Set 9:
Bremsstrahlung
Bremsstrahlung can be viewed as Thomson scattering of an electron with the virtual photons in the Coulomb field of the ion.

For relativistic electrons, Coulomb field is essentially Lorentz contracted and also transformed into nearly equal transverse $B$ field.
Coulomb Field

- To a moving electron, the Coulomb field of an ion is Lorentz transformed from a spherically symmetric field to a pulse resembling a radiation field.

- From the general formula for the Lorentz transformation of the field Coulomb $E$ field and $B = 0$

\[
\begin{align*}
E'_\parallel &= E_\parallel & B'_\parallel &= 0 \\
E'_\perp &= \gamma E_\perp & B'_\perp &= -\gamma \beta \times \mathbf{E}
\end{align*}
\]
Coulomb Field

- Given an electron moving in the $x$ direction it will see a field of the form

$$E'_x = \frac{qx}{r^3} = \frac{q\gamma(x' - vt')}{r^3}, \quad B'_x = 0$$

$$E'_y = \frac{q\gamma y}{r^3} = \frac{q\gamma y'}{r^3}, \quad B'_y = -\frac{q\gamma\beta z'}{r^3}$$

$$E'_z = \frac{q\gamma z}{r^3} = \frac{q\gamma z'}{r^3}, \quad B'_z = \frac{q\gamma\beta y'}{r^3}$$

$$r^2 = \gamma^2(x' - vt')^2 + y'^2 + z'^2$$

- Take the electron to be at $x' = 0$, $z' = 0$ and $y' = b$ then the fields are

$$\mathbf{E}' = \frac{q\gamma}{(\gamma^2 v^2 t'^2 + b^2)^{3/2}}(-vt', b, 0)$$

$$\mathbf{B}' = (0, 0, \beta E'_y)$$
Rest Frame Radiation

- Take an ion of charge \( q = -Ze \), the acceleration of the electron in the rest frame

\[
d'_\parallel = -\frac{eE'_x}{m} = \frac{Ze^2 \gamma t'}{m(\gamma^2 v^2 t'^2 + b^2)^{3/2}}
\]

\[
d'_\perp = -\frac{eE'_y}{m} = -\frac{Ze^2 \gamma b}{m(\gamma^2 v^2 t'^2 + b^2)^{3/2}}
\]

- Power is radiated via the Larmor formula and frequency content comes from the Fourier moments

\[
P' = \frac{2e^2 a'^2}{3c^3}
\]

\[
\frac{dW'}{d\omega'} = \frac{8\pi e^2}{3c^3} |a'(\omega')|^2
\]
Rest Frame Radiation

- Frequency spectrum

\[
a(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} a(t) e^{i\omega t} dt
\]

\[
a_{\perp}(\omega') = -\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega't'} dt' \frac{Ze^2\gamma b}{m(\gamma^2 v'^2 t'^2 + b^2)^{3/2}}
\]

\[
= -\frac{1}{2\pi} \frac{Ze^2\gamma b}{m} \frac{b}{\gamma v} \frac{1}{b^3} \left[ \int_{-\infty}^{\infty} \frac{e^{ixy}}{(x^2 + 1)^{3/2}} dx \right] \quad [x = \gamma vt/b]
\]

\[
= -\frac{1}{2\pi m} \frac{Ze^2}{bv} \left[ 2yK_1(y) \right] \quad [y = \omega'b/\gamma v]
\]

\[
a_{\parallel}(\omega') = -\frac{1}{2\pi m} \frac{Ze^2}{bv} \frac{1}{\gamma} \left[ 2iyK_0(y) \right]
\]
Bremsstrahlung

• The modified Bessel functions $K_0, K_1$ have a characteristic high frequency cut off for $y \gg 1$ or $\omega' \gg \gamma v/b$ with

$$\lim_{y \to 0} y K_1(y) = 1 \quad \lim_{y \to \infty} y K_1(y) = \sqrt{\frac{\pi y}{2}} e^{-y}$$

$$\lim_{y \to 0} y K_0(y) = 0 \quad \lim_{y \to \infty} y K_0(y) = \sqrt{\frac{\pi y}{2}} e^{-y}$$

• More power comes out in the $\perp$ term especially for $\gamma \gg 1$, thus since $dW/d\omega$ is a Lorentz invariant

$$\frac{dW}{d\omega} = \frac{dW'}{d\omega'} \approx \frac{8\pi e^2}{3c^3} \frac{1}{4\pi^2 m^2} \left( \frac{Z^2 e^4}{b^2 v^2} \right) 4y^2 K_1^2(y)$$

$$= \frac{8e^6 Z^2}{3\pi m^2 c^3 b^2 v^2} y^2 K_1^2(y) \quad [\omega = \gamma \omega']$$
Virtual Quanta

