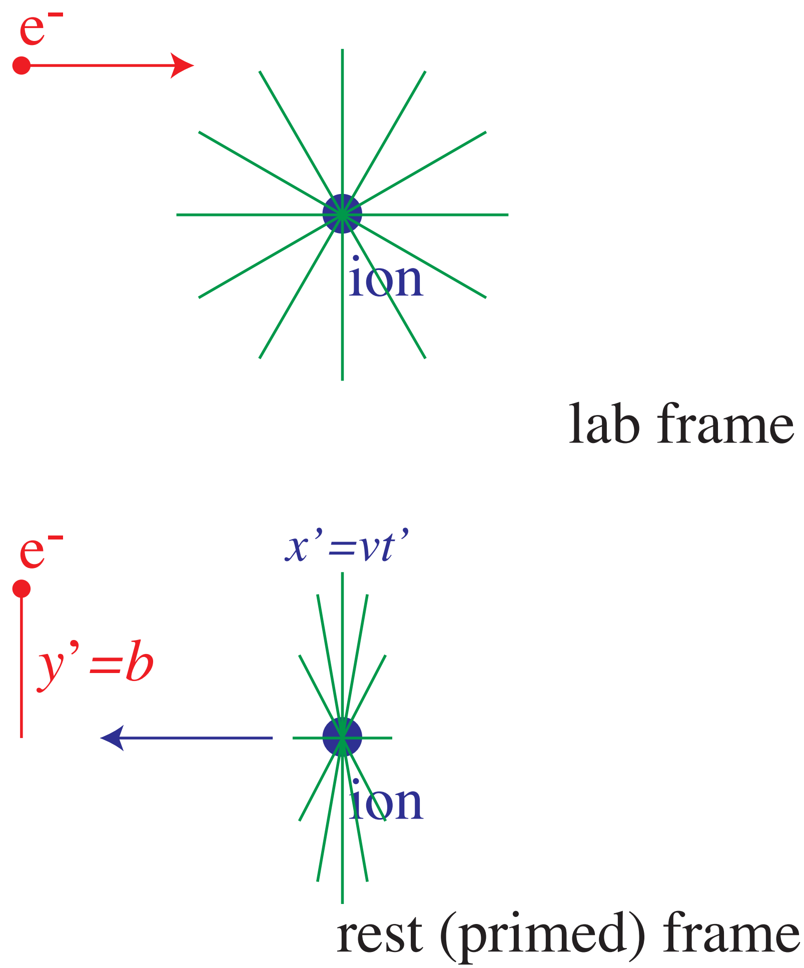
Set 9:

Bremsstrahlung

Coulomb Field

- Bremsstrahlung can be viewed as Thomson scattering of an electron with the *virtual* photons in the Coulomb field of the ion
- For relativistic electrons, Coulomb field is essentially Lorentz contracted and also transformed into nearly equal transverse B field



Coulomb Field

- To a moving electron, the Coulomb field of an ion is Lorentz transformed from a spherically symmetric field to a pulse resembling a radiation field
- From the general formula for the Lorentz transformation of the field Coulomb E field and $B = 0$

$$E'_{\|} = E_{\|} \qquad B'_{\|} = 0$$

$$E'_{\perp} = \gamma E_{\perp} \qquad B'_{\perp} = -\gamma\beta \times \mathbf{E}$$

Coulomb Field

- Given an electron moving in the x direction it will see a field of the form

$$E'_x = \frac{qx}{r^3} = \frac{q\gamma(x' - vt')}{r^3}, \qquad B'_x = 0$$

$$E'_y = \frac{q\gamma y}{r^3} = \frac{q\gamma y'}{r^3}, \qquad B'_y = -\frac{q\gamma\beta z'}{r^3}$$

$$E'_z = \frac{q\gamma z}{r^3} = \frac{q\gamma z'}{r^3}, \qquad B'_z = \frac{q\gamma\beta y'}{r^3}$$

