliary angles $\hat{\mathbf{n}}_f\cdot\mathbf{v}_i=v_i\cos\alpha_f$ and $\hat{\mathbf{n}}_i\cdot\mathbf{v}_i=v_i\cos\alpha_i$

$$E_iE_f(\cos\theta-1)+E_f\gamma_i mc^2(\beta_i\cos\alpha_f-1)=E_i\gamma_i mc^2(\beta_i\cos\alpha_i-1)$$

$$E_f=\frac{E_i\gamma_i mc^2(\beta_i\cos\alpha_i-1)}{E_i(\cos\theta-1)+\gamma_i mc^2(\beta_i\cos\alpha_i-1)}$$

- So the change in photon energy is given by

$$\frac{E_f}{E_i}=\frac{1-\beta_i\cos\alpha_i}{1-\beta_i\cos\alpha_f+\frac{E_i}{\gamma mc^2}(1-\cos\theta)}$$

Recoil Effect

- Two ways of changing the energy: Doppler boost β_i from incoming electron velocity and E_i comparable to γmc^2
- Take the incoming electron rest frame $\beta_i = 0$ and $\gamma = 1$

$$\left.\frac{E_f}{E_i}\right|_{\rm rest} = \frac{1}{1 + \frac{E_i}{mc^2}(1 - \cos\theta)}$$

- Since $-1 \le \cos\theta \le 1$, $E_f \le E_i$, energy is lost from the recoil except for purely forward scattering via comparing the incoming photon energy E_i and the electron rest mass mc^2
- The backwards scattering limit is easy to derive

$$|\mathbf{q}_f| = m|\mathbf{v}_f| = 2\frac{E_i}{c}\,, \qquad \Delta E = \frac{1}{2}mv_f^2 = \frac{1}{2}m\left(\frac{2E_i}{mc}\right)^2 = \frac{2E_i}{mc^2}E_i$$

$$E_f = E_i - \Delta E = (1 - \frac{2E_i}{mc^2})E_i \approx \frac{E_i}{1 + \frac{2E_i}{mc^2}}$$

Compton Wavelength

- Alternate phrasing in terms of length scales $E = h\nu = hc/\lambda$ so

$$\left.\frac{\lambda_i}{\lambda_f}\right|_{\rm rest} = \frac{1}{1 + \frac{hc}{\lambda_i mc^2}(1 - \cos\theta)}$$

$$\left.\frac{\lambda_f}{\lambda_i} - 1\right|_{\rm rest} = \frac{h}{mc\lambda_i}(1 - \cos\theta) \equiv \frac{\lambda_c}{\lambda_i}(1 - \cos\theta)$$

where the Compton wavelength is $\lambda_c = h/mc \approx 2.4 \times 10^{-10}$ cm.

- Doppler effect: consider the limit of $\beta_i \ll 1$ then expand to first order

$$\frac{E_f}{E_i} = 1 - \beta_i \cos\alpha_i + \beta_i \cos\alpha_f - \frac{E_i}{mc^2}(1 - \cos\theta)$$

however averaging over angles the Doppler shifts don't change the energies

Second Order Doppler Shift

- To second order in the velocities, the Doppler shift transfers energy from the electron to the photon in opposition to the recoil

$$\frac{E_f}{E_i} = 1 - \beta_i \cos\alpha_i + \beta_i \cos\alpha_f + \beta_i^2 \cos^2\alpha_f - \frac{E_i}{mc^2}$$

$$\langle \frac{E_f}{E_i} \rangle \approx 1 + \frac{1}{3}\beta_i^2 - \frac{E_i}{mc^2}$$

- For a thermal distribution of velocities

$$\frac{1}{2}m\langle v^2 \rangle = \frac{3kT}{2} \qquad \beta_i^2 \approx \frac{3kT}{mc^2} \to \langle \frac{E_f}{E_i} - 1 \rangle \sim \frac{kT - E_i}{mc^2}$$

so that if $E_i \ll kT$ the photon gains energy and $E_i \gg kT$ it loses energy $\to$ this is a thermalization process

