Set 10:

Synchrotron

Lorentz Force

- Acceleration of electrons due to the magnetic field gives rise to synchrotron radiation
- Lorentz force

$$\frac{dP^\mu}{d\tau} = \frac{e}{c} F^\mu{}_\nu U^\nu \,, \qquad F^\mu{}_\nu = \begin{pmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}$$

$$\frac{d}{d\tau} \gamma m \begin{pmatrix} c \\ v_x \\ v_y \\ v_z \end{pmatrix} = \frac{e}{c} \begin{pmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix} \gamma \begin{pmatrix} c \\ v_x \\ v_y \\ v_z \end{pmatrix}$$

Lorentz Force

$$\frac{d}{d\tau}\gamma m \begin{pmatrix} c \\ v_x \\ v_y \\ v_z \end{pmatrix} = \frac{e}{c}\gamma \begin{pmatrix} \mathbf{E}\cdot\mathbf{v} \\ E_x c + B_z v_y - B_y v_z \\ E_y c - B_z v_x + B_x v_z \\ E_z c + B_y v_x - B_x v_y \end{pmatrix}$$

$$\frac{d}{d\tau}\gamma m \begin{pmatrix} c \\ \mathbf{v} \end{pmatrix} = \frac{e}{c}\gamma \begin{pmatrix} \mathbf{E}\cdot\mathbf{v} \\ \mathbf{E}c + \mathbf{v}\times\mathbf{B} \end{pmatrix} \qquad [d\tau = dt/\gamma]$$

$$\frac{d}{dt}\gamma m \begin{pmatrix} c \\ \mathbf{v} \end{pmatrix} = \frac{e}{c} \begin{pmatrix} \mathbf{E}\cdot\mathbf{v} \\ \mathbf{E}c + \mathbf{v}\times\mathbf{B} \end{pmatrix}$$

Energy-Momentum Equations

- Energy and momentum equations

$$\frac{d}{dt}\gamma mc^2 = e\mathbf{E}\cdot\mathbf{v}$$
$$\frac{d}{dt}\gamma m\mathbf{v} = e(\mathbf{E} + \frac{\mathbf{v}}{c}\times\mathbf{B})$$

- Consider a B field with no E field

$$\frac{d}{dt}\gamma mc^2 = 0, \qquad \frac{d}{dt}\gamma m\mathbf{v} = e\frac{\mathbf{v}}{c}\times\mathbf{B}$$

where the former ignores radiation energy losses

- Then the velocity is constant

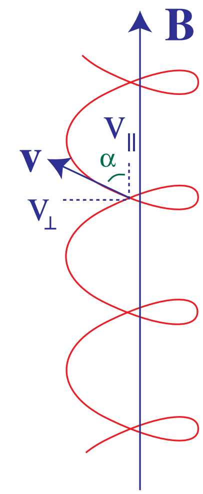
$$\frac{d\gamma}{dt} = 0 \quad \rightarrow \gamma = \text{const} \quad \rightarrow |v| = \text{const}$$

Cyclical Motion

- Define components $\mathbf{v}_{\|} \parallel \mathbf{B}$, $\mathbf{v}_{\perp} \perp \mathbf{B}$. Momentum equation becomes

$$\frac{d\mathbf{v}_{\|}}{dt} = 0 \qquad \frac{d\mathbf{v}_{\perp}}{dt} = \frac{e}{\gamma mc}\mathbf{v}_{\perp} \times \mathbf{B}$$

  so that $v_{\|}$ =const. $v^2 = v_{\|}^2 + v_{\perp}^2$ =const. and thus $v_{\perp}$ =const.



- Acceleration is orthogonal to $v_{\perp}$ with

$$|a_{\perp}| = \frac{e}{\gamma mc}|v_{\perp}|B$$

Power

- Equation of motion solved by

$$\mathbf{v}_\perp(t) = v_\perp \begin{pmatrix} \sin(\omega_B t + \delta) \\ \cos(\omega_B t + \delta) \end{pmatrix}, \qquad \omega_B = \frac{eB}{\gamma m c}$$

- Radiated power

$$P = \frac{2e^2}{3c^3}\gamma^4 a_\perp^2 = \frac{2}{3}\frac{e^4 v_\perp^2 B^2}{m^2 c^5}\gamma^2 = \frac{1}{4\pi}\frac{v_\perp^2 B^2}{c}\gamma^2 \sigma_T\,, \qquad \left[\sigma_T = \frac{8\pi}{3}\frac{e^4}{m^2 c^4}\right]$$

