

*Astro 321*

Lecture Notes **Set 4**

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# Horizon Problem

- The horizon in a decelerating universe scales as  $\eta \propto a^{(1+3w)/2}$ ,  $w > -1/3$ . For example in a matter dominated universe

$$\eta \propto a^{1/2}$$

- CMB decoupled at  $a_* = 10^{-3}$  so subtends an angle on the sky

$$\frac{\eta_*}{\eta_0} = a_*^{1/2} \approx 0.03 \approx 2^\circ$$

- So why is the CMB sky isotropic to  $10^{-5}$  in temperature if it is composed of  $\sim 10^4$  causally disconnected regions
- If smooth by fiat, why are there  $10^{-5}$  fluctuations correlated on superhorizon scales

# Flatness & Relic Problems

- Flatness problem: why is the radius of curvature larger than the observable universe. (Before the CMB determinations, why is it at least comparable to observable universe  $|\Omega_K| \lesssim \Omega_m$ )
- Also phrased as a coincidence problem: since  $\rho_K \propto a^{-2}$  and  $\rho_m \propto a^{-3}$ , why would they be comparable today – modern version is dark energy coincidence  $\rho_\Lambda = \text{const.}$
- Relic problem – why don't relics like monopoles dominate the energy density
- Inflation is a theory that solves all three problems at once and also supplies a source for density perturbations

# Accelerating Expansion

- In a matter or radiation dominated universe, the horizon grows as a power law in  $a$  so that there is no way to establish causal contact on a scale longer than the inverse Hubble length ( $1/aH$ , comoving coordinates) at any given time: general for a decelerating universe

$$\eta = \int d \ln a \frac{1}{aH(a)}$$

- $H^2 \propto \rho \propto a^{-3(1+w)}$ ,  $aH \propto a^{-(1+3w)/2}$ , critical value of  $w = -1/3$ , the division between acceleration and deceleration
- In an accelerating universe, the Hubble length shrinks in comoving coordinates and so the horizon gets its contribution at the earliest times, e.g. in a cosmological constant universe, the horizon saturates to a constant value

# Causal Contact

- Note confusion in nomenclature: the true horizon always grows meaning that one always sees more and more of the universe. The Hubble length decreases: the difference in conformal time, the distance a photon can travel between two epochs denoted by the scale factor decreases. Regions that were in causal contact, leave causal contact.
- Horizon problem solved if the universe was in an acceleration phase up to  $\eta_i$  and the conformal time since then is shorter than the total conformal age

$$\eta_0 \gg \eta_0 - \eta_i$$

total distance  $\gg$  distance traveled since inflation  
apparent horizon

# Flatness & Relic

- Comoving radius of curvature is constant and can even be small compared to the full horizon  $R \ll \eta_0$  yet still  $\eta_0 \gg R \gg \eta_0 - \eta_i$
- In physical coordinates, the rapid expansion of the universe makes the current observable universe much smaller than the curvature scale
- Likewise, the number density of relics formed before the accelerating (or inflationary) epoch is diluted to make them rare in the current observable volume
- Common to reference time to the end of inflation  $\tilde{\eta} \equiv \eta - \eta_i$ . Here conformal time is negative during inflation and its value (as a difference in conformal time) reflects the comoving Hubble length - defines leaving the horizon as  $k|\tilde{\eta}| = 1$

# Exponential Expansion

- If the accelerating component has equation of state  $w = -1$ ,  $\rho = \text{const.}$ ,  $H = H_i \text{ const.}$  so that  $a \propto \exp(Ht)$

$$\begin{aligned}\tilde{\eta} &= - \int_a^{a_i} d \ln a \frac{1}{aH} = \frac{1}{aH_i} \Big|_a^{a_i} \\ &\approx -\frac{1}{aH_i} \quad (a_i \gg a)\end{aligned}$$