$$r^2 = \gamma^2(x' - vt')^2 + y'^2 + z'^2$$

- Take the electron to be at $x' = 0$, $z' = 0$ and $y' = b$ then the fields are

$$\mathbf{E}' = \frac{q\gamma}{(\gamma^2 v^2 t'^2 + b^2)^{3/2}}(-vt', b, 0)$$

$$\mathbf{B}' = (0, 0, \beta E'_y)$$

Rest Frame Radiation

- Take an ion of charge $q = -Ze$, the acceleration of the electron in the rest frame

$$a'_{\parallel} = -\frac{eE'_x}{m} = \frac{Ze^2 v\gamma t'}{m(\gamma^2 v^2 t'^2 + b^2)^{3/2}}$$

$$a'_{\perp} = -\frac{eE'_y}{m} = -\frac{Ze^2\gamma b}{m(\gamma^2 v^2 t'^2 + b^2)^{3/2}}$$

- Power is radiated via the Larmor formula and frequency content comes from the Fourier moments

$$P' = \frac{2e^2 a'^2}{3c^3}$$

$$\frac{dW'}{d\omega'} = \frac{8\pi e^2}{3c^3}|a'(\omega')|^2$$

Rest Frame Radiation

- Frequency spectrum

$$a(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} a(t) e^{i\omega t} dt$$

$$a'_{\perp}(\omega') = -\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega' t'} dt' \frac{Ze^2 \gamma b}{m(\gamma^2 v'^2 t'^2 + b^2)^{3/2}}$$

$$= -\frac{1}{2\pi} \frac{Ze^2 \gamma b}{m} \frac{b}{\gamma v} \frac{1}{b^3} \left[\int_{-\infty}^{\infty} \frac{e^{ixy}}{(x^2+1)^{3/2}} dx \right] \qquad [x = \gamma v t / b]$$

$$= -\frac{1}{2\pi m} \frac{Ze^2}{bv} [2yK_1(y)] \qquad [y = \omega' b / \gamma v]$$

$$a'_{\parallel}(\omega') = -\frac{1}{2\pi m} \frac{Ze^2}{bv} \frac{1}{\gamma} [2iyK_0(y)]$$

Bremsstrahlung

- The modified Bessel functions K_0 K_1 have a characteristic high frequency cut off for $y \gg 1$ or $\omega' \gg \gamma v/b$ with

$$\lim_{y\to 0} yK_1(y) = 1 \qquad \lim_{y\to\infty} yK_1(y) = \sqrt{\frac{\pi y}{2}}e^{-y}$$

$$\lim_{y\to 0} yK_0(y) = 0 \qquad \lim_{y\to\infty} yK_0(y) = \sqrt{\frac{\pi y}{2}}e^{-y}$$

- More power comes out in the $\perp$ term especially for $\gamma \gg 1$, thus since $dW/d\omega$ is a Lorentz invariant

$$\frac{dW}{d\omega} = \frac{dW'}{d\omega'} \approx \frac{8\pi e^2}{3c^3}\frac{1}{4\pi^2 m^2}\left(\frac{Z^2 e^4}{b^2 v^2}\right) 4y^2 K_1^2(y)$$

$$= \frac{8e^6 Z^2}{3\pi m^2 c^3 b^2 v^2} y^2 K_1^2(y) \qquad [\omega = \gamma\omega']$$

Virtual Quanta

- Bremsstrahlung can be viewed as Thomson scattering of virtual particles: the Coulomb field looks like a pulse of radiation quantified by its electric field:

$$\left.\frac{dW'}{dA\,d\omega'}\right|_{\rm virtual} = c|E(\omega)|^2 = \frac{cm^2}{e^2}|a_\perp(\omega')|^2$$

$$= \frac{cm^2}{e^2}\left\{\frac{1}{2\pi m}\frac{Ze^2}{bv}\,[2yK_1(y)]\right\}^2$$

$$= \frac{Z^2e^2c}{\pi^2b^2v^2}[yK_1(y)]^2$$

$$\left.\frac{dW'}{d\omega'}\right|_{\rm rad} = \sigma_T \left.\frac{dW'}{dA\,d\omega'}\right|_{\rm virtual} = \frac{8\pi}{3}\left(\frac{e^2}{mc^2}\right)^2 \frac{Z^2e^2c}{\pi^2b^2v^2}[yK_1(y)]^2$$

$$\left.\frac{dW}{d\omega}\right|_{\rm rad} = \frac{8}{3\pi}\frac{e^6Z^2}{m^2c^3b^2v^2}[yK_1(y)]^2$$

