Astro 321

Lecture Notes Set 2

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Distribution Function

- The distribution function f gives the number of particles per unit phase space $d^3x d^3q$ where q is the momentum (conventional to work in physical coordinates)
- Consider a box of volume $V = L^3$. Periodicity implies that the allowed momentum states are given by $q_i = n_i 2\pi/L$ so that the density of states is

$$dN_s = g\frac{V}{(2\pi)^3}d^3q$$

  where g is the degeneracy factor (spin/polarization states)
- The distribution function $f(\mathbf{x}, \mathbf{q}, t)$ describes the particle occupancy of these states, i.e.

$$N = \int dN_s f = gV \int \frac{d^3q}{(2\pi)^3} f$$

Bulk Properties

- Integrals over the distribution function define the bulk properties of the collection of particles
- Number density

$$n(\mathbf{x}, t) \equiv N/V = g \int \frac{d^3q}{(2\pi)^3} f$$

- Energy density

$$\rho(\mathbf{x}, t) = g \int \frac{d^3q}{(2\pi)^3} E(q) f$$

where $E^2 = q^2 + m^2$

Bulk Properties

- Pressure: particles bouncing off a surface of area A in a volume spanned by L_x: per momentum state

$$p_q = \frac{F}{A} = \frac{N_{\rm part}}{A}\frac{\Delta q}{\Delta t}$$

$$(\Delta q = 2|q_x|, \quad \Delta t = 2L_x/v_x)$$

$$= \frac{N_{\rm part}}{V}|q_x||v_x| = f\frac{|q||v|}{3} = f\frac{q^2}{3E}$$

so that summed over states

$$p(\mathbf{x},t) = g\int \frac{d^3q}{(2\pi)^3}\frac{|q|^2}{3E(q)} f$$

- Likewise anisotropic stress (vanishes in the background)

$$\pi^i{}_j(\mathbf{x},t) = g\int \frac{d^3q}{(2\pi)^3}\frac{3q^iq_j - q^2\delta^i{}_j}{3E(q)} f$$

Observable Properties

- Only get to measure luminous properties of the universe. For photons mass $m = 0$, $g = 2$ (units: $J\ m^{-3}$)

$$\rho(\mathbf{x}, t) = 2 \int \frac{d^3q}{(2\pi)^3} q f = 2 \int dq d\Omega \left(\frac{q}{2\pi} \right)^3 f$$

- Spectral energy density (per unit frequency $q = h\nu = \hbar 2\pi\nu = 2\pi\nu$, solid angle)

$$u_\nu = \frac{d\rho}{d\nu d\Omega} = 2(2\pi)\nu^3 f$$

- Photons travelling at speed of light so that $u_\nu = I_\nu = 4\pi\nu^3 f$ the specific intensity or brightness, energy flux across a surface, units of W m^{-2} Hz^{-1} sr^{-1}

Observable Properties

- Integrate over frequencies for total intensity

$$I = \int d\nu I_\nu = \int d\ln\nu\, \nu I_\nu$$

νI_ν often plotted since it shows peak under a log plot; I and νI_ν have units of W m^{-2} sr^{-1} and is independent of choice of frequency unit

- Flux density: integrate over the solid angle of a radiation source, units of W m^{-2} Hz^{-1} or Jansky = 10^{-26} W m^{-2} Hz^{-1}

$$S_\nu = \int_{\rm source} I_\nu d\Omega$$

a.k.a. spectral energy distribution

Observable Properties

- Flux integrate over frequency, units of W m^{-2}

$$S = \int d\ln\nu\, \nu S_\nu$$

- Flux in a frequency band S_b measured in terms of magnitudes (optical), set to some standard zero point per band

$$m_b - m_{\rm norm} = 2.5\log_{10}(S_{\rm norm}/S_b) \approx \ln(S_{\rm norm}/S_b)$$

- Luminosity: integrate over area assuming isotropic emission or beaming factor, units of W

$$L = 4\pi d_L^2 S$$

Extragalactic Light

- Looking at background radiation νI_ν peaks in the microwave mm-cm region, and has the distribution of a perfect black body $f = 1/(e^{q/T} - 1), T = 2.725 \pm 0.002K$ or $n_\gamma = 410$ cm^{-3}, $\Omega_\gamma = 2.47 \times 10^{-5} h^{-2}$. This is the cosmic microwave background.
- Strong support for hot big bang – densities high enough so that interactions can create a thermal distribution of photons that has since redshifted into the microwave

