Astro 321

Lecture Notes Set 7

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Closed Universe

- Friedman equation in a closed universe

$$\frac{1}{a}\frac{da}{dt} = H_0\left(\Omega_m a^{-3} + (1-\Omega_m)a^{-2}\right)^{1/2}$$

- Parametric solution in terms of a development angle $\theta = H_0\eta(\Omega_m - 1)^{1/2}$, scaled conformal time η

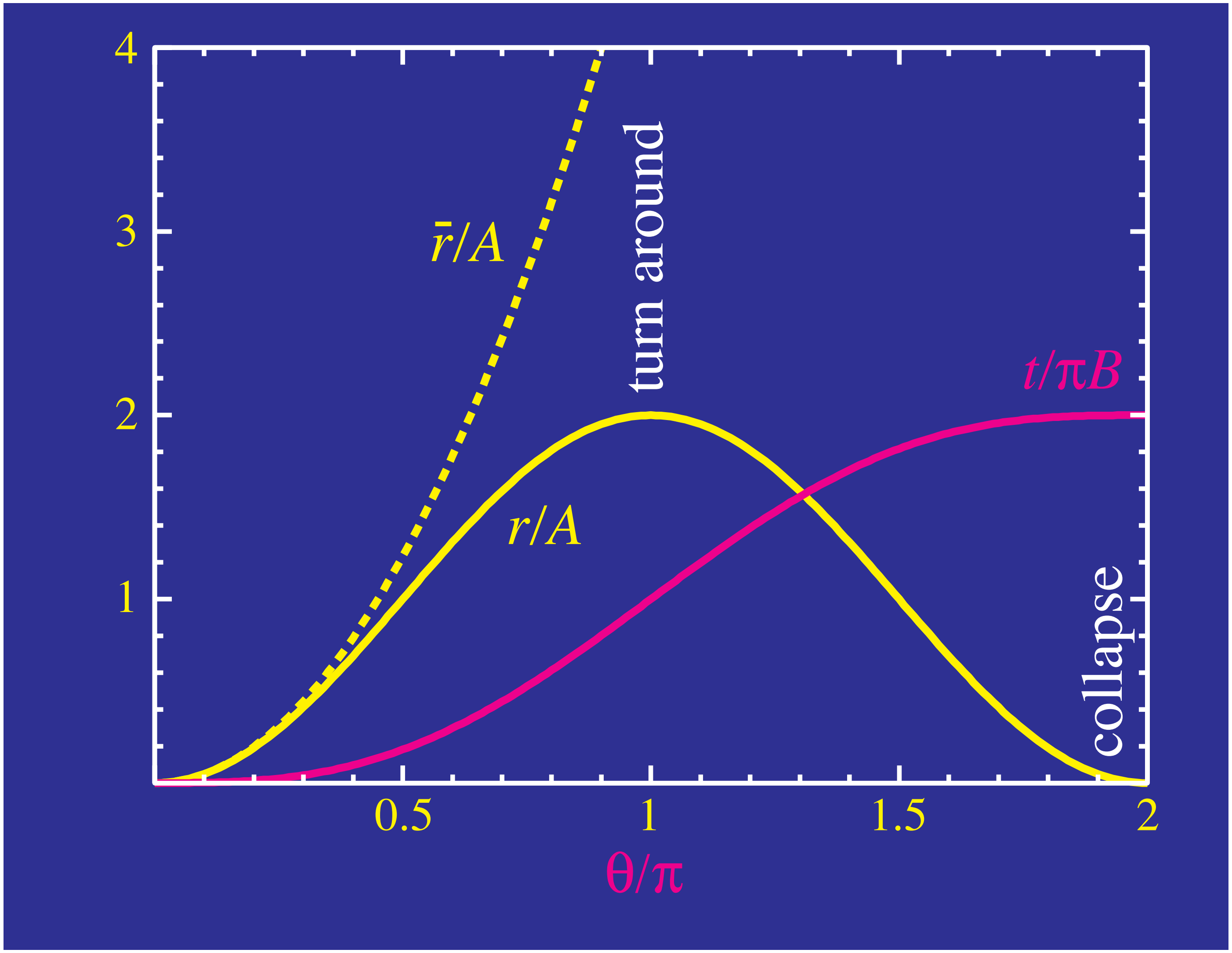
$$\begin{aligned} r(\theta) &= A(1-\cos\theta) \\ t(\theta) &= B(\theta - \sin\theta) \end{aligned}$$

where $A = r_0\Omega_m/2(\Omega_m - 1)$, $B = H_0^{-1}\Omega_m/2(\Omega_m - 1)^{3/2}$.