- Synchrotron emission can be viewed as Compton scattering off of virtual B photons. Defining the pitch angle α between $\mathbf{B}$ and $\mathbf{v}$,

$$v_\perp = v \sin\alpha \qquad \langle \sin^2\alpha \rangle = \int \sin^2\alpha \frac{d\Omega}{4\pi} = \frac{2}{3}$$

$$\langle P_{\rm synch} \rangle = \frac{1}{6\pi}\frac{v^2 B^2}{c}\gamma^2 \sigma_T$$

Virtual Photons

- The energy density associated with the magnetic field is $u_B = B^2/8\pi$

$$\langle P_{\rm synch}\rangle = \frac{4}{3}\sigma_T \frac{v^2}{c}\gamma^2 u_B = \frac{4}{3}\sigma_T c\beta^2\gamma^2 u_B$$

- Power in Compton scattering

$$\langle P_{\rm comp}\rangle = \frac{4}{3}\sigma_T c\beta^2\gamma^2 u_\gamma$$

so that synchrotron power= Compton power with incident energy density replaced by the magnetic field energy density

$$\frac{\langle P_{\rm synch}\rangle}{\langle P_{\rm comp}\rangle} = \frac{u_B}{u_\gamma}$$

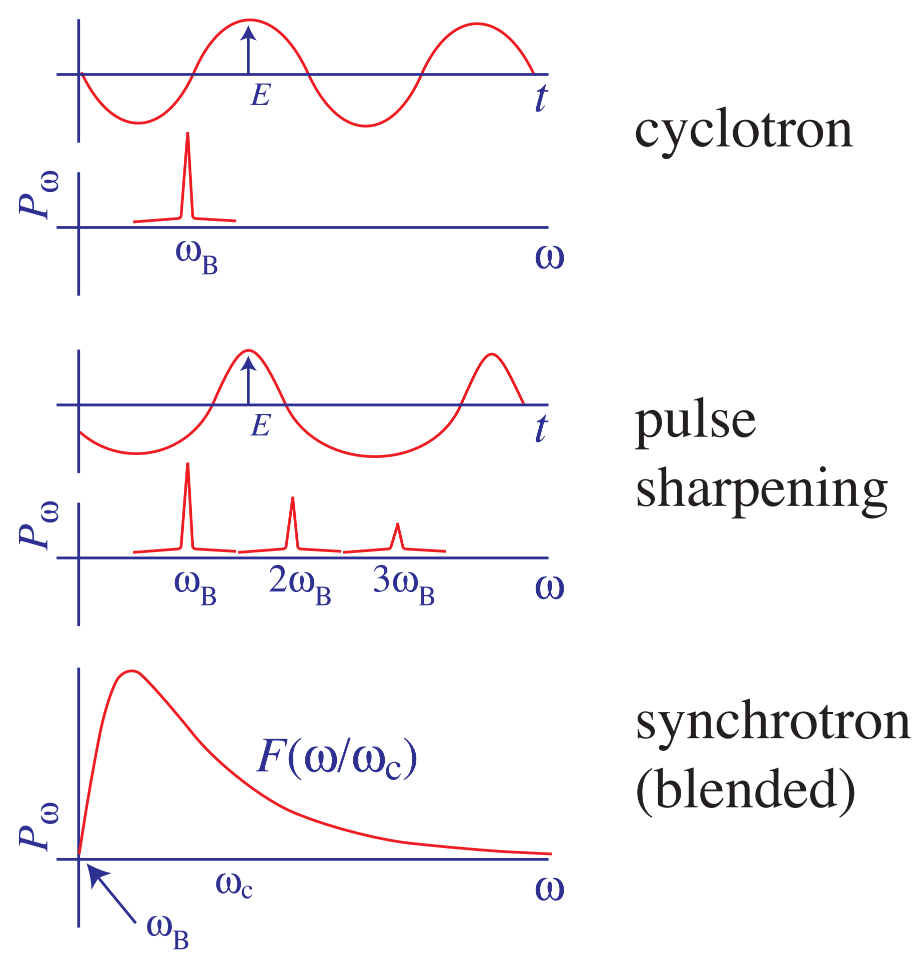
- Synchrotron radiation produces photons for inverse Compton scattering - removes energy from electrons - self regulation process