Liouville Equation

- Liouville theorem states that the phase space distribution function is conserved along a trajectory in the absence of particle interactions

$$\frac{Df}{Dt} = \left[\frac{\partial}{\partial t} + \frac{d\mathbf{q}}{dt}\frac{\partial}{\partial \mathbf{q}} + \frac{d\mathbf{x}}{dt}\frac{\partial}{\partial \mathbf{x}}\right] f = 0$$

subtlety in expanding universe is that the de Broglie wavelength of particles changes with the expansion so that

$$q \propto a^{-1}$$

- Homogeneous and isotropic limit

$$\frac{\partial f}{\partial t} + \frac{dq}{dt}\frac{\partial f}{\partial q} = \frac{\partial f}{\partial t} - H(a)\frac{\partial f}{\partial \ln q} = 0$$

Energy Density Evolution

- Integrate Liouville equation over $g\int d^3q/(2\pi)^3 E$ to form

$$\begin{aligned}
\frac{\partial\rho}{\partial t} &= H(a)g\int\frac{d^3q}{(2\pi)^3}Eq\frac{\partial}{\partial q}f \\
&= H(a)g\int\frac{d\Omega}{(2\pi)^3}\int dq q^3 E\frac{\partial}{\partial q}f \\
&= -H(a)g\int\frac{d\Omega}{(2\pi)^3}\int dq\frac{d(q^3E)}{dq}f \\
&= -H(a)g\int\frac{d\Omega}{(2\pi)^3}\int dq(3q^2E + q^3\frac{dE}{dq})f \\
&\quad (\frac{dE}{dq} = \frac{d(q^2+m^2)^{1/2}}{dq} = \frac{1}{2}\frac{2q}{E} = \frac{q}{E}) \\
&= -3H(a)g\int\frac{d^3q}{(2\pi)^3}(E + \frac{q^2}{3E})f = -3H(a)(\rho + p)
\end{aligned}$$

as derived previously from energy conservation

Boltzmann Equation

- Boltzmann equation says that Liouville theorem must be modified to account for collisions

$$\frac{Df}{Dt} = C[f]$$

- If collisions are sufficiently rapid, distribution will tend to thermal equilibrium form

Kompaneets Example

- Collision term for photons under Compton scattering with free electrons $\gamma' + e^{-\prime} \rightarrow \gamma + e^{-}$

$$C[f] = \frac{1}{2E(q)} \int Dq_e Dq_e' Dq' (2\pi)^4 \delta^{(4)}(q + q_e - q' - q_e')$$
$$[f_e(q_e')f(q')(1 + f(q)) - f_e(q_e)f(q)(1 + f(q'))]|M|^2$$

where stimulated emission included, Pauli blocking neglected, Lorentz invariant phase space element

$$Dq = \frac{d^3q}{(2\pi)^3} \frac{1}{2E(q)}$$

and the matrix element for scattering through an angle β in the electron rest frame, averaged over polarization states, is

$$|M|^2 = 2(4\pi)^2\alpha^2 \left[\frac{q'}{q} + \frac{q}{q'} - \sin^2\beta\right]$$

Kompaneets Example

- Thermalization of photons in the presence of a “bath” of electrons at temperature T_e (Maxwell-Boltzmann distributed electrons)

$$C[f] = \frac{d\tau}{dt}\frac{1}{m_e q^2}\frac{\partial}{\partial q}\left[q^4\left(T_e\frac{\partial f}{\partial q} + f(1+f)\right)\right]$$

where the scattering rate is given by

$$\frac{d\tau}{dt} = x_e n_e \sigma_T \qquad \sigma_T = \frac{8\pi\alpha^2}{3m_e^2} = 6.65\times 10^{-25}\mathrm{cm}^{-2}$$