- Turn around at $\theta = \pi$, $r = 2A$, $t = B\pi$.
- Collapse at $\theta = 2\pi$, $r \to 0$, $t = 2\pi B$

Spherical Collapse

- Parametric Solution:



Correspondence

- Eliminate cosmological correspondence in A and B in terms of enclosed mass M

$$M = \frac{4\pi}{3} r_0^3 \Omega_m \rho_c = \frac{4\pi}{3} r_0^3 \Omega_m \frac{3H_0^2}{8\pi G}$$

- Related as $A^3 = GMB^2$, and to initial perturbation

$$\begin{aligned} \lim_{\theta\to 0} r(\theta) &= A\left(\frac{1}{2}\theta^2 - \frac{1}{24}\theta^4\right) \\ \lim_{\theta\to 0} t(\theta) &= B\left(\frac{1}{6}\theta^3 - \frac{1}{120}\theta^5\right) \end{aligned}$$

- Leading Order: $r = A\theta^2/2$, $t = B\theta^3/6$

$$r = \frac{A}{2}\left(\frac{6t}{B}\right)^{2/3}$$

Next Order

- Leading order is unperturbed matter dominated expansion
  $r \propto a \propto t^{2/3}$
- Iterate r and t solutions

$$\lim_{\theta \to 0} t(\theta) = \frac{\theta^3}{6} B \left[1 - \frac{1}{20} \left(\frac{6t}{B} \right)^{2/3} \right]$$

$$\theta \approx \left(\frac{6t}{B} \right)^{1/3} \left[1 + \frac{1}{60} \left(\frac{6t}{B} \right)^{2/3} \right]$$

Next Order

- Substitute back into $r(\theta)$

$$
\begin{aligned}
r(\theta) &= A\frac{\theta^2}{2}\left(1-\frac{\theta^2}{12}\right) \\
&= \frac{A}{2}\left(\frac{6t}{B}\right)^{2/3}\left[1-\frac{1}{20}\left(\frac{6t}{B}\right)^{2/3}\right] \\
&= \frac{1}{2}(6t)^{2/3}(GM)^{1/3}\left[1-\frac{1}{20}\left(\frac{6t}{B}\right)^{2/3}\right]
\end{aligned}
$$

Density Correspondence

- Density

$$\rho_m = \frac{M}{\frac{4}{3}\pi r^3}$$

$$= \frac{1}{6\pi t^2 G}\left[1 + \frac{3}{20}\left(\frac{6t}{B}\right)^{2/3}\right]$$

- Density perturbation

$$\delta \equiv \frac{\rho_m - \bar{\rho}_m}{\bar{\rho}_m} \approx \frac{3}{20}\left(\frac{6t}{B}\right)^{2/3}$$

Density Correspondence

- Time $\longrightarrow$ scale factor

$$
\begin{aligned}
t &= \frac{2}{3H_0\Omega_m^{1/2}} a^{3/2} \\
\delta &= \frac{3}{20} a \left(4/B H_0 \Omega_m^{1/2}\right)^{2/3}
\end{aligned}
$$

- A and B constants $\longrightarrow$ initial cond.

$$
\begin{aligned}
B &= \frac{1}{2H_0\Omega_m^{1/2}} \left(\frac{3}{5}\frac{a_i}{\delta_i}\right)^{3/2} \\
A &= \frac{3}{10}\frac{r_i}{\delta_i}
\end{aligned}
$$

Spherical Collapse Relations

- Scale factor $a \propto t^{2/3}$

$$a = \left(\frac{3}{4}\right)^{2/3} \left(\frac{3}{5}\frac{a_i}{\delta_i}\right) (\theta - \sin\theta)^{2/3}$$

- At collapse $\theta = 2\pi$

$$a_{\rm col} = \left(\frac{3}{4}\right)^{2/3} \left(\frac{3}{5}\frac{a_i}{\delta_i}\right) (2\pi)^{2/3} \approx 1.686\frac{a_i}{\delta_i}$$

- Perturbation collapses when linear theory predicts $\delta_c \equiv 1.686$

Virialization

- A real density perturbation is neither spherical nor homogeneous
- Shell crossing if δ_i doesn't monotonically decrease
- Collapse does not proceed to a point but reaches virial equilibrium

$$
\begin{aligned}
U &= -2K \\
E &= U + K = U(r_{\rm max}) = \frac{1}{2}U(r_{\rm vir}) \\
r_{\rm vir} &= \frac{1}{2}r_{\rm max}
\end{aligned}
$$