Frequency Spectrum

- The electron gyrates at a frequency

$$\omega_B = \frac{eB}{\gamma m c}$$

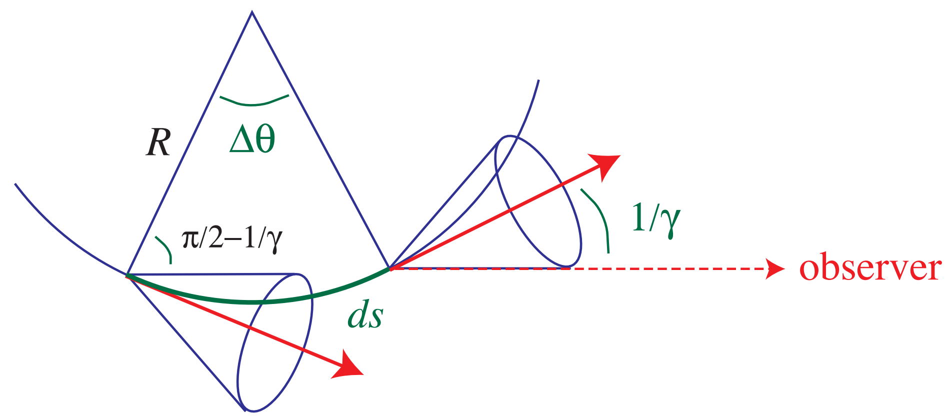
- Non relativistic, or cyclotron limit, given by the dipole approximation. The electric field of the radiation follows a sinusoid: the frequency structure is a near delta function at ω_B



Frequency Spectrum

- As the velocities become relativistic beaming sharpens the time profile while remaining periodic
- Appears as a series of delta functions at integer multiples of ω_B
- The beaming sets the width of the pulses and hence a cut off in the frequency spectrum of $\omega_c \sim 1/\Delta t$
- In the ultrarelativistic limit, the frequency content extends to $\omega_c \gg \omega_B$ and forms a continuum due to the range of γ's

Frequency Spectrum



- The profile width is given by beaming by determining the length of time the emission is observable. Take the ultra-relativistic limit where the radiation makes a cone of angle $\alpha \sim 1/\gamma$
- The observer first sees the radiation when the velocity (tangent to the spiral) is $1/\gamma$ from the line of sight and continues until it is $1/\gamma$ on the other side
- These two points and the center form an equilateral triangle

Frequency Cutoff

- The angle along the spiral traversed during the emission + 2 angles

$$\Delta\theta + 2\left(\frac{\pi}{2} - \frac{1}{\gamma}\right) = \pi$$
$$\Delta\theta = \frac{2}{\gamma}$$

- The arclength traversed given a gyration radius R combined with velocity gives the duration of emission

$$\Delta s = R\Delta\theta = v\Delta t_{\rm em} \quad \rightarrow \Delta t_{\rm em} = \frac{R\Delta\theta}{v} = \frac{2R}{\gamma v}$$

Frequency Cutoff

- Now eliminate the gyration radius R by expressing the angular speed as

$$\frac{\Delta\theta}{\Delta t_{\rm em}} = \frac{v}{R}$$

and extracting the angular speed from the centripetal acceleration

$$\gamma m \frac{\Delta \mathbf{v}}{\Delta t_{\rm em}} = \gamma m v \frac{\Delta\theta}{\Delta t_{\rm em}} = \frac{e}{c}\mathbf{v} \times \mathbf{B} = e\frac{v}{c} B \sin\alpha \qquad [\Delta v = v \Delta\theta]$$

Frequency Cutoff

- Reexpress using $\Delta\theta/\Delta t_{\rm em} = v/R$

$$
\begin{aligned}
\gamma m v^2 \frac{1}{R} &= e\frac{v}{c} B \sin\alpha \\
R &= \frac{\gamma m v c}{e B \sin\alpha} = \frac{v}{\omega_B \sin\alpha} \\
\Delta t_{\rm em} &= \frac{2}{\gamma \omega_B \sin\alpha}
\end{aligned}
$$

- The arrival time is further shortened in that light must travel across the path difference between beginning and end of the emission

$$
\Delta t_{\rm trav} = \frac{\Delta s}{c} = \frac{2}{\gamma c} \frac{v}{\omega_B \sin\alpha}
$$

Frequency Distribution

- The emission only beats the particle by a small amount leading to a smaller observed pulse duration

$$\Delta t = \Delta t_{\rm em} - \Delta t_{\rm trav} = \frac{2}{\gamma \omega_B \sin\alpha}\left(1 - \frac{v}{c}\right)$$

$$\Delta t \approx \frac{1}{\gamma^3 \omega_B \sin\alpha} = \frac{mc}{\gamma^2 eB \sin\alpha} \qquad \left[1 - \frac{v}{c} \approx \frac{1}{2\gamma^2}\right]$$