- From $\partial f/\partial t = C[f]$, can check that particle number is conserved: $\partial n/\partial t = 0$
- Setting $C[f_{\rm eq}] = 0$ returns a diff. eq. solved by the (equilbrium or Bose-Einstein) distribution

$$f_{\rm eq} = \frac{1}{e^{(q-\mu)/T_e} - 1}$$

Kompaneets Example

- Verify

$$\frac{\partial f_{\rm eq}}{\partial q/T_e} = -\frac{e^{(q-\mu)/T_e}}{[e^{(q-\mu)/T_e}-1]^2}$$
$$= -f_{\rm eq}\frac{e^{(q-\mu)/T_e}}{e^{(q-\mu)/T_e}-1}$$
$$= -f_{\rm eq}(1+f_{\rm eq})$$

- μ is the chemical potential; from number density integral we see that it represents a way of changing number density at equilibrium - i.e. unavoidable if particle number is conserved in the collisional process
- The equilibrium distribution comes about through general considerations of statistical equilibrium.

Poor Man's Boltzmann Equation

- Non expanding medium

$$\frac{\partial f}{\partial t} = \Gamma\left(f - f_{\rm eq}\right)$$

where Γ is some rate for collisions

- Add in expansion in a homogeneous medium

$$\frac{\partial f}{\partial t} + \frac{dq}{dt}\frac{\partial f}{\partial q} = \Gamma\left(f - f_{\rm eq}\right)$$

$$\left(q \propto a^{-1} \rightarrow \frac{1}{q}\frac{dq}{dt} = -\frac{1}{a}\frac{da}{dt} = H\right)$$

$$\frac{\partial f}{\partial t} - H\frac{\partial f}{\partial \ln q} = \Gamma\left(f - f_{\rm eq}\right)$$

- So equilibrium will be maintained if collision rate exceeds expansion rate $\Gamma > H$

Thermodynamic Equilibrium

- Consider a gas of particles in thermal and diffusive contact with a reservoir of temperature T. Then the relative probability of being in a state with energy E_i and particle number N_i is given by the Gibbs factor (μ= chemical potential, non-vanishing even in equilibrium if collisions do not change particle number)

$$P(E_i, N_i) \propto \exp[-(E_i - \mu N_i)/T]$$

- The mean occupation of the state defines the distribution function

$$f \equiv \frac{\sum N_i P(E_i, N_i)}{\sum P(E_i, N_i)}$$

- The energy, allowing for a zero point, is $E_i = (N_i + 1/2)E$ where E is the particle energy.

Bose-Einstein / Fermi-Dirac

- For fermions, occupation is either 0 or 1

$$f = \frac{\exp[-(E-\mu)/T]}{1+\exp[-(E-\mu)/T]} = \frac{1}{\exp[(E-\mu)/T]+1}$$

- For bosons, infinite sum gives

$$f = \frac{1}{\exp[(E-\mu)/T]-1}$$

- For the nonrelativistic limit $E = m + \frac{1}{2}q^2/m$, $E/T \gg 1$ so both distributions go to the Maxwell-Boltzmann distribution

$$f = \exp[-(m-\mu)/T]\exp(-q^2/2mT)$$

Non-Relativistic Bulk Properties

- Number density

$$
\begin{aligned}
n &= g e^{-(m-\mu)/T} \frac{4\pi}{(2\pi)^3} \int_0^\infty q^2 dq \exp(-q^2/2mT) \\
&= g e^{-(m-\mu)/T} \frac{2^{3/2}}{2\pi^2} (mT)^{3/2} \int_0^\infty x^2 dx \exp(-x^2) \\
&= g (\frac{mT}{2\pi})^{3/2} e^{-(m-\mu)/T}
\end{aligned}
$$

- Energy density $E = m \rightarrow \rho = mn$
- Pressure $q^2/3E = q^2/3m \rightarrow p = nT$, ideal gas law