since $U \propto r^{-1}$. Thus $\theta_{\rm vir} = \frac{3}{2}\pi$

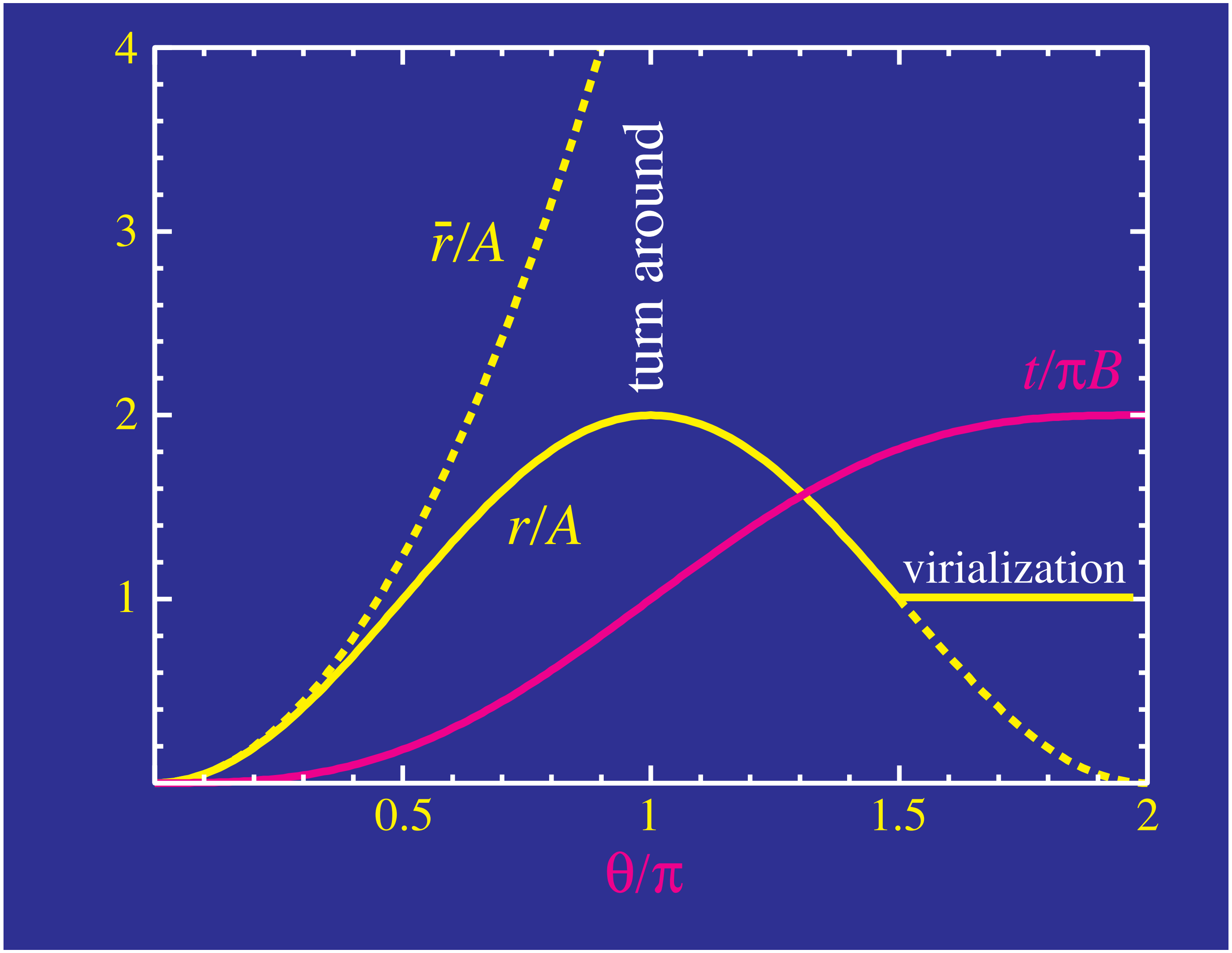
- Overdensity at virialization

$$
\frac{\rho_m(\theta = 3\pi/2)}{\bar{\rho}_m(\theta = 2\pi)} = 18\pi^2 \approx 178
$$

- Threshold $\Delta_v = 178$ often used to define a collapsed object

Virialization

- Schematic Picture:



The Mass Function

- Spherical collapse predicts the end state as virialized halos given an initial density perturbation
- Initial density perturbation is a Gaussian random field
- Compare the variance in the linear density field to threshold $\delta_c = 1.686$ to determine collapse fraction
- Combine to form the mass function, the number density of halos in a range dM around M.
- Halo density defined entirely by linear theory
- Fudge the result to get the right answer compared with simulations (a la Press-Schechter)!

Press-Schechter Formalism

- Smooth linear density density field on mass scale M with tophat

$$R = \left(\frac{3M}{4\pi}\right)^{1/3}$$

- Result is a Gaussian random field with variance $\sigma^2(M)$
- Fluctuations above the threshold δ_c correspond to collapsed regions. The fraction in halos $> M$ becomes

$$\frac{1}{\sqrt{2\pi}\sigma(M)} \int_{\delta_c}^{\infty} d\delta \exp\left(-\frac{\delta^2}{2\sigma^2(M)}\right) = \frac{1}{2}\mathrm{erfc}\left(\frac{\nu}{\sqrt{2}}\right)$$

  where $\nu \equiv \delta_c/\sigma(M)$
- Problem: even as $\sigma(M) \to \infty$, $\nu \to 0$, collapse fraction $\to 1/2$ – only overdense regions participate in spherical collapse.
- Multiply by an ad hoc factor of 2!