Energy Transfer

- A more proper treatment of the radiative transfer equation (below) for the distribution gives the Kompaneets equation and

$$\frac{E_f}{E_i} - 1 = \frac{4kT - E_i}{mc^2}$$

where the coefficient reflects the fact that averaged over the Wien tail $\langle E_i \rangle = 3kT$ and $\langle E_i^2 \rangle = 12(kT)^2$ so that

$$E_f - E_i \propto 4kTE_i - E_i^2 = 0$$

in thermodynamic equilibrium. For multiple scattering events one finds that the energy transfer goes as

$$4\frac{k(T_e - T_\gamma)}{mc^2}\tau$$

Inverse Compton Scattering

- Relativistic electrons boost energy of low energy photons by a potentially enormous amount - a way of getting γ rays in astrophysics

$$\frac{E_f}{E_i} = \frac{1 - \beta_i \cos \alpha_i}{1 - \beta_i \cos \alpha_f + \frac{E_i}{\gamma_i mc^2}(1 - \cos \theta)}$$

- Take $\beta_i \to 1$ and $\gamma \gg 1$ but $E_i < \gamma_i mc^2$ so that the scattering is Thomson in the rest frame
- Qualitative: α_i incoming angle wrt electron velocity can be anything; BUT outgoing α_f is strongly beamed in direction of velocity with $\alpha_f \sim 1/\gamma \ll 1$ or $\cos \alpha_f \approx 1$ so

$$\frac{E_f}{E_i} \approx \frac{1 - \beta_i \cos \alpha_i}{1 - \beta_i} \qquad \left[\frac{1}{\gamma_i^2} = (1 - \beta)(1 + \beta) \approx 2(1 - \beta)\right]$$

$$\approx (1 - \beta_i \cos \alpha_i) 2\gamma_i^2$$

Inverse Compton Scattering

- Since the incoming angle is random, a typical photon gains energy by a factor of γ_i^2!
- From the electron's perspective: see a bunch of (beamed) photons coming head on boosted in energy by a factor of γ_i and scatter out roughly isotropic *preserving energy* in the rest frame. Reverse the boost and pick up another factor of γ_i in the lab frame beamed into the direction of motion.
- Limit to energy transfer - total energy of the electron in lab frame

$$\begin{aligned} E_f - E_i \approx E_f &< \gamma_i m c^2 \\ \gamma_i^2 E_i &< \gamma_i m c^2 \\ \gamma_i E_i &< m c^2 \end{aligned}$$

Inverse Compton Scattering

- Maximum energy boost

$$E_f = \gamma_i(\gamma_i E_i) < \gamma_i mc^2 = \gamma_i(511\mathrm{keV})$$

  so that E_f can be an enormous energy and the scattering is still Thomson in the rest frame

- Is it consistent to neglect recoil compared with $1/\gamma_i^2$ in the lab frame? The condition becomes

$$\mathrm{Recoil} = \frac{E_i}{\gamma_i mc^2} \ll \frac{1}{\gamma_i^2}$$
$$E_i\gamma_i \ll mc^2$$

  Yes until the maximum energy is reached.

Inverse Compton Scattering

- With $\gamma_i = 10^3$

$$\begin{array}{lll} \text{radio} & 1\text{GHz} = 10^9\text{Hz} \rightarrow 10^{15}\text{Hz} \approx 300\text{nm} & \text{UV} \\ \text{optical} & 4 \times 10^{14}\text{Hz} \rightarrow 4 \times 10^{20}\text{Hz} \approx 1.6\text{MeV} & \text{gamma ray} \end{array}$$

- These energies are less than the maximal $\gamma_i(512\text{keV})$ = 512MeV so still Thomson in rest frame.

Single Scattering

- Consider a distribution of photons (and electrons) in the lab frame. Characterize the radiative transfer in the optically thin single scattering regime
- Consider the energy density (A is a constant)

$$u = 2\int \frac{d^3p}{(2\pi\hbar)^2} E f = A \int f p^3 dp d\Omega$$

- In the electron rest frame the energy density transforms with the aid of the Lorentz transformations

$$p' = p\gamma(1-\beta\mu)$$

$$d\Omega' = \frac{d\Omega}{\gamma^2(1-\beta\mu)^2} \qquad \rightarrow p'^2 d\Omega' = p^2 d\Omega$$

$$f' = f$$

Single Scattering

- Therefore energy density transforms as

$$u' = A \int f' {p'}^3 dp' d\Omega' = A \int f p' p^2 dp' d\Omega$$
$$= A \int \gamma^2 (1 - \beta\mu)^2 f p^3 dp d\Omega$$