Non-Relativistic Bremsstrahlung

- For non-relativistic velocities $\gamma = 1$ and $\omega' \approx \omega$
- $yK_1(y)$ can be approximated by a step function at $y = 1$ or $\omega = v/b$

$$\frac{dW}{d\omega} = \frac{8}{3\pi} \frac{e^6 Z^2}{m^2 c^3 b^2 v^2} \times \begin{cases} 1 & \omega < v/b, \\ 0 & \omega > v/b. \end{cases}$$

- Spectrum is flat out to a cut off frequency. The interaction takes place over a temporal extent Δt which defines the range of frequencies $\rightarrow$ for low ω looks like a δ function

$$v\Delta t = b \quad \rightarrow \Delta t = \frac{b}{v} \quad \rightarrow \omega_{\mathrm{max}} \approx \frac{v}{b}$$

Non-Relativistic Bremsstrahlung

- The amplitude is related to the change in velocity through the dipole formula

$$\ddot{d} = e\dot{v} \quad \rightarrow_{FT} \quad -\omega^2 d(\omega) = \frac{e}{2\pi}\int \dot{v}e^{i\omega t}dt$$

$$\approx \frac{e}{2\pi}\Delta v \qquad \omega < v/b, e^{i\omega t} \sim 1$$

$$\frac{dW}{d\omega} = \frac{8\pi}{3c^3}\frac{e^2}{4\pi^2}\Delta v^2$$

- The total change in velocity can be found by integrating the acceleration

$$\Delta v = \int a_\perp dt = -\int dt \frac{Ze^2 b}{m(v^2t^2 + b^2)^{3/2}} = -\frac{Ze^2}{mbv}\int \frac{dx}{(1+x)^{3/2}}$$

$$= -\frac{2Ze^2}{mbv} \qquad [x = vt/b]$$

Non-Relativistic Bremsstrahlung

- The flat energy distribution for $\omega < v/b$ then becomes as before

$$\frac{dW}{d\omega} = \frac{8}{3\pi}\frac{e^6 Z^2}{m^2 c^3 b^2 v^2}$$

- To get the total bremsstrahlung emission given an ion density n_i and electron density n_e, consider that

$$\Big[n_e v = \text{electron flux}\Big] \times \Big[2\pi b db = d(\text{area})\Big] = \frac{dN_e}{dt}$$

$$\frac{dW}{d\omega dV dt} = (\text{electron flux}) \int d(\text{area}) \frac{dW}{d\omega}(\text{ion density})$$

$$= n_e n_i 2\pi v \int_{b_{\min}}^{b_{\max}} \frac{8}{3\pi}\frac{e^6 Z^2}{m^2 c^3 b^2 v^2} b db$$

$$= \frac{16}{3}\frac{e^6 Z^2}{m^2 c^3 v} n_e n_i \ln\frac{b_{\max}}{b_{\min}}$$

Gaunt Factor

- The dependence on maximum and minimum scale is logarithmic and so only a crude specification is needed
- The maximum impact parameter is set by the condition of $\omega_{\rm max}$

$$\omega < \frac{v}{b} \rightarrow b < \frac{v}{\omega}$$

- The minimum impact parameter comes from two sources: violation of the Born approximation that the acceleration is evaluated on the unperturbed path