- In particular, the current horizon scale  $H_0 \tilde{\eta}_0 \approx 1$  exited the horizon during inflation at

$$\begin{aligned}\tilde{\eta}_0 &\approx H_0^{-1} = \frac{1}{a_H H_i} \\ a_H &= \frac{H_0}{H_i}\end{aligned}$$

# Sufficient Inflation

- Current horizon scale must have exited the horizon during inflation so that the start of inflation could not be after  $a_H$ . How long before the end of inflation must it have begun?

$$\frac{a_H}{a_i} = \frac{H_0}{H_i a_i}$$
$$\frac{H_0}{H_i} = \sqrt{\frac{\rho_c}{\rho_i}}, \quad a_i = \frac{T_{\text{CMB}}}{T_i}$$

- $\rho_c^{1/4} = 3 \times 10^{-12} \text{ GeV}$ ,  $T_{\text{CMB}} = 3 \times 10^{-13} \text{ GeV}$

$$\frac{a_H}{a_i} = 3 \times 10^{-29} \left( \frac{\rho_i^{1/4}}{10^{14} \text{ GeV}} \right)^{-2} \left( \frac{T_i}{10^{10} \text{ GeV}} \right)$$
$$\ln \frac{a_i}{a_H} = 65 + 2 \ln \left( \frac{\rho_i^{1/4}}{10^{14} \text{ GeV}} \right) - \ln \left( \frac{T_i}{10^{10} \text{ GeV}} \right)$$

# Perturbation Generation

- Horizon scale  $\tilde{\eta}$  during inflation acts like an event horizon - things leaving causal contact
- Particle creation similar to Hawking radiation from a black hole with hubble length replacing the BH horizon

$$T_H \approx H_i$$

- Because  $H_i$  remains roughly constant during inflation the result is a scale invariant spectrum of fluctuations due to zero-point fluctuations becoming classical
- Fluctuations in the field driving inflation (inflaton) carry the energy density of the universe and so their zero point fluctuations are net energy density or curvature fluctuations
- Any other light field (gravitational waves, etc...) will also carry scale invariant perturbations but are iso-curvature fluctuations

# Slow Roll Inflation

- Single minimally coupled scalar field rolling slowly in a nearly flat potential
- Scalar field equation of motion  $V' \equiv dV/d\phi$

$$\partial_\mu \partial^\mu \phi + V'(\phi) = 0$$

so that in the background  $\phi = \phi_0$  and

$$\ddot{\phi}_0 + 2\frac{\dot{a}}{a}\dot{\phi}_0 + a^2 V' = 0$$

$$\frac{d^2 \phi_0}{dt^2} + 3H \frac{d\phi_0}{dt} + V' = 0$$

- Simply the continuity equation with the associations

$$\rho_\phi = \frac{1}{2}a^{-2}\dot{\phi}_0^2 + V \quad p_\phi = \frac{1}{2}a^{-2}\dot{\phi}_0^2 - V$$

# Slow Roll Parameters

- Net energy is dominated by potential energy and so acts like a cosmological constant  $w \rightarrow -1$
- First slow roll parameter

$$\epsilon = \frac{3}{2}(1 + w) = \frac{1}{16\pi G} \left( \frac{V'}{V} \right)^2$$

- Second slow roll parameter  $d^2\phi_0/dt^2 \approx 0$ , or  $\ddot{\phi}_0 \approx (\dot{a}/a)\dot{\phi}_0$

$$\delta = \frac{\ddot{\phi}_0}{\dot{\phi}_0} \left( \frac{\dot{a}}{a} \right)^{-1} - 1 = \epsilon - \frac{1}{8\pi G} \frac{V''}{V}$$

- Slow roll condition  $\epsilon, \delta \ll 1$  corresponds to a very flat potential