- Bremsstrahlung can be viewed as Thomson scattering of virtual particles: the Coulomb field looks like a pulse of radiation quantified by its electric field:

\[
\frac{dW'}{dA d\omega'} \bigg|_{\text{virtual}} = c |E(\omega)|^2 = \frac{cm^2}{e^2} |a_\perp (\omega')|^2 \\
= \frac{cm^2}{e^2} \left\{ \frac{1}{2\pi m} \frac{Z e^2}{bv} \right\}^2 \left[ 2yK_1(y) \right]^2 \\
= \frac{Z^2 e^2 c}{\pi^2 b^2 v^2} \left[ yK_1(y) \right]^2 \\
\frac{dW'}{d\omega'} \bigg|_{\text{rad}} = \sigma_T \frac{dW'}{dA d\omega'} \bigg|_{\text{virtual}} = \frac{8\pi}{3} \left( \frac{e^2}{mc^2} \right)^2 \frac{Z^2 e^2 c}{\pi^2 b^2 v^2} \left[ yK_1(y) \right]^2 \\
\frac{dW}{d\omega} \bigg|_{\text{rad}} = \frac{8}{3\pi} \frac{e^6 Z^2}{m^2 c^3 b^2 v^2} \left[ yK_1(y) \right]^2
\]
Non-Relativistic Bremsstrahlung

- For non-relativistic velocities $\gamma = 1$ and $\omega' \approx \omega$
- $yK_1(y)$ can be approximated by a step function at $y = 1$ or $\omega = \frac{v}{b}$

$$\frac{dW}{d\omega} = \frac{8}{3\pi} \frac{e^6 Z^2}{m^2 c^3 b^2 v^2} \times \left\{ \begin{array}{ll} 1 & \omega < \frac{v}{b}, \\ 0 & \omega > \frac{v}{b}. \end{array} \right.$$

- Spectrum is flat out to a cut off frequency. The interaction takes place over a temporal extent $\Delta t$ which defines the range of frequencies $\rightarrow$ for low $\omega$ looks like a $\delta$ function

$$v\Delta t = b \rightarrow \Delta t = \frac{b}{v} \rightarrow \omega_{\text{max}} \approx \frac{v}{b}$$
Non-Relativistic Bremsstrahlung

- The amplitude is related to the change in velocity through the dipole formula

\[ \dd = e \dot{v} \rightarrow_{FT} -\omega^2 d(\omega) = \frac{e}{2\pi} \int \dot{v} e^{i\omega t} dt \]

\[ \approx \frac{e}{2\pi} \Delta v \quad \omega < v/b, \quad e^{i\omega t} \sim 1 \]

\[ \frac{dW}{d\omega} = \frac{8\pi}{3c^3} \frac{e^2}{4\pi^2} \Delta v^2 \]

- The total change in velocity can be found by integrating the acceleration

\[ \Delta v = \int a_\perp dt = - \int dt \frac{Ze^2b}{m(v^2t^2 + b^2)^{3/2}} = - \frac{Ze^2}{mbv} \int \frac{dx}{(1 + x)^{3/2}} \]

\[ = -\frac{2Ze^2}{mbv} \quad [x = vt/b] \]
Non-Relativistic Bremsstrahlung

- The flat energy distribution for $\omega < v/b$ then becomes as before

$$\frac{dW}{d\omega} = \frac{8}{3\pi} \frac{e^6 Z^2}{m^2 c^3 b^2 v^2}$$

- To get the total bremsstrahlung emission given an ion density $n_i$ and electron density $n_e$, consider that

$$[n_e v = \text{electron flux}] \times [2\pi b db = d(\text{area})] = \frac{dN_e}{dt}$$

$$\frac{dW}{d\omega \, dV \, dt} = (\text{electron flux}) \int d(\text{area}) \frac{dW}{d\omega} (\text{ion density})$$

$$= n_e n_i 2\pi v \int_{b_{\min}}^{b_{\max}} \frac{8}{3\pi} \frac{e^6 Z^2}{m^2 c^3 b^2 v^2} b \, db$$

$$= \frac{16}{3} \frac{e^6 Z^2}{m^2 c^3 v} n_e n_i \ln \frac{b_{\max}}{b_{\min}}$$
Gaunt Factor

- The dependence on maximum and minimum scale is logarithmic and so only a crude specification is needed.