$$\Delta v \sim \frac{Ze^2}{mbv} = v$$
$$b \sim \frac{Ze^2}{mv^2}$$

Gaunt Factor

- Or a violation of the uncertainty principle

$$\Delta x \Delta q > \hbar \quad \rightarrow bmv > \hbar \quad \rightarrow b > \frac{h}{mv}$$

$$b_{\min} = \max\left(\frac{Ze^2}{mv^2}, \frac{h}{mv}\right)$$

- These factors are collected in to a Gaunt factor

$$\frac{dW}{d\omega dV dt} = \frac{16\pi e^6 Z^2}{3\sqrt{3}c^3 m^2 v} n_e n_i g_{\mathrm{ff}}(v,\omega)$$

$$g_{\mathrm{ff}}(v,\omega) = \frac{\sqrt{3}}{\pi} \ln\left(\frac{b_{\max}}{b_{\min}}\right)$$

in practice g_{ff} is calibrated and tabulated

Thermal Bremsstrahlung

- Integrate over thermal distribution of electron velocities

$$n_e = \int \frac{d^3q}{(2\pi\hbar)^3} f_e \propto \int v^2 e^{-mv^2/2kT} dv$$

require that the kinetic energy be sufficient to emit a photon $\frac{1}{2}mv^2 > h\nu \to v_{\rm min}$ to get any radiation

- Qualitative: replace v with thermal velocity $\sim (kT/m)^{1/2}$

$$\frac{dW}{d\nu dV dt} \to \frac{\int_{v_{\rm min}}^{\infty} \frac{dW}{d\nu dV dt} v^2 e^{-mv^2/2kT} dv}{\int_0^{\infty} v^2 e^{-mv^2/2kT} dv}$$

$$= \frac{2^5 \pi Z^2 e^6}{3m^2c^3} \left(\frac{2\pi m}{3kT} \right)^{1/2} n_e n_i e^{-h\nu/kT} \bar{g}_{\rm ff}$$

where $\bar{g}_{\rm ff}$ is the velocity averaged Gaunt factor. Notice the spectrum is flat out to $h\nu \sim kT$

Thermal Emission and Absorption

- Emission coefficient

$$\frac{dW}{dt\,dV\,d\nu} = 4\pi j_\nu^{\rm ff}$$

$$j_\nu^{\rm ff} = \frac{2^3 Z^2 e^6}{3m^2c^3}\left(\frac{2\pi m}{3kT}\right)^{1/2} n_e n_i e^{-h\nu/kT}\bar{g}_{\rm ff}$$

- Kirchoff's law $S_\nu = j_\nu/\alpha_\nu$ gives the thermal absorption coefficient

$$j_\nu^{\rm ff} = \alpha_\nu^{\rm ff} B_\nu(T) = \alpha_\nu^{\rm ff}\frac{2h\nu^3/c^2}{e^{h\nu/kT}-1}$$

$$\alpha_\nu^{\rm ff} = \frac{4Z^2e^6}{3m^2ch}\left(\frac{2\pi m}{3kT}\right)^{1/2} n_e n_i \nu^{-3}(1-e^{-h\nu/kT})\bar{g}_{\rm ff}$$

- Note the steep spectrum in frequency: low frequency absorption is very efficient and yields a high optical depth $I_\nu \to S_\nu$

Radiative Transfer

- Alternately, non-relativistic bremsstrahlung can be characterized by a collision term like the Kompaneets equation ($k = \hbar = c = 1$, $x = h\nu/kT_e$)

$$C_{\rm ff}[f] = \sqrt{\frac{2}{\pi}} \left(\frac{T_e}{m} \right)^{-1/2} Z^2 \alpha T_e^{-3} n_i n_e \sigma_T g_{\rm ff} \frac{e^{-x}}{x^3} [1 - (e^x - 1) f]$$

note that emission and absorption is balanced only if $f = 1/(e^x - 1)$, a true blackbody (no chemical potential)

Galactic Bremsstrahung

- $H\alpha$ ($n = 3 \rightarrow 2$, hydrogen) as a tracer of ionized gas and hence bremsstrahlung emission in galaxy