- Assume that f is isotropic (in lab frame) then the energy densities are related as

$$u' = \int d\Omega \gamma^2 (1 - \beta\mu)^2 A \int f p^3 dp$$
$$= \int d\Omega \gamma^2 (1 - \beta\mu)^2 \frac{u}{4\pi}$$
$$= \gamma^2 (1 + \frac{1}{3}\beta^2) u$$

Single Scattering

- In the rest frame the emitted power is given by the Thomson scattering strength $\sigma_T = P/S$ and power is a Lorentz invariant

$$\frac{dE_{\rm rad}}{dt} = \frac{dE'}{dt'} = c\sigma_T u'$$
$$= c\sigma_T \gamma^2 [1 + \frac{1}{3}\beta^2] u$$

- The total power accounts for the "absorbed" energy of the incoming photons

$$\frac{dE}{dt} = c\sigma_T[\gamma^2(1+\frac{1}{3}\beta^2) - 1]u\,, \qquad \gamma^2 - 1 = \gamma^2\beta^2$$
$$= \frac{4}{3}\sigma_T c\gamma^2\beta^2 u$$

Single Scattering

- If we assume a thermal distribution of electrons $\langle \beta^2 \rangle = 3kT/mc^2$ we get

$$\frac{du}{dt} = \frac{4kT}{mc^2} c\sigma_T n_e u \qquad \rightarrow \frac{du}{d\tau} = \frac{4kT}{mc^2} u$$

- Assume a power law distribution of electron energies $dE = mc^2 d\gamma$ confined to a range $\gamma_{\rm min} \leq \gamma \leq \gamma_{\rm max}$

$$n_e = \int n_{e,\gamma} d\gamma \qquad n_{e,\gamma} = K\gamma^{-p}$$

- Change in the radiation energy density given $\gamma \gg 1$ ($\beta \approx 1$)

$$\begin{aligned} \frac{du}{dt} &= \frac{d^2E}{dtdV} = \int \frac{dE}{dt} n_{e,\gamma} d\gamma \\ &= \frac{4}{3}\sigma_T c u \frac{K}{3-p}(\gamma_{\rm max}^{3-p} - \gamma_{\rm min}^{3-p}) \end{aligned}$$

Single Scattering

- Energy spectrum in the power law case

$$\frac{d^2E}{dtdV} \propto \int \gamma^{2-p} d\gamma$$

and the scattered photon energy $E_f \propto \gamma^2 E_i$

$$\frac{d^2E}{dtdV\,dE_f} = \frac{d\gamma}{dE_f}\frac{d^2E}{dtdV\,d\gamma} \propto \frac{1}{\gamma}\gamma^{2-p} = \gamma^{1-p} \propto E_f^{(1-p)/2}$$

so the scattered spectrum is also a power law E_f^{-s} but with a power law index $s = (p-1)/2$.

Radiative Transfer Equation

- For full radiative transfer, we must go beyond the single scattering, optically thin limit. Generally (set $\hbar = c = k = 1$ and neglect Pauli blocking and polarization)

$$\frac{\partial f}{\partial t} = \frac{1}{2E(p_f)} \int \frac{d^3 p_i}{(2\pi)^3} \frac{1}{2E(p_i)} \int \frac{d^3 q_f}{(2\pi)^3} \frac{1}{2E(q_f)} \int \frac{d^3 q_i}{(2\pi)^3} \frac{1}{2E(q_i)}$$
$$\times (2\pi)^4 \delta(p_f + q_f - p_i - q_i) |M|^2$$
$$\times \{ f_e(q_i) f(p_i) [1 + f(p_f)] - f_e(q_f) f(p_f) [1 + f(p_i)] \}$$

where the matrix element is calculated in field theory and is Lorentz invariant. In terms of the rest frame $\alpha = e^2/\hbar c$ (c.f. Klein Nishina Cross Section)