Ultra-Relativistic Bulk Properties

- Chemical potential $\mu = 0$, $\zeta(3) \approx 1.202$
- Number density

$$n_{\text{boson}} = gT^3 \frac{\zeta(3)}{\pi^2} \qquad \zeta(n+1) \equiv \frac{1}{n!} \int_0^\infty \frac{x^n}{e^x - 1}$$

$$n_{\text{fermion}} = \frac{3}{4} gT^3 \frac{\zeta(3)}{\pi^2}$$

- Energy density

$$\rho_{\text{boson}} = gT^4 \frac{3}{\pi^2} \zeta(4) = gT^4 \frac{\pi^2}{30}$$

$$\rho_{\text{fermion}} = \frac{7}{8} gT^4 \frac{3}{\pi^2} \zeta(4) = \frac{7}{8} gT^4 \frac{\pi^2}{30}$$

- Pressure $q^2/3E = E/3 \rightarrow p = \rho/3$, $w_r = 1/3$

Entropy Density

- First law of thermodynamics

$$dS = \frac{1}{T}(d\rho(T)V + p(T)dV)$$

so that

$$\left.\frac{\partial S}{\partial V}\right|_T = \frac{1}{T}[\rho(T) + p(T)]$$

$$\left.\frac{\partial S}{\partial T}\right|_V = \frac{V}{T}\frac{d\rho}{dT}$$

- Since $S(V,T) \propto V$ is extensive

$$S = \frac{V}{T}[\rho(T) + p(T)] \quad \sigma = S/V = \frac{1}{T}[\rho(T) + p(T)]$$

Entropy Density

- Integrability condition $dS/dV dT = dS/dT dV$ relates the evolution of entropy density

$$\frac{d\sigma}{dT} = \frac{1}{T}\frac{d\rho}{dT}$$

$$\frac{d\sigma}{dt} = \frac{1}{T}\frac{d\rho}{dt} = \frac{1}{T}[-3(\rho+p)]\frac{d\ln a}{dt}$$

$$\frac{d\ln\sigma}{dt} = -3\frac{d\ln a}{dt} \qquad \sigma \propto a^{-3}$$

comoving entropy density is conserved in thermal equilibrium

- For ultra relativisitic bosons $s_{\rm boson} = 3.602 n_{\rm boson}$; for fermions factor of $7/8$ from energy density.

$$g_* = \sum_{\rm bosons} g_b + \frac{7}{8}\sum g_f$$

Neutrino Freezeout

- Neutrino equilibrium maintained by weak interactions, e.g. $e^+ + e^- \leftrightarrow \nu + \bar{\nu}$
- Weak interaction cross section $T_{10} = T/10^{10}K \sim T/1\mathrm{MeV}$

$$\sigma_w \sim G_F^2 E_\nu^2 \approx 4 \times 10^{-44}\, T_{10}^2 \mathrm{cm}^2$$

- Rate $\Gamma = n_\nu \sigma_w = H$ at $T_{10} \sim 3$ or $t \sim 0.2$s
- After neutrino freezeout, electrons and positrons annihilate dumping their entropy into the photons
- Before g_*: $\gamma, e^+, e^- = 2 + \frac{7}{8}(2+2) = \frac{11}{2}$
- After g_*: γ = 2; so conservation of entropy gives

$$g_* T^3\Big|_{\mathrm{initial}} = g_* T^3\Big|_{\mathrm{final}} \qquad T_\nu = \left(\frac{4}{11}\right)^{1/3} T_\gamma$$

Relic Neutrinos

- Relic number density (zero chemical potential; now required by oscillations & BBN)

$$n_\nu = n_\gamma \frac{3}{4}\frac{4}{11} = 112 \mathrm{cm}^{-3}$$

- Relic energy density assuming one species with finite m_ν:

$\rho_\nu = m_\nu n_\nu$

$$\rho_\nu = 112 \frac{m_\nu}{\mathrm{eV}}\,\mathrm{eV\,cm}^{-3} \qquad \rho_c = 1.05 \times 10^4 h^2\,\mathrm{eV cm}^{-3}$$

$$\Omega_\nu h^2 = \frac{m_\nu}{93.7\mathrm{eV}}$$

- Candidate for dark matter? an eV mass neutrino goes non relativistic around $z \sim 1000$ and retains a substantial velocity dispersion σ_ν.