- The maximum impact parameter is set by the condition of $\omega_{\text{max}}$

  $$\omega < \frac{v}{b} \rightarrow b < \frac{v}{\omega}$$

- The minimum impact parameter comes from two sources: violation of the Born approximation that the acceleration is evaluated on the unperturbed path.

  $$\Delta v \sim \frac{Z e^2}{mbv} = v$$

  $$b \sim \frac{Z e^2}{mv^2}$$
## Gaunt Factor

- Or a violation of the uncertainty principle

\[ \Delta x \Delta q > \hbar \quad \rightarrow \quad bmv > \hbar \quad \rightarrow \quad b > \frac{\hbar}{mv} \]

\[ b_{\text{min}} = \max \left( \frac{Ze^2}{mv^2}, \frac{h}{mv} \right) \]

- These factors are collected into a Gaunt factor

\[
\frac{dW}{d\omega dV dt} = \frac{16\pi e^6 Z^2}{3\sqrt{3} c^3 m^2 v} n_e n_i g_{\text{ff}}(v, \omega)
\]

\[ g_{\text{ff}}(v, \omega) = \frac{\sqrt{3}}{\pi} \ln \left( \frac{b_{\text{max}}}{b_{\text{min}}} \right) \]

in practice \( g_{\text{ff}} \) is calibrated and tabulated
Thermal Bremsstrahlung

- Integrate over thermal distribution of electron velocities

\[ n_e = \int \frac{d^3q}{(2\pi \hbar)^3} f_e \propto \int v^2 e^{-mv^2/2kT} dv \]

require that the kinetic energy be sufficient to emit a photon
\[ \frac{1}{2}mv^2 > \hbar \nu \rightarrow v_{\text{min}} \]
to get any radiation

- Qualitative: replace \( v \) with thermal velocity \( \sim (kT/m)^{1/2} \)

\[ \frac{dW}{d\nu dV dt} \rightarrow \frac{\int_\nu^{\min} \frac{dW}{dV dt} v^2 e^{-mv^2/2kT} dv}{\int_0^\infty v^2 e^{-mv^2/2kT} dv} \]

\[ = \frac{2^5 \pi Z^2 e^6}{3m^2 c^3} \left( \frac{2\pi m}{3kT} \right)^{1/2} n_e n_i e^{-\hbar \nu/kT} \bar{g}_{ff} \]

where \( \bar{g}_{ff} \) is the velocity averaged Gaunt factor. Notice the spectrum is flat out to \( \hbar \nu \sim kT \)
Thermal Emission and Absorption

- Emission coefficient

\[
\frac{dW}{dtdV d\nu} = 4\pi j^\text{ff}_\nu
\]

\[
j^\text{ff}_\nu = \frac{2^3 Z^2 e^6}{3 m^2 c^3} \left(\frac{2\pi m}{3kT}\right)^{1/2} n_e n_i e^{-h\nu/kT} \bar{g}_\text{ff}
\]

- Kirchoff’s law \( S_\nu = j_\nu/\alpha_\nu \) gives the thermal absorption coefficient

\[
j^\text{ff}_\nu = \alpha^\text{ff}_\nu B_\nu(T) = \alpha^\text{ff}_\nu \frac{2h\nu^3/c^2}{e^{h\nu/kT} - 1}
\]

\[
\alpha^\text{ff}_\nu = \frac{4Z^2 e^6}{3m^2 c h} \left(\frac{2\pi m}{3kT}\right)^{1/2} n_e n_i \nu^{-3} (1 - e^{-h\nu/kT}) \bar{g}_\text{ff}
\]

- Note the steep spectrum in frequency: low frequency absorption is very efficient and yields a high optical depth \( I_\nu \to S_\nu \)
Alternately, non-relativistic bremsstrahlung can be characterized by a collision term like the Kompaneets equation \((k = \hbar = c = 1, x = h\nu/kT_e)\)

\[
C_{ff}[f] = \sqrt{\frac{2}{\pi}} \left(\frac{T_e}{m}\right)^{-1/2} Z^2 \alpha T_e^{-3} n_i n_e \sigma_T g_{ff} \frac{e^{-x}}{x^3} [1 - (e^x - 1)f]
\]

Note that emission and absorption is balanced only if \(f = 1/(e^x - 1)\), a true blackbody (no chemical potential).
Galactic Bremsstrahlung

- $H\alpha (n = 3 \rightarrow 2$, hydrogen) as a tracer of ionized gas and hence bremsstrahlung emission in galaxy