Set 10:
Synchrotron
Lorentz Force

- Acceleration of electrons due to the magnetic field gives rise to synchrotron radiation
- Lorentz force

\[
\frac{dP^\mu}{d\tau} = \frac{e}{c} F^\mu_\nu U^\nu, \quad F^\mu_\nu = \begin{pmatrix}
0 & E_x & E_y & E_z \\
E_x & 0 & B_z & -B_y \\
E_y & -B_z & 0 & B_x \\
E_z & B_y & -B_x & 0
\end{pmatrix}
\]

\[
\frac{d}{d\tau} \gamma m \begin{pmatrix}
c \\
v_x \\
v_y \\
v_z \\
\end{pmatrix} = \frac{e}{c} \begin{pmatrix}
0 & E_x & E_y & E_z \\
E_x & 0 & B_z & -B_y \\
E_y & -B_z & 0 & B_x \\
E_z & B_y & -B_x & 0
\end{pmatrix} \begin{pmatrix}
c \\
v_x \\
v_y \\
v_z \\
\end{pmatrix}
\]
Lorentz Force

\[
\frac{d}{d\tau} \gamma m \left( \begin{array}{c} c \\ v_x \\ v_y \\ v_z \end{array} \right) = \frac{e}{c} \left( \begin{array}{c} E \cdot v \\ E_x c + B_z v_y - B_y v_z \\ E_y c - B_z v_x + B_x v_z \\ E_z c + B_y v_x - B_x v_y \end{array} \right)
\]

\[
\frac{d}{d\tau} \gamma m \left( \begin{array}{c} c \\ v \end{array} \right) = \frac{e}{c} \left( \begin{array}{c} E \cdot v \\ E c + v \times B \end{array} \right)
\]

\[
\frac{d}{dt} \gamma m \left( \begin{array}{c} c \\ v \end{array} \right) = \frac{e}{c} \left( \begin{array}{c} E \cdot v \\ E c + v \times B \end{array} \right)
\]

\[d\tau = dt/\gamma\]
Energy-Momentum Equations

- Energy and momentum equations

\[ \frac{d}{dt} \gamma mc^2 = e \mathbf{E} \cdot \mathbf{v} \]
\[ \frac{d}{dt} \gamma m \mathbf{v} = e (\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}) \]

- Consider a $B$ field with no $E$ field

\[ \frac{d}{dt} \gamma mc^2 = 0, \quad \frac{d}{dt} \gamma m \mathbf{v} = e \frac{\mathbf{v}}{c} \times \mathbf{B} \]

where the former ignores radiation energy losses

- Then the velocity is constant

\[ \frac{d\gamma}{dt} = 0 \quad \rightarrow \quad \gamma = \text{const} \quad \rightarrow \quad |v| = \text{const} \]
Cyclical Motion

- Define components $v_\parallel \parallel B$, $v_\perp \perp B$. Momentum equation becomes

$$\frac{dv_\parallel}{dt} = 0 \quad \frac{dv_\perp}{dt} = \frac{e}{\gamma mc} v_\perp \times B$$

so that $v_\parallel = \text{const.}$ $v^2 = v_\parallel^2 + v_\perp^2 = \text{const.}$ and thus $v_\perp = \text{const.}$

- Acceleration is orthogonal to $v_\perp$ with

$$|a_\perp| = \frac{e}{\gamma mc} |v_\perp| B$$
Power

- Equation of motion solved by

\[ v_\perp(t) = v_\perp \begin{pmatrix} \sin(\omega_B t + \delta) \\ \cos(\omega_B t + \delta) \end{pmatrix}, \quad \omega_B = \frac{eB}{\gamma mc} \]

- Radiated power

\[
P = \frac{2e^2}{3c^3} \gamma^4 a_\perp = \frac{2}{3} \frac{e^4 v_\perp^2 B^2}{m^2 c^5} \gamma^2 = \frac{1}{4\pi} \frac{v_\perp^2 B^2}{c} \gamma^2 \sigma_T, \quad \left[ \sigma_T = \frac{8\pi}{3} \frac{e^4}{m^2 c^4} \right]
\]

- Synchrotron emission can be viewed as Compton scattering off of virtual $B$ photons. Defining the pitch angle $\alpha$ between $B$ and $v$,

\[ v_\perp = v \sin \alpha \quad \langle \sin^2 \alpha \rangle = \int \sin^2 \alpha \frac{d\Omega}{4\pi} = \frac{2}{3} \]

\[ \langle P_{\text{synch}} \rangle = \frac{1}{6\pi} \frac{v^2 B^2}{c} \gamma^2 \sigma_T \]
Virtual Photons