$$|M|^2 = 2(4\pi)^2 \alpha^2 \left[\frac{E(p_i)}{E(p_f)} + \frac{E(p_f)}{E(p_i)} - \sin^2 \beta \right]$$

with β as the rest frame scattering angle

Kompaneets Equation

- The Kompaneets equation is the radiative transfer equation in the limit that electrons are thermal

$$f_e = e^{-(m-\mu)/T_e} e^{-q^2/2mT_e} \qquad \left[n_e = e^{-(m-\mu)/T_e} \left(\frac{mT_e}{2\pi} \right)^{3/2} \right]$$

$$= \left(\frac{2\pi}{mT_e} \right)^{3/2} n_e e^{-q^2/2mT_e}$$

and assume that the energy transfer is small (non-relativistic electrons, $E_i \ll m$

$$\frac{E_f - E_i}{E_i} \ll 1 \qquad [\mathcal{O}(T_e/m, E_i/m)]$$

Kompaneets Equation

- Kompaneets equation (restoring $\hbar$, c k)

$$\frac{\partial f}{\partial t} = n_e \sigma_T c \left(\frac{kT_e}{mc^2} \right) \frac{1}{x^2} \frac{\partial}{\partial x} \left[x^4 \left(\frac{\partial f}{\partial x} + f(1+f) \right) \right] \qquad x = \hbar\omega / kT_e$$

- Equilibrium solution must be a Bose-Einstein distribution
  $\partial f / \partial t = 0$

$$\left[x^4 \left(\frac{\partial f}{\partial x} + f(1+f) \right) \right] = K$$

$$\frac{\partial f}{\partial x} + f(1+f) = \frac{K}{x^4}$$

Kompaneets Equation

Assume that as $x \to 0$, $f \to 0$ then $K = 0$ and

$$\frac{df}{dx} = -f(1+f) \qquad \to \frac{df}{f(1+f)} = dx$$

$$\ln \frac{f}{1+f} = -x + c \qquad \to \frac{f}{1+f} = e^{-x+c}$$

$$f = \frac{e^{-x+c}}{1-e^{-x+c}} = \frac{1}{e^{x-c}-1}$$

Kompaneets Equation

- More generally, no evolution in the number density

$$
\begin{aligned}
n_\gamma &\propto \int d^3p f \propto \int dx x^2 f \\
\frac{\partial n_\gamma}{\partial t} &\propto \int dx x^2 \frac{1}{x^2} \frac{\partial}{\partial x} \left[x^4 \left(\frac{\partial f}{\partial x} + f(1+f) \right) \right] \\
&\propto x^4 \left[\frac{\partial f}{\partial x} + f(1+f) \right]_0^\infty = 0
\end{aligned}
$$

- Energy evolution $R \equiv n_e \sigma_T c (kT_e/mc^2)$

$$
u = 2 \int \frac{d^3p}{(2\pi\hbar)^3} E f = 2 \int \frac{p^3 dpc}{2\pi^2 \hbar^3} f = \left[\frac{(kT_e)^4}{c^4 \hbar^3} \frac{1}{\pi^2} \equiv A \right] \int x^3 dx f
$$

$$
\frac{\partial u}{\partial t} = AR \int dx x \frac{\partial}{\partial x} \left[x^4 \left(\frac{\partial f}{\partial x} + f(1+f) \right) \right]
$$

Kompaneets Equation

$$\frac{\partial u}{\partial t} = -AR \int dx x^4 \left(\frac{\partial f}{\partial x} + f(1+f) \right)$$
$$= AR \int dx 4x^3 f - AR \int dx x^4 f(1+f)$$
$$= 4 n_e \sigma_T c \frac{kT_e}{mc^2} u - AR \int dx x^4 f(1+f)$$

Change in energy is difference between Doppler and recoil

- If f is a Bose-Einstein distribution at temperature T_γ

$$\frac{\partial f}{\partial x_\gamma} = -f(1+f) \qquad x_\gamma = \frac{pc}{kT_\gamma}$$

$$AR \int dx x^4 f(1+f) = -AR \int dx x^4 \frac{\partial f}{\partial x_\gamma} = AR \int dx 4x^3 \frac{dx}{dx_\gamma} f$$