# Perturbations

- Linearize perturbation  $\phi = \phi_0 + \phi_1$  then

$$\ddot{\phi}_1 + 2\frac{\dot{a}}{a}\dot{\phi}_1 + k^2\phi_1 + a^2V''\phi_1 = 0$$

in slow roll inflation  $V''$  term negligible

- Implicitly assume that the spatial metric fluctuations (curvature  $\zeta$ ) vanishes, otherwise covariant derivatives pick these up – formal justification is work in that frame and transform back.
- Curvature represents a warping of the scale factor  $a \rightarrow (1 + \zeta)a$  or  $\delta a/a = \zeta$

$$\zeta = \frac{\delta a}{a} = \frac{\dot{a}}{a}\delta\eta = \frac{\dot{a}}{a}\frac{\phi_1}{\dot{\phi}_0}$$

a change in the field value  $\phi_1$  defines a change in the epoch that inflation ends, imprinting a curvature fluctuation

# Slow-Roll Evolution

- Rewrite in  $u \equiv a\phi$  to remove expansion damping

$$\ddot{u} + \left[ k^2 - 2 \left( \frac{\dot{a}}{a} \right)^2 \right] u = 0$$

- or for conformal time measured from the end of inflation

$$\tilde{\eta} = \eta - \eta_{\text{end}}$$

$$\tilde{\eta} = \int_{a_{\text{end}}}^a \frac{da}{Ha^2} \approx -\frac{1}{aH}$$

- Compact, slow-roll equation:

$$\ddot{u} + \left[ k^2 - \frac{2}{\tilde{\eta}^2} \right] u = 0$$

# Slow Roll Limit

- Slow roll equation has the exact solution:

$$u = A\left(k \pm \frac{i}{\tilde{\eta}}\right)e^{\mp ik\tilde{\eta}}$$

- For  $|k\tilde{\eta}| \gg 1$  (early times, inside Hubble length) behaves as free oscillator

$$\lim_{|k\tilde{\eta}| \rightarrow \infty} u = Ake^{\mp ik\tilde{\eta}}$$

- Normalization  $A$  will be set by origin in quantum fluctuations of free field

# Slow Roll Limit

- For  $|k\tilde{\eta}| \ll 1$  (late times,  $\gg$  Hubble length) fluctuation freezes in

$$\lim_{|k\tilde{\eta}| \rightarrow 0} u = \pm \frac{i}{\tilde{\eta}} A = \pm i H a A$$

$$\phi_1 = \pm i H A$$

$$\zeta = \mp i H A \left( \frac{\dot{a}}{a} \right) \frac{1}{\dot{\phi}_0}$$

- Slow roll replacement

$$\left( \frac{\dot{a}}{a} \right)^2 \frac{1}{\dot{\phi}_0^2} = \frac{8\pi G a^2 V}{3} \frac{3}{2a^2 V \epsilon} = \frac{4\pi G}{\epsilon} = \frac{4\pi}{\epsilon m_{\text{pl}}^2}$$

- Bardeen curvature power spectrum

$$\Delta_{\zeta}^2 \equiv \frac{k^3 |\zeta|^2}{2\pi^2} = \frac{2k^3}{\pi} \frac{H^2}{\epsilon m_{\text{pl}}^2} A^2$$

# Quantum Fluctuations

- Simple harmonic oscillator  $\ll$  Hubble length

$$\ddot{u} + k^2 u = 0$$

- Quantize the simple harmonic oscillator

$$\hat{u} = u(k, \eta) \hat{a} + u^*(k, \eta) \hat{a}^\dagger$$

where  $u(k, \eta)$  satisfies classical equation of motion and the creation and annihilation operators satisfy

$$[a, a^\dagger] = 1, \quad a|0\rangle = 0$$

- Normalize wavefunction  $[\hat{u}, d\hat{u}/d\eta] = i$

$$u(k, \eta) = \frac{1}{\sqrt{2k}} e^{-ik\tilde{\eta}}$$

# Quantum Fluctuations

- Zero point fluctuations of ground state

$$\begin{aligned}\langle u^2 \rangle &= \langle 0 | u^\dagger u | 0 \rangle \\ &= \langle 0 | (u^* \hat{a}^\dagger + u \hat{a})(u \hat{a} + u^* \hat{a}^\dagger) | 0 \rangle \\ &= \langle 0 | \hat{a} \hat{a}^\dagger | 0 \rangle |u(k, \tilde{\eta})|^2 \\ &= \langle 0 | [\hat{a}, \hat{a}^\dagger] + \hat{a}^\dagger \hat{a} | 0 \rangle |u(k, \tilde{\eta})|^2 \\ &= |u(k, \tilde{\eta})|^2 = \frac{1}{2k}\end{aligned}$$