- The energy density associated with the magnetic field is
  \[ u_B = \frac{B^2}{8\pi} \]

\[ \langle P_{\text{synch}} \rangle = \frac{4}{3} \sigma_T \frac{v^2}{c} \gamma^2 u_B = \frac{4}{3} \sigma_T c \beta^2 \gamma^2 u_B \]

- Power in Compton scattering

\[ \langle P_{\text{comp}} \rangle = \frac{4}{3} \sigma_T c \beta^2 \gamma^2 u_\gamma \]

so that synchrotron power = Compton power with incident energy density replaced by the magnetic field energy density

\[ \frac{\langle P_{\text{synch}} \rangle}{\langle P_{\text{comp}} \rangle} = \frac{u_B}{u_\gamma} \]

- Synchrotron radiation produces photons for inverse Compton scattering - removes energy from electrons - self regulation process
The electron gyrates at a frequency
\[ \omega_B = \frac{eB}{\gamma mc} \]

Non relativistic, or cyclotron limit, given by the dipole approximation. The electric field of the radiation follows a sinusoid: the frequency structure is a near delta function at \( \omega_B \).
**Frequency Spectrum**

- As the velocities become relativistic beaming sharpens the time profile while remaining periodic.
- Appears as a series of delta functions at integer multiples of $\omega_B$.
- The beaming sets the width of the pulses and hence a cut off in the frequency spectrum of $\omega_c \sim 1/\Delta t$.
- In the ultrarelativistic limit, the frequency content extends to $\omega_c \gg \omega_B$ and forms a continuum due to the range of $\gamma$’s.
• The profile width is given by beaming by determining the length of time the emission is observable. Take the ultra-relativistic limit where the radiation makes a cone of angle $\alpha \sim 1/\gamma$

• The observer first sees the radiation when the velocity (tangent to the spiral) is $1/\gamma$ from the line of sight and continues until it is $1/\gamma$ on the other side

• These two points and the center form an equilateral triangle
Frequency Cutoff

- The angle along the spiral traversed during the emission + 2 angles

\[ \Delta \theta + 2 \left( \frac{\pi}{2} - \frac{1}{\gamma} \right) = \pi \]

\[ \Delta \theta = \frac{2}{\gamma} \]

- The arclength traversed given a gyration radius \( R \) combined with velocity gives the duration of emission

\[ \Delta s = R \Delta \theta = v \Delta t_{em} \quad \rightarrow \quad \Delta t_{em} = \frac{R \Delta \theta}{v} = \frac{2R}{\gamma v} \]
• Now eliminate the gyration radius $R$ by expressing the angular speed as

$$\frac{\Delta \theta}{\Delta t_{em}} = \frac{v}{R}$$

and extracting the angular speed from the centripetal acceleration

$$\gamma m \frac{\Delta v}{\Delta t_{em}} = \gamma m v \frac{\Delta \theta}{\Delta t_{em}} = \frac{e}{c} v \times B = \frac{e}{c} v B \sin \alpha \quad [\Delta v = v \Delta \theta]$$
Frequency Cutoff

- Reexpress using $\Delta \theta / \Delta t_{em} = v / R$

$$
\gamma mv^2 \frac{1}{R} = e \frac{v}{c} B \sin \alpha
$$

$$
R = \frac{\gamma mvc}{eB \sin \alpha} = \frac{v}{\omega_B \sin \alpha}
$$

$$
\Delta t_{em} = \frac{2}{\gamma \omega_B \sin \alpha}
$$

- The arrival time is further shortened in that light must travel across the path difference between beginning and end of the emission

$$
\Delta t_{trav} = \frac{\Delta s}{c} = \frac{2}{\gamma c \omega_B \sin \alpha} \frac{v}{\omega_B \sin \alpha}
$$
The emission only beats the particle by a small amount leading to a smaller observed pulse duration

\[ \Delta t = \Delta t_{\text{em}} - \Delta t_{\text{trav}} = \frac{2}{\gamma \omega_B \sin \alpha} \left( 1 - \frac{v}{c} \right) \]

\[ \Delta t \approx \frac{1}{\gamma^3 \omega_B \sin \alpha} = \frac{mc}{\gamma^2 eB \sin \alpha} \left[ 1 - \frac{v}{c} \approx \frac{1}{2\gamma^2} \right] \]

Define a critical frequency

\[ \omega_c = \frac{3}{2\Delta t} = \frac{3}{2} \gamma^2 \frac{eB}{mc} \sin \alpha \]
Frequency Spectrum

- Scaling relations: the frequency spectrum a function of $\omega/\omega_c$

$$P_\omega = CF(\omega/\omega_c)$$

and the total power is known

$$P = \frac{\sigma_T v^2 B^2}{4\pi} \frac{\gamma^2 \sin^2 \alpha}{c} = \int_0^\infty P_\omega d\omega$$

- Solve for $C$ up to an order unity coefficient $\int F(x)dx = N$

$$P = C\omega_c \int_0^\infty F(x)dx = CN\frac{3}{2} \gamma^2 \frac{eB}{mc} \sin \alpha$$

$$C = \frac{\sigma_T v^2 B^2}{4\pi} \frac{\gamma^2 \sin^2 \alpha}{c} \frac{N}{2} \frac{2\gamma eB}{mc} \sin \alpha = \frac{\sigma_T v^2 B \sin \alpha}{6\pi} \frac{mc}{N}$$

$$= \frac{4}{9} B \sin \alpha \frac{e^3}{mc^2} \frac{1}{N} \left[ \sigma_T = \frac{8\pi}{3} \frac{e^4}{m^2c^4} \right]$$
Electron Distribution

- Put it together

\[ P_\omega = \frac{4}{9} B \sin \alpha \frac{e^3}{mc^2} \frac{1}{N} F(\omega/\omega_c) \quad \alpha = \text{polar angle wrt } B \]

- Power law electron distribution \( n_{e,\gamma} \propto \gamma^{-p} \)

\[ j_\nu \propto \int d\gamma n_{e,\gamma} P_\omega \propto \int n_{e,\gamma} F(\omega/\omega_c) d\gamma \propto \int F(\omega/\omega_c) \gamma^{-p} d\gamma \]

- Transform to \( x = \omega/\omega_c = \omega/A\gamma^2 \)

\[ dx = -\frac{2\omega}{A\gamma^3} d\gamma \rightarrow \gamma^{-p} d\gamma = \frac{\gamma^{-p+3} A}{2\omega} dx = \left( \frac{\omega}{Ax} \right)^{(-p+3)/2} \frac{A}{2\omega} dx \]

\[ j_\nu \propto \omega^{(-p+1)/2} \]

- Just like inverse Compton scattering, the spectrum of radiation has a power law \( \omega^{-s} \) with \( s = (p - 1)/2 \)
Full Calculation

- A detailed integration of the orbits similar to bremsstrahlung yields

\[
\frac{dW}{dt\,d\omega} \equiv P_\omega = \frac{\sqrt{3} \, e^3 B \sin \alpha}{2\pi \, mc^2} F(\omega/\omega_c)
\]

\[
F(x) = x \int_x^\infty K_{5/3}(y) \, dy
\]

with asymptotic behavior

\[
\lim_{x \ll 1} F(x) = \frac{4\pi}{\sqrt{3}\Gamma(1/3)} \left( \frac{x}{2} \right)^{1/3}
\]

\[
\lim_{x \gg 1} F(x) = \left( \frac{\pi}{2} \right)^{1/2} e^{-x} x^{1/2}
\]

and a maximum at \( x = 0.29 \)
Polarization

- Polarization in direction perpendicular to $\mathbf{B}$ vs complement with a component parallel

$$\Pi = \frac{P_{\perp,\omega} - P_{\parallel,\omega}}{P_{\perp,\omega} + P_{\parallel,\omega}} = \frac{G(x)}{F(x)}$$

$$G(x) = xK_{2/3}(x)$$

- The polarization is linear (when integrated over pitch angles) and integrated over frequencies for a single $\gamma$ is 75%

- For a power law distribution of electron energies

$$\Pi = \frac{p + 1}{p + \frac{7}{3}}$$
Radiative Transfer

- Can think of the emission and absorption as the sum over a discrete set of states or lines with $h\nu = E_2 - E_1$

$$j_\nu = \frac{h\nu}{4\pi} \sum_{E_2} \sum_{E_1} n_2 A_{21} \phi_{21}(\nu) = \frac{1}{4\pi} \sum_{E_2} P_\nu n_2 = \frac{1}{4\pi} \sum_{E_2} 2\pi P_\omega n_2$$

$$h\nu \sum_{E_1} A_{21} \phi_{21}(\nu) = 2\pi P_\omega$$

where $\phi_{21}(\nu)$ the profile enforces energy conservation

$h\nu = E_2 - E_1$

$$\alpha_\nu = \frac{h\nu}{4\pi} \sum_{E_1} \sum_{E_2} [n_1 B_{12} - n_2 B_{21}] \phi_{21}(\nu)$$

$$\alpha_\nu = \frac{h\nu}{4\pi} \sum_{E_1} \sum_{E_2} [n_1 - n_2] \frac{c^2}{2h\nu^3} A_{21} \phi_{21}(\nu)$$
Synchrotron Self Absorption

• So that

\[ \alpha_\nu = \frac{c^2}{4h\nu^3} \sum_{E_2} [n_1 - n_2] P_\omega \]

• Now switch to the continuous phase space distribution

\[ \sum_{E} n \rightarrow g_e \int \frac{d^3 q}{(2\pi \hbar)^3} f_e \]

\[ \alpha_\nu = \frac{c^2}{4h\nu^3} g_e \int \frac{d^3 q}{(2\pi \hbar)^3} [f_e(E - h\nu) - f_e(E)] P_\omega \]

\[ j_\nu = \frac{1}{2} g_e \int \frac{d^3 q}{(2\pi \hbar)^3} f_e P_\omega = \frac{1}{4\pi} g_e \int \frac{d^3 q}{(2\pi \hbar)^3} f_e P_\nu \]
Synchrotron Self Absorption

- Check: for a thermal distribution $f_e = e^{-(E-\mu)/kT}$ then

$$f_e(E - h\nu) - f_e(E) = e^{-(E-\mu)/kT}(e^{h\nu/kT} - 1)$$

$$\frac{j_\nu}{\alpha_\nu} = \frac{c^2}{2h\nu^3} \sqrt{\frac{1}{e^{h\nu/kT} - 1}} = B_\nu$$

- In the limit that $h\nu \ll E$ then

$$\alpha_\nu \approx -\frac{c^2}{4\nu^2} g_e \int \frac{d^3q}{(2\pi \hbar)^3} \frac{\partial f_e}{\partial E} P_\omega$$

- For a power law distribution of relativistic electrons

$$n_e = \int d\gamma n_{e,\gamma} = g_e \int \frac{d^3q}{(2\pi \hbar)^3} f_e = g_e \int \frac{q^2 dq}{2\pi^2 \hbar^3} f_e \propto \int \gamma^2 d\gamma f_e$$

- Thus $f_e \propto \gamma^{-2} n_{e,\gamma}$
Synchrotron Self Absorption

- So that for a power law $n_{e,\gamma} \propto \gamma^{-p}$ and $\partial f_{e}/\partial \gamma \propto \gamma^{-3-p}$

- Combine with phase space integration

$$q^2 dq \propto \gamma^2 d\gamma \text{ and } P_\omega \propto F(x) \text{ and } x = \omega/\omega_c = \omega/A\gamma^2$$

$$\alpha_\nu \propto \nu^{-2} \int d\gamma \gamma^{-p-1} F(\omega/\omega_c) \propto \nu^{-p/2-2}$$

- Compare to emission integral

$$j_\nu \propto \int n_{e,\gamma} F(\omega/\omega_c) d\gamma \propto \int F(\omega/\omega_c) \gamma^{-p} d\gamma \propto \nu^{-p/2+1/2},$$

- Source function has extra $\nu^{5/2}$: $\nu^2$ from absorption formula and $\nu^{1/2}$ from extra $\gamma$

$$S_\nu = \frac{j_\nu}{\alpha_\nu} \propto \nu^{5/2}$$
Synchrotron Self Absorption

- At high synchrotron absorption optical depth (low frequencies) $I_\nu \rightarrow S_\nu \propto \nu^{5/2}$
- At low optical depth $I_\nu \propto j_\nu \propto \nu^{-(p-1)/2}$
- The synchrotron spectrum has a characteristic turn over between the regimes
Galactic Synchrotron

- HASLAM 408MHz synchrotron map
Galactic Synchrotron

- WMAP 23GHz emission and polarization map