Kompaneets Equation

- Radiative transfer equation for energy density

$$\frac{\partial u}{\partial t} = 4n_e\sigma_T c\frac{kT_e}{mc^2}\left[1 - \frac{T_\gamma}{T_e}\right]u$$

$$\frac{1}{u}\frac{\partial u}{\partial t} = 4n_e\sigma_T c\frac{k(T_e - T_\gamma)}{mc^2}$$

which is our original form from the energy-momentum conservation argument

- The analogue to the optical depth for energy transfer is the Compton y parameter

$$d\tau = n_e\sigma_T ds = n_e\sigma_t cdt$$

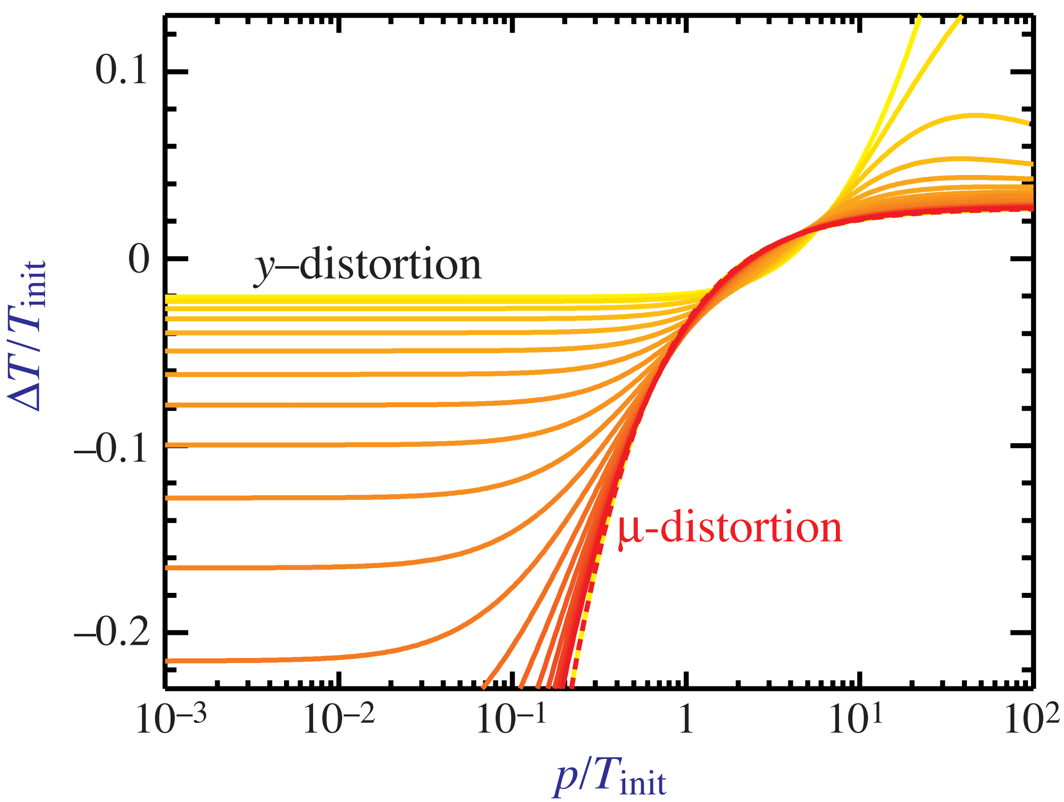
$$dy = \frac{k(T_e - T_\gamma)}{mc^2}d\tau$$

Kompaneets Equation

- Example: hot X-ray cluster with $kT \sim$ keV and the CMB: $T_e \gg T_\gamma$
- Inverse Compton scattering transfers energy to the photons while conserving the photon number
- Optically thin conditions: low energy photons boosted to high energy leaving a deficit in the number density in the RJ tail and an enhancement in the Wien tail called a Compton-y distortion — see problem set
- Compton scattering off high energy electrons can give low energy photons a large boost in energy but cannot create the photons in the first place