- Classical equation of motion take this quantum fluctuation outside horizon where it freezes in. Slow roll equation
- So  $A = (2k^3)^{1/2}$  and curvature power spectrum

$$\Delta_\zeta^2 \equiv \frac{1}{\pi} \frac{H^2}{\epsilon m_{\text{pl}}^2}$$

# Tilt

- Curvature power spectrum is scale invariant to the extent that  $H$  is constant
- Scalar spectral index

$$\begin{aligned}\frac{d \ln \Delta_{\zeta}^2}{d \ln k} &\equiv n_S - 1 \\ &= 2 \frac{d \ln H}{d \ln k} - \frac{d \ln \epsilon}{d \ln k}\end{aligned}$$

- Evaluate at horizon crossing where fluctuation freezes

$$\begin{aligned}\frac{d \ln H}{d \ln k} \Big|_{-k\tilde{\eta}=1} &= \frac{k}{H} \frac{dH}{d\tilde{\eta}} \Big|_{-k\tilde{\eta}=1} \frac{d\tilde{\eta}}{dk} \Big|_{-k\tilde{\eta}=1} \\ &= \frac{k}{H} (-aH^2\epsilon) \Big|_{-k\tilde{\eta}=1} \frac{1}{k^2} = -\epsilon\end{aligned}$$

where  $aH = -1/\tilde{\eta} = k$

# Tilt

- Evolution of  $\epsilon$

$$\frac{d \ln \epsilon}{d \ln k} = -\frac{d \ln \epsilon}{d \ln \tilde{\eta}} = -2(\delta + \epsilon) \frac{\dot{a}}{a} \tilde{\eta} = 2(\delta + \epsilon)$$

- Tilt in the slow-roll approximation

$$n_S = 1 - 4\epsilon - 2\delta$$

# Gravitational Waves

- Gravitational wave amplitude satisfies Klein-Gordon equation ( $K = 0$ ), same as scalar field

$$\ddot{H}_T^{(\pm 2)} + 2\frac{\dot{a}}{a}\dot{H}_T^{(\pm 2)} + k^2 H_T^{(\pm 2)} = 0.$$

- Acquires quantum fluctuations in same manner as  $\phi$ . Lagrangian sets the normalization

$$\phi_1 \rightarrow H_T^{(\pm 2)} \sqrt{\frac{3}{16\pi G}}$$

- Scale-invariant gravitational wave amplitude (each component: NB more traditional notation  $H_T^{(\pm 2)} = (h_+ \pm ih_\times)/\sqrt{6}$ )

$$\Delta_H^2 = \frac{16\pi G}{3 \cdot 2\pi^2} \frac{H^2}{2} = \frac{4}{3\pi} \frac{H^2}{m_{\text{pl}}^2}$$

# Gravitational Waves

- Gravitational wave power  $\propto H^2 \propto V \propto E_i^4$  where  $E_i$  is the energy scale of inflation
- Tensor tilt:

$$\frac{d \ln \Delta_H^2}{d \ln k} \equiv n_T = 2 \frac{d \ln H}{d \ln k} = -2\epsilon$$

- Consistency relation between tensor-scalar ratio and tensor tilt

$$\frac{\Delta_H^2}{\Delta_\zeta^2} = \frac{4}{3}\epsilon = -\frac{2}{3}\epsilon$$

- Measurement of scalar tilt and gravitational wave amplitude constrains inflationary model in the slow roll context
- Comparison of tensor-scalar ratio and tensor tilt tests the idea of slow roll itself