- Define a critical frequency

$$\omega_c = \frac{3}{2\Delta t} = \frac{3}{2}\gamma^2 \frac{eB}{mc}\sin\alpha$$

Frequency Spectrum

- Scaling relations: the frequency spectrum a function of ω/ω_c

$$P_\omega = CF(\omega/\omega_c)$$

and the total power is known

$$P = \frac{\sigma_T}{4\pi}\frac{v^2 B^2}{c}\gamma^2 \sin^2\alpha = \int_0^\infty P_\omega d\omega$$

- Solve for C up to an order unity coefficient $\int F(x)dx = N$

$$P = C\omega_c \int_0^\infty F(x)dx = CN\frac{3}{2}\gamma^2\frac{eB}{mc}\sin\alpha$$

$$C = \frac{\sigma_T}{4\pi}\frac{v^2 B^2}{c}\frac{\gamma^2\sin^2\alpha}{N\frac{3}{2}\gamma^2\frac{eB}{mc}\sin\alpha} = \frac{\sigma_T}{6\pi}\frac{v^2 B}{c}\frac{mc}{Ne}\sin\alpha$$

$$= \frac{4}{9}B\sin\alpha\frac{e^3}{mc^2}\frac{1}{N} \qquad \left[\sigma_T = \frac{8\pi}{3}\frac{e^4}{m^2c^4}\right]$$

Electron Distribution

- Put it together

$$P_\omega = \frac{4}{9} B \sin\alpha \frac{e^3}{mc^2} \frac{1}{N} F(\omega/\omega_c)$$

α =polar angle wrt **B**

- Power law electron distribution $n_{e,\gamma} \propto \gamma^{-p}$

$$j_\nu \propto \int d\gamma n_{e,\gamma} P_\omega \propto \int n_{e,\gamma} F(\omega/\omega_c) d\gamma \propto \int F(\omega/\omega_c) \gamma^{-p} d\gamma$$

- Transform to $x = \omega/\omega_c = \omega/A\gamma^2$

$$dx = -\frac{2\omega}{A\gamma^3} d\gamma \to \gamma^{-p} d\gamma = \frac{\gamma^{-p+3} A}{2\omega} dx = \left(\frac{\omega}{Ax}\right)^{(-p+3)/2} \frac{A}{2\omega} dx$$

$$j_\nu \propto \omega^{(-p+1)/2}$$

- Just like inverse Compton scattering, the spectrum of radiation has a power law ω^{-s} with $s = (p-1)/2$

Full Calculation

- A detailed integration of the orbits similar to bremsstrahlung yields

$$\frac{dW}{dtd\omega} \equiv P_\omega = \frac{\sqrt{3}}{2\pi}\frac{e^3 B \sin\alpha}{mc^2} F(\omega/\omega_c)$$

$$F(x) = x\int_x^\infty K_{5/3}(y)dy$$

with asymptotic behavior

$$\lim_{x\ll 1} F(x) = \frac{4\pi}{\sqrt{3}\Gamma(1/3)}\left(\frac{x}{2}\right)^{1/3}$$

$$\lim_{x\gg 1} F(x) = \left(\frac{\pi}{2}\right)^{1/2} e^{-x} x^{1/2}$$

and a maximum at $x = 0.29$

Polarization

- Polarization in direction perpendicular to $\mathbf{B}$ vs complement with a component parallel

$$\Pi = \frac{P_{\perp,\omega} - P_{\|,\omega}}{P_{\perp,\omega} + P_{\|,\omega}} = \frac{G(x)}{F(x)} \qquad G(x) = xK_{2/3}(x)$$