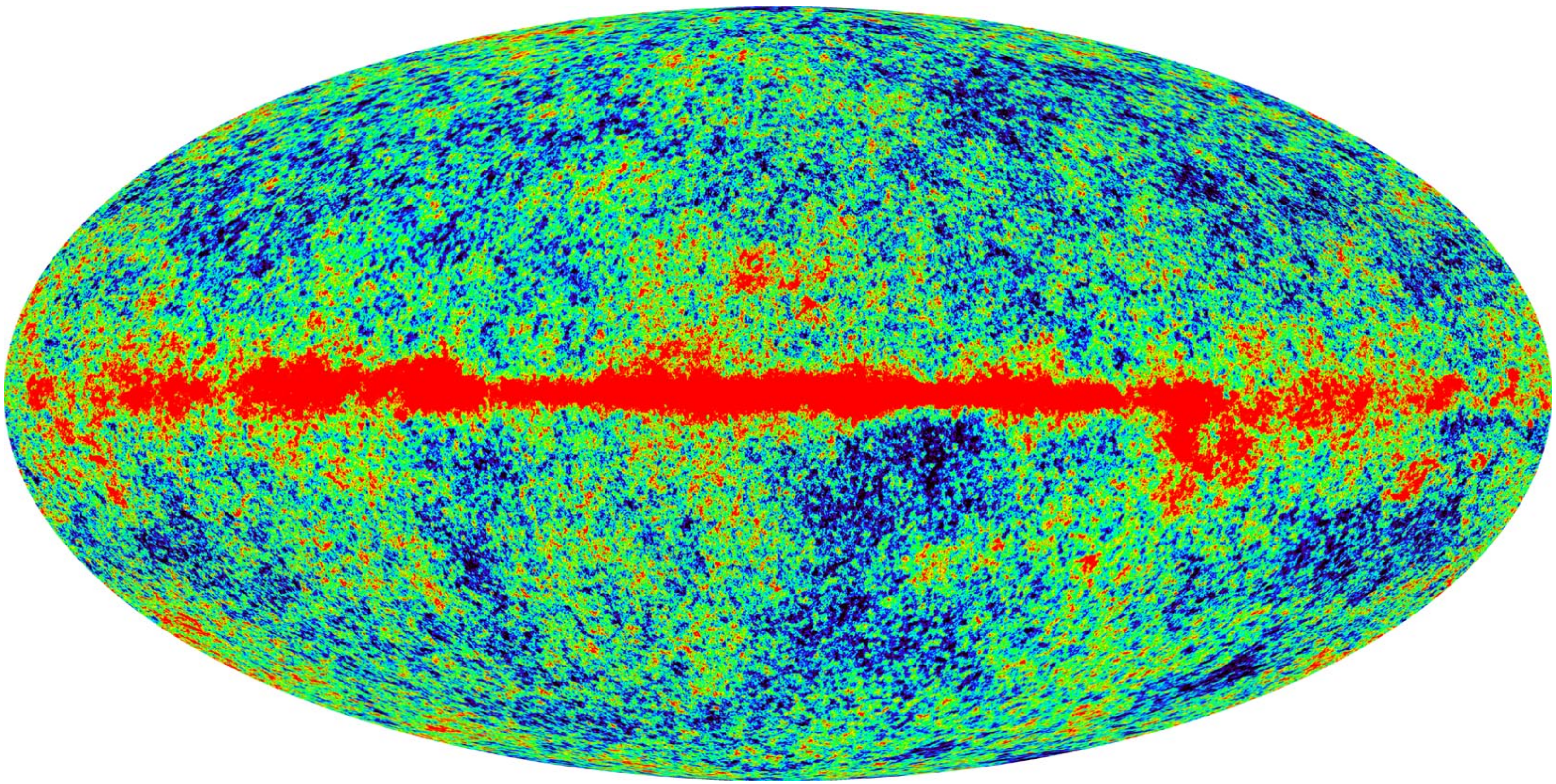
Kompaneets Equation

- Numerical solution of the Kompaneets equation going from a Compton-y distortion to a chemical potential distortion of a blackbody



Compton Scattering Map

- WMAP measured the Compton "emission" of photons, i.e. the Compton scattering of the CMB at $z \sim 1000$. At 61 GHz:



- Red band at equator is galactic contamination from the next two processes.

Compton Scattering Map

- Gamma ray bubble from center of galaxy thought to be inverse Compton from synchrotron radiation seen at radio frequencies.