Press-Schechter Mass Function

- Differentiate in M to find fraction in range dM and multiply by ρ_m/M the number density of halos if all of the mass were composed of such halos $\to$ differential number density of halos

$$\frac{dn}{d\ln M} = \frac{\rho_m}{M}\frac{d}{d\ln M}\mathrm{erfc}\left(\frac{\nu}{\sqrt{2}}\right)$$
$$= \sqrt{\frac{2}{\pi}}\frac{\rho_m}{M}\frac{d\ln\sigma^{-1}}{d\ln M}\nu\exp(-\nu^2/2)$$

- High mass: exponential cut off above M_* where $\sigma(M_*) = \delta_c$

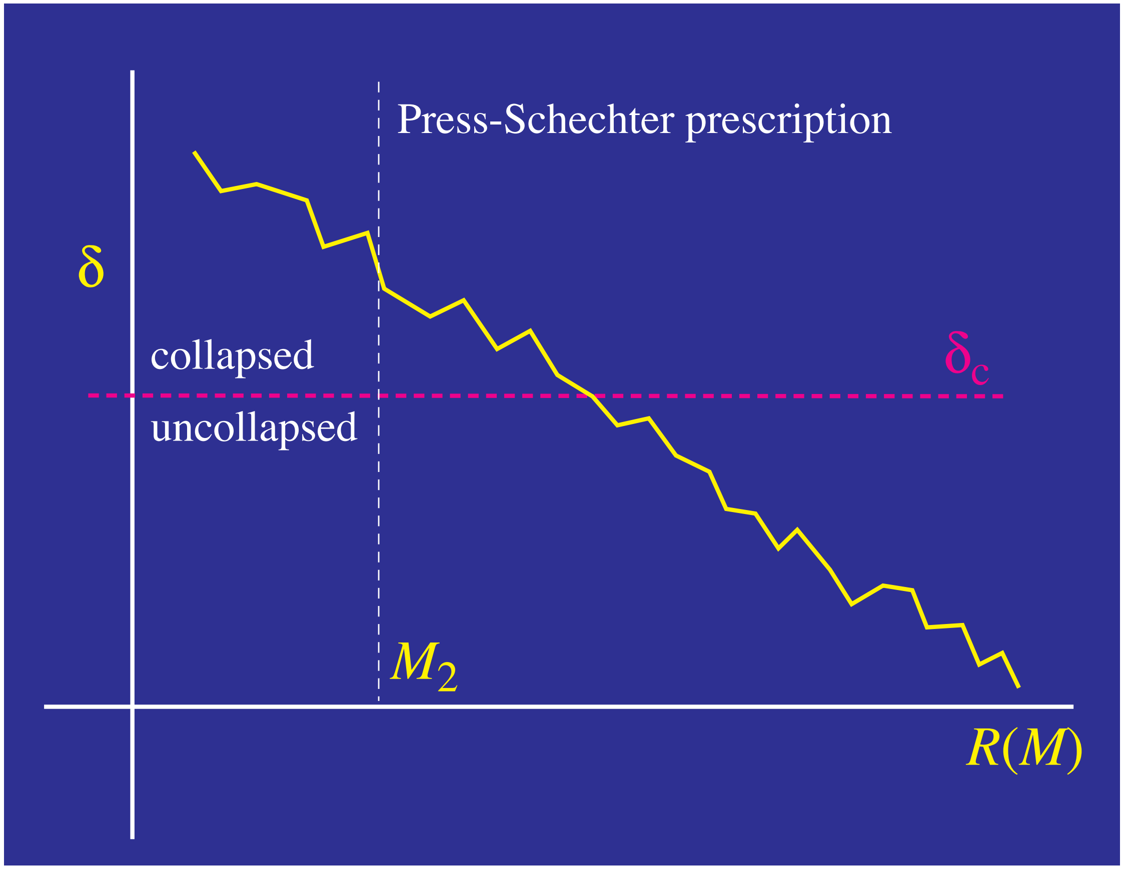
$$M_* \sim 10^{13}h^{-1}M_\odot \quad \text{today}$$

- Low mass divergence: (too many for the observations?)