- The polarization is linear (when integrated over pitch angles) and integrated over frequencies for a single γ is 75%
- For a power law distribution of electron energies

$$\Pi = \frac{p+1}{p+\frac{7}{3}}$$

Radiative Transfer

- Can think of the emission and absorption as the sum over a discrete set of states or lines with $h\nu = E_2 - E_1$

$$j_\nu = \frac{h\nu}{4\pi} \sum_{E_2} \sum_{E_1} n_2 A_{21} \phi_{21}(\nu) = \frac{1}{4\pi} \sum_{E_2} P_\nu n_2 = \frac{1}{4\pi} \sum_{E_2} 2\pi P_\omega n_2$$

$$h\nu \sum_{E_1} A_{21} \phi_{21}(\nu) = 2\pi P_\omega$$

where $\phi_{21}(\nu)$ the profile enforces energy conservation $h\nu = E_2 - E_1$

$$\alpha_\nu = \frac{h\nu}{4\pi} \sum_{E_1} \sum_{E_2} [n_1 B_{12} - n_2 B_{21}] \phi_{21}(\nu)$$

$$\alpha_\nu = \frac{h\nu}{4\pi} \sum_{E_1} \sum_{E_2} [n_1 - n_2] \frac{c^2}{2h\nu^3} A_{21} \phi_{21}(\nu)$$

Synchrotron Self Absorption

- So that

$$\alpha_\nu = \frac{c^2}{4h\nu^3} \sum_{E_2} [n_1 - n_2] P_\omega$$

- Now switch to the continuous phase space distribution

$$\sum_E n \rightarrow g_e \int \frac{d^3q}{(2\pi\hbar)^3} f_e$$

$$\alpha_\nu = \frac{c^2}{4h\nu^3} g_e \int \frac{d^3q}{(2\pi\hbar)^3} [f_e(E - h\nu) - f_e(E)] P_\omega$$

$$j_\nu = \frac{1}{2} g_e \int \frac{d^3q}{(2\pi\hbar)^3} f_e P_\omega = \frac{1}{4\pi} g_e \int \frac{d^3q}{(2\pi\hbar)^3} f_e P_\nu$$

Synchrotron Self Absorption

- Check: for a thermal distribution $f_e = e^{-(E-\mu)/kT}$ then

$$f_e(E - h\nu) - f_e(E) = e^{-(E-\mu)/kT}(e^{h\nu/kT} - 1)$$

$$\frac{j_\nu}{\alpha_\nu} = \frac{c^2}{2h\nu^3}\frac{1}{e^{h\nu/kT} - 1} = B_\nu$$

- In the limit that $h\nu \ll E$ then

$$\alpha_\nu \approx -\frac{c^2}{4\nu^2} g_e \int \frac{d^3q}{(2\pi\hbar)^3}\frac{\partial f_e}{\partial E} P_\omega$$

- For a power law distribution of relativistic electrons

$$n_e = \int d\gamma n_{e,\gamma} = g_e \int \frac{d^3q}{(2\pi\hbar)^3} f_e = g_e \int \frac{q^2 dq}{2\pi^2\hbar^3} f_e \propto \int \gamma^2 d\gamma f_e$$

- Thus $f_e \propto \gamma^{-2} n_{e,\gamma}$

Synchrotron Self Absorption

- So that for a power law $n_{e,\gamma} \propto \gamma^{-p}$ and $\partial f_e/\partial\gamma \propto \gamma^{-3-p}$
- Combine with phase space integration
  $q^2 dq \propto \gamma^2 d\gamma$ and $P_\omega \propto F(x)$ and $x = \omega/\omega_c = \omega/A\gamma^2$

$$\alpha_\nu \propto \nu^{-2} \int d\gamma \gamma^{-p-1} F(\omega/\omega_c) \propto \nu^{-p/2-2}$$

- Compare to emission integral

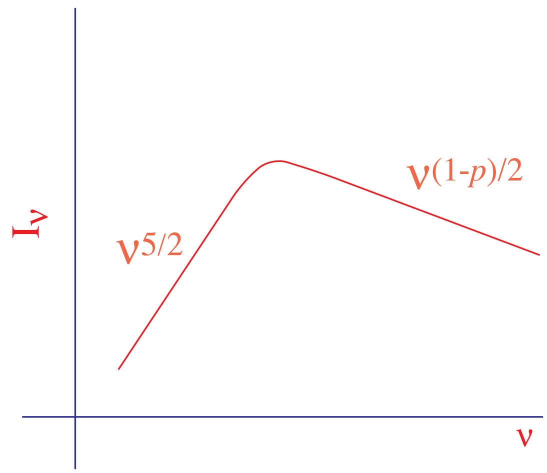
$$j_\nu \propto \int n_{e,\gamma} F(\omega/\omega_c) d\gamma \propto \int F(\omega/\omega_c)\gamma^{-p} d\gamma \propto \nu^{-p/2+1/2},$$