Hot Dark Matter

- Momenta for a nonrelativistic species redshifts like temperature for a relativistic one, so average momentum is still given by

$$\langle q \rangle = 3T_\nu = m\sigma_\nu$$

$$\sigma_\nu = 3\left(\frac{m_\nu}{1\text{eV}}\right)^{-1}\left(\frac{T_\nu}{1\text{eV}}\right) = 3\left(\frac{m_\nu}{1\text{eV}}\right)^{-1}\left(\frac{T_\nu}{10^4\text{K}}\right)$$

$$= 6\times 10^{-4}\left(\frac{m_\nu}{1\text{eV}}\right)^{-1} = 200\text{km/s}\left(\frac{m_\nu}{1\text{eV}}\right)^{-1}$$

- on order the rotation velocity of galactic halos and higher at higher redshift - small objects can’t form: top down structure formation – not observed – must not constitute the bulk of the dark matter

Cold Dark Matter

- Problem with neutrinos is they decouple while relativistic and hence have a comparable number density to photons - for a reasonable energy density, the mass must be small
- The equilibrium distribution for a non-relativistic species declines exponentially beyond the mass threshold

$$n = g(\frac{mT}{2\pi})^{3/2} e^{-m/T}$$

- Freezeout when annihilation rate equal expansion rate $\Gamma \propto \sigma_A$, increasing annihilation cross section decreases abundance
- Appropriate candidates supplied by supersymmetry
- Alternate solution: keep light particle but not created in thermal equilibrium, axion dark matter

Big Bang Nucleosynthesis

- Most of light element synthesis can be understood through nuclear statistical equilibrium and reaction rates
- Equilibrium abundance of species with mass number A and charge Z (Z protons and $A - Z$ neutrons)

$$n_A = g_A(\frac{m_A T}{2\pi})^{3/2} e^{(\mu_A - m_A)/T}$$

- In chemical equilibrium with protons and neutrons

$$\mu_A = Z\mu_p + (A - Z)\mu_n$$

$$n_A = g_A(\frac{m_A T}{2\pi})^{3/2} e^{-m_A/T} e^{(Z\mu_p + (A-Z)\mu_n)/T}$$

Big Bang Nucleosynthesis

- Eliminate chemical potentials with n_p, n_n

$$e^{\mu_p/T} = \frac{n_p}{g_p}\left(\frac{2\pi}{m_p T}\right)^{3/2} e^{m_p/T}$$

$$e^{\mu_n/T} = \frac{n_n}{g_n}\left(\frac{2\pi}{m_n T}\right)^{3/2} e^{m_n/T}$$

$$n_A = g_A g_p^{-Z} g_n^{Z-A} (\frac{m_A T}{2\pi})^{3/2} \left(\frac{2\pi}{m_p T}\right)^{3Z/2} \left(\frac{2\pi}{m_n T}\right)^{3(A-Z)/2}$$

$$\times\, e^{-m_A/T} e^{(Z\mu_p+(A-Z)\mu_n)/T} n_p^Z n_n^{A-Z}$$

$$(g_p = g_n = 2; m_p \approx m_n = m_b = m_A/A)$$

$$(B_A = Zm_p + (A-Z)m_n - m_A$$

$$= g_A 2^{-A}\left(\frac{2\pi}{m_b T}\right)^{3(A-1)/2} A^{3/2} n_p^Z n_n^{A-Z} e^{B_A/T}$$

Big Bang Nucleosynthesis

- Convenient to define abundance fraction

$$X_A \equiv A\frac{n_A}{n_b} = Ag_A 2^{-A}\left(\frac{2\pi}{m_b T}\right)^{3(A-1)/2} A^{3/2} n_p^Z n_n^{A-Z} n_b^{-1} e^{B_A/T}$$

$$= Ag_A 2^{-A}\left(\frac{2\pi n_b^{2/3}}{m_b T}\right)^{3(A-1)/2} A^{3/2} e^{B_A/T} X_p^Z X_n^{A-Z}$$

$$(n_\gamma = \frac{2}{\pi^2}T^3\zeta(3) \qquad \eta_{b\gamma} \equiv n_b/n_\gamma)$$

$$= A^{5/2} g_A 2^{-A}\left[\left(\frac{2\pi T}{m_b}\right)^{3/2}\frac{2\zeta(3)\eta_{b\gamma}}{\pi^2}\right]^{A-1} e^{B_A/T} X_p^Z X_n^{A-Z}$$

- Deuterium $A = 2$, $Z = 1$, $g_2 = 3$, $B_2 = 2.225$ MeV

$$X_2 = \frac{3}{\pi^2}\left(\frac{4\pi T}{m_b}\right)^{3/2}\eta_{b\gamma}\zeta(3)e^{B_2/T}X_p X_n$$

Deuterium

- Deuterium "bottleneck" is mainly due to the low baryon-photon number of the universe $\eta_{b\gamma} \sim 10^{-9}$, secondarily due to the low binding energy B_2
- $X_2/X_pX_n \approx \mathcal{O}(1)$ at $T \approx 100$keV or 10^9 K, much lower than the binding energy B_2
- Most of the deuterium formed then goes through to helium via $\mathrm{D} + \mathrm{D} \to {}^3\mathrm{He} + p$, ${}^3\mathrm{He} + \mathrm{D} \to {}^4\mathrm{He} + n$
- Deuterium freezes out as number abundance becomes too small to maintain reactions $n_D =$ const. The deuterium freezeout fraction $n_D/n_b \propto \eta_{b\gamma}^{-1} \propto (\Omega_b h^2)^{-1}$ and so is fairly sensitive to the baryon density.
- Observations of the ratio in quasar absorption systems give $\Omega_b h^2 \approx 0.02$

Helium

- Essentially all neutrons around during nucleosynthesis end up in Helium
- In equilibrium, the neutron-to-proton ratio is determined by the mass difference $Q = m_n - m_p = 1.293$ MeV

$$\frac{n_n}{n_p} = \exp[-Q/T]$$

- Equilibrium is maintained through weak interactions, e.g. $n \leftrightarrow p + e^- + \bar{\nu}$ with rate

$$\frac{\Gamma}{H} \approx \frac{T}{0.8\mathrm{MeV}}$$

- Freezeout fraction

$$\frac{n_n}{n_p} = \exp[-1.293/0.8] \approx 0.2$$

Helium

- Finite lifetime of neutrons brings this to $\sim 1/7$ by 10^9K
- Helium mass fraction

$$Y_{\rm He} = \frac{4n_{He}}{n_b} = \frac{4(n_n/2)}{n_n + n_p}$$
$$= \frac{2n_n/n_p}{1 + n_n/n_p} \approx \frac{2/7}{8/7} \approx \frac{1}{4}$$

- Depends mainly on the expansion rate during BBN - measure number of relativistic species
- Traces of ^{7}Li as well. Measured abundances in reasonable agreement with deuterium measure $\Omega_b h^2 = 0.02$

Recombination

- Statistical equilibrium says that neutral hydrogen will form sometime after the temperature drops below the binding energy of hydrogen
- Apply equiliburium distribution

$$n_i = g_i \left(\frac{m_i T}{2\pi} \right)^{3/2} e^{-m_i/T}$$

to the $e^- + p \leftrightarrow H$ system: Saha Equation

$$\begin{aligned} \frac{n_e n_p}{n_H n_b} &= \frac{x_e^2}{1 - x_e} \\ &= \frac{1}{n_b} \left(\frac{m_e T}{2\pi} \right)^{3/2} e^{-B/T} \end{aligned}$$

where $B = m_e + m_p - m_H = 13.6\text{eV}$

Recombination

- But again the photon-baryon ratio is very low

$$\eta_{b\gamma} \equiv n_b/n_\gamma \approx 3 \times 10^{-8}\Omega_b h^2$$

- Eliminate in favor of $\eta_{b\gamma}$ and B/T through

$$n_\gamma = 0.244T^3\,, \qquad \frac{m_e}{B} = 3.76 \times 10^4$$

- Big coefficient

$$\frac{{x_e}^2}{1 - x_e} = 3.16 \times 10^{15} \left(\frac{B}{T}\right)^{3/2} e^{-B/T}$$

$T = 1/3\text{eV} \to x_e = 0.7$, $T = 0.3\text{eV} \to x_e = 0.2$

- Further delayed by inability to maintain equilibrium since net is through 2γ process and redshifting out of line