$$\frac{dn}{d\ln M} \propto\sim M^{-1}$$

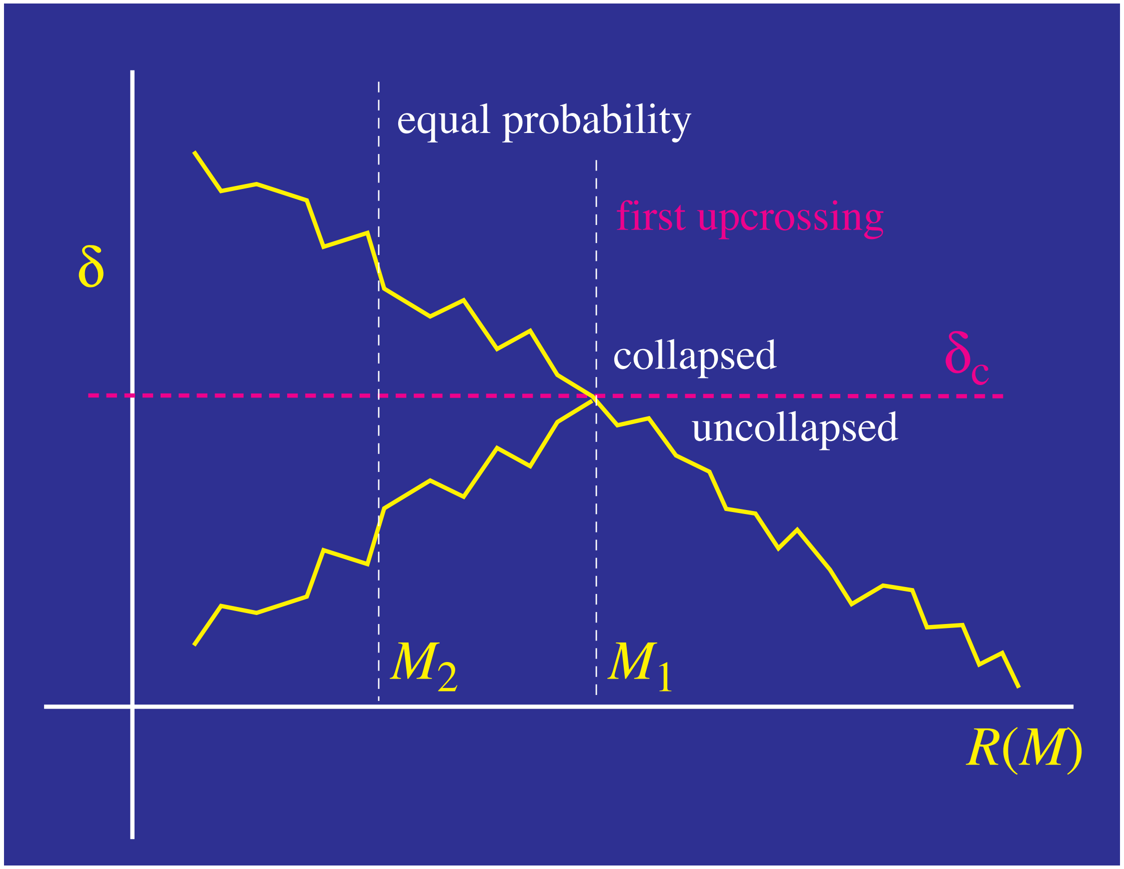
Extended Press-Schechter Formalism

- A region that is underdense when smoothed on the scale M may be overdense on a scale of a larger M
- If smoothing is a tophat in k-space, independence of k-modes implies fluctuation executes a random walk



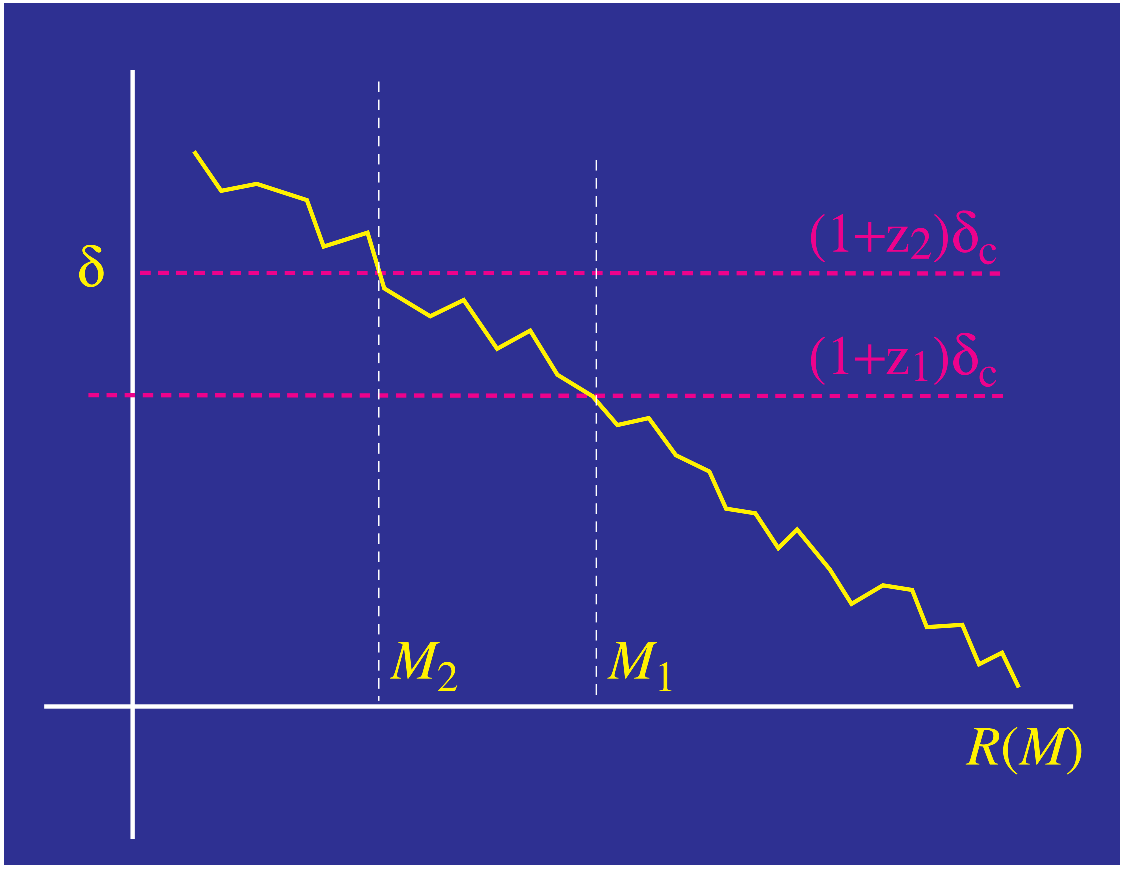
Extended Press-Schechter Formalism

- For each trajectory that lies above threshold at M_2, there is an equivalent trajectory that is its mirror image reflected around δ_c
- Press-Schechter ignored this branch. It supplies the missing factor of 2



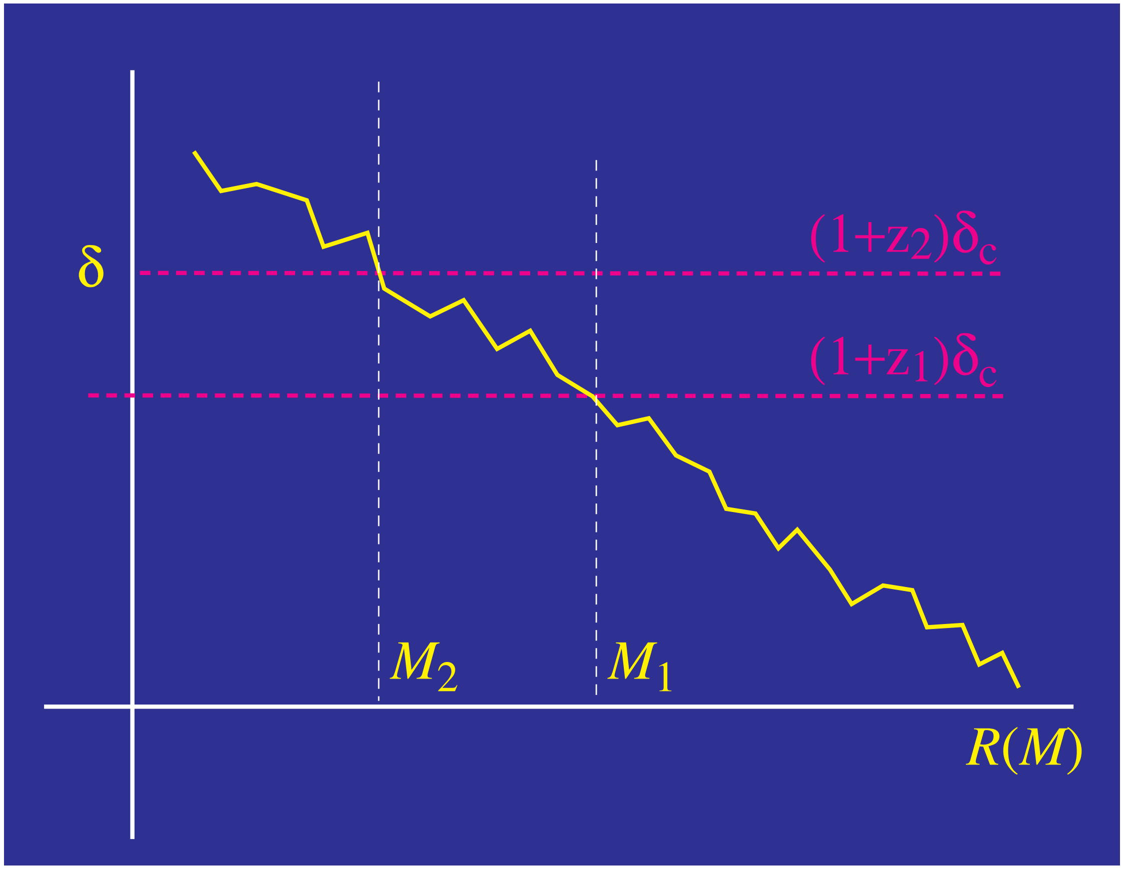
Conditional Mass Function

- Extended Press-Schechter also gives the conditional mass function, useful for merger histories.
- Given a halo of mass M_1 exists at z_1, what is the probability that it was part of a halo of mass M_2 at z_2



Conditional Mass Function

- Same as before but with the origin translated.
- Conditional mass function is mass function with δ_c and $\sigma^2(M)$ shifted



Magic "2" resolved?

- Spherical collapse is defined for a real-space not k-space smoothing. Random walk is only a qualitative explanation.
- Modern approach: think of spherical collapse as motivating a fitting form for the mass function

$$\nu \exp(-\nu^2/2) \rightarrow A[1 + (a\nu^2)^{-p}]\sqrt{a\nu^2} \exp(-a\nu^2/2)$$

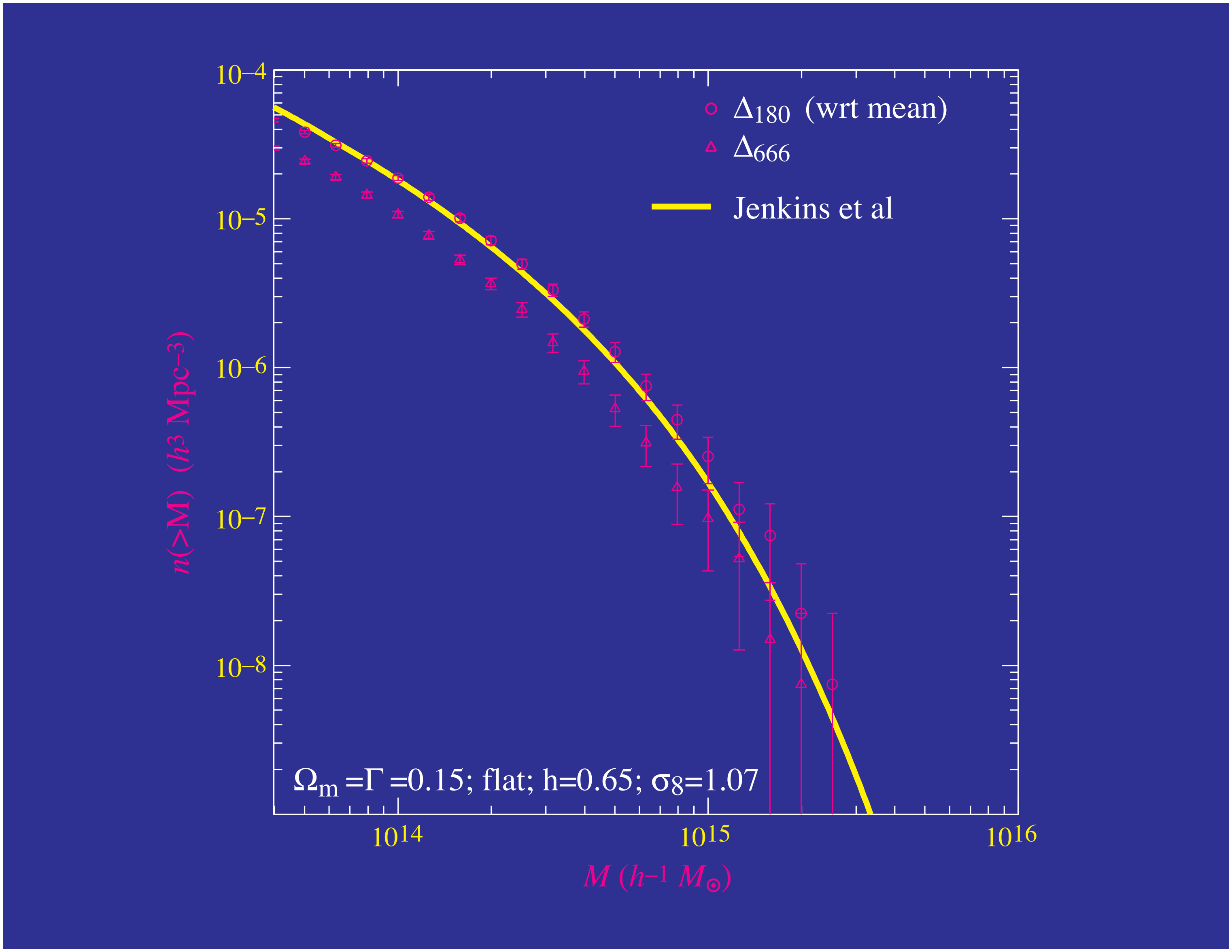
Sheth-Torman 1999, $a = 0.75$, $p = 0.3$. or a completely empirical fitting

$$\frac{dn}{d\ln M} = 0.301 \frac{\rho_m}{M} \frac{d\ln \sigma^{-1}}{d\ln M} \exp[-|\ln \sigma^{-1} + 0.64|^{3.82}]$$

Jenkins et al 2001. Choice is tied up with the question: what is the mass of a halo?

Numerical Mass Function

- Example of difference in mass definition (from Hu & Kravstov 2002)



Halo Bias

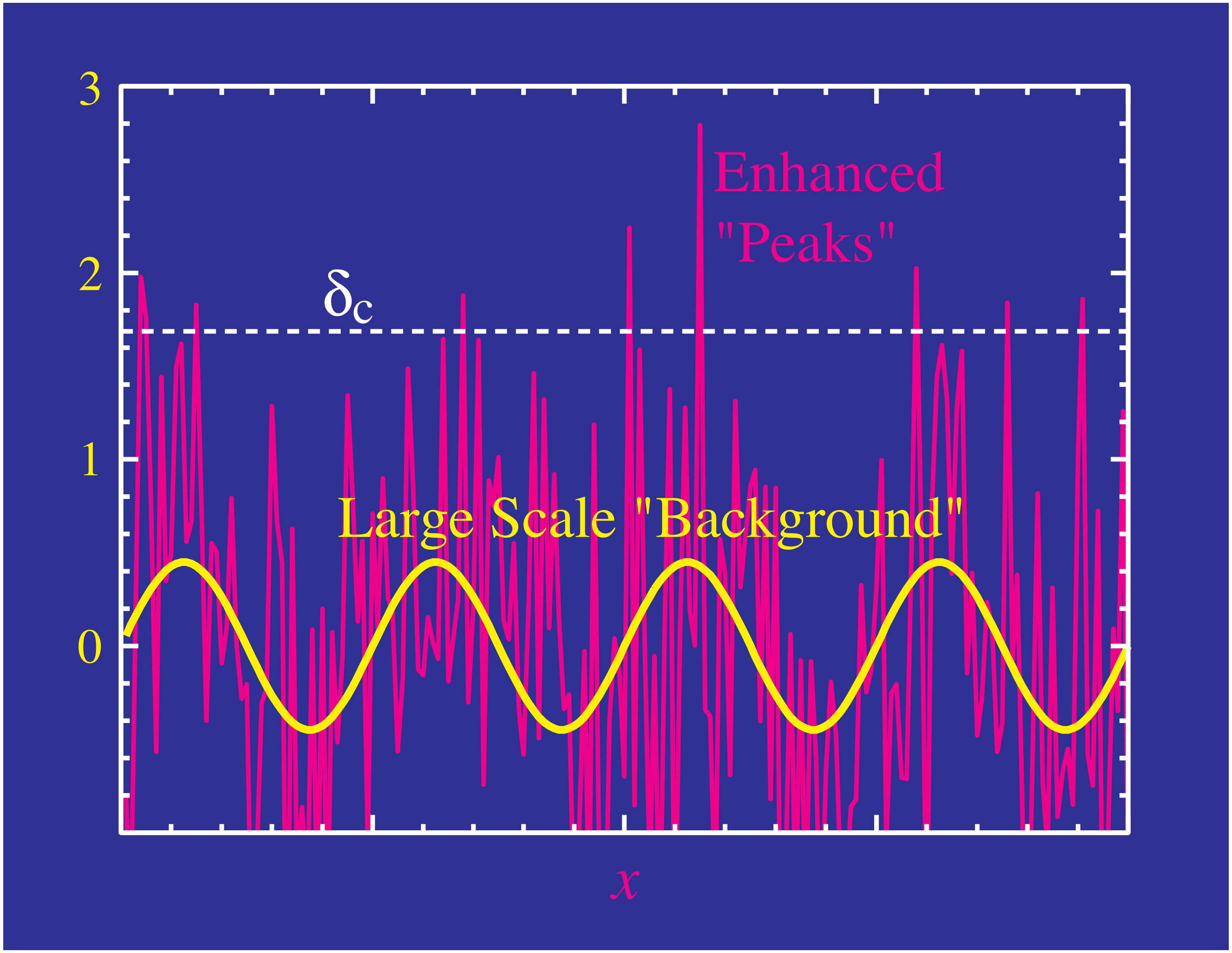
- If halos are formed without regard to the underlying density fluctuation and move under the gravitational field then their number density is an unbiased tracer of the dark matter density fluctuation

$$\left(\frac{\delta n}{n}\right)_{\rm halo} = \left(\frac{\delta\rho}{\rho}\right)$$

- However spherical collapse says the probability of forming a halo depends on the initial density field
- Large scale density field acts as “background” enhancement of probability of forming a halo or “peak”
- Peak-Background Split (Mo & White 1997)

Peak-Background Split

- Schematic Picture:



Perturbed Mass Function

- Density fluctuation split

$$\delta = \delta_b + \delta_p$$

- Lowers the threshold for collapse

$$\delta_{cp} = \delta_c - \delta_b$$

so that $\nu = \delta_{cp}/\sigma$

- Taylor expand number density $n_M \equiv dn/d\ln M$

$$n_M + \frac{dn_M}{d\nu}\frac{d\nu}{d\delta_b}\delta_b \ldots = n_M \left[1 + \frac{(\nu^2-1)}{\sigma\nu}\right]$$

if mass function is given by Press-Schechter

$$n_M \propto \nu \exp(-\nu^2/2)$$

Halo Bias

- Halos are biased tracers of the "background" dark matter field with a bias $b(M)$ that is given by spherical collapse and the form of the mass function

$$\frac{\delta n_M}{n_M} = [1 + b(M)]\,\delta$$

- For Press-Schechter

$$b(M) = 1 + \frac{\nu^2 - 1}{\delta_c}$$