- Source function has extra $\nu^{5/2}$: ν^2 from absorption formula and $\nu^{1/2}$ from extra γ

$$S_\nu = \frac{j_\nu}{\alpha_\nu} \propto \nu^{5/2}$$

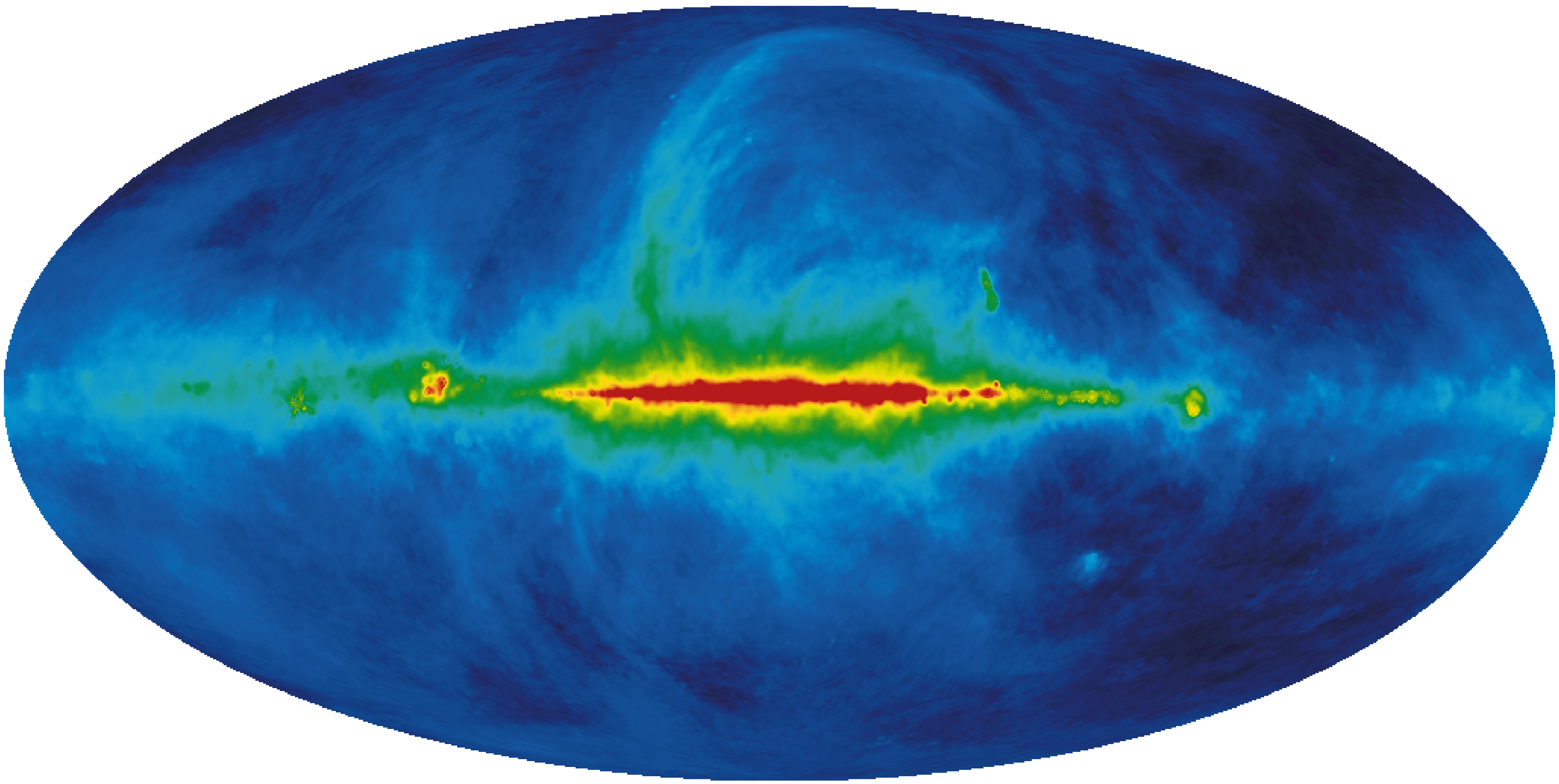
Synchrotron Self Absorption

- At high synchrotron absorption optical depth (low frequencies) $I_\nu \rightarrow S_\nu \propto \nu^{5/2}$
- At low optical depth $I_\nu \propto j_\nu \propto \nu^{-(p-1)/2}$
- The synchrotron spectrum has a characteristic turn over between the regimes



Galactic Synchrotron

- HASLAM 408MHz synchrotron map



Galactic Synchrotron

- WMAP 23GHz emission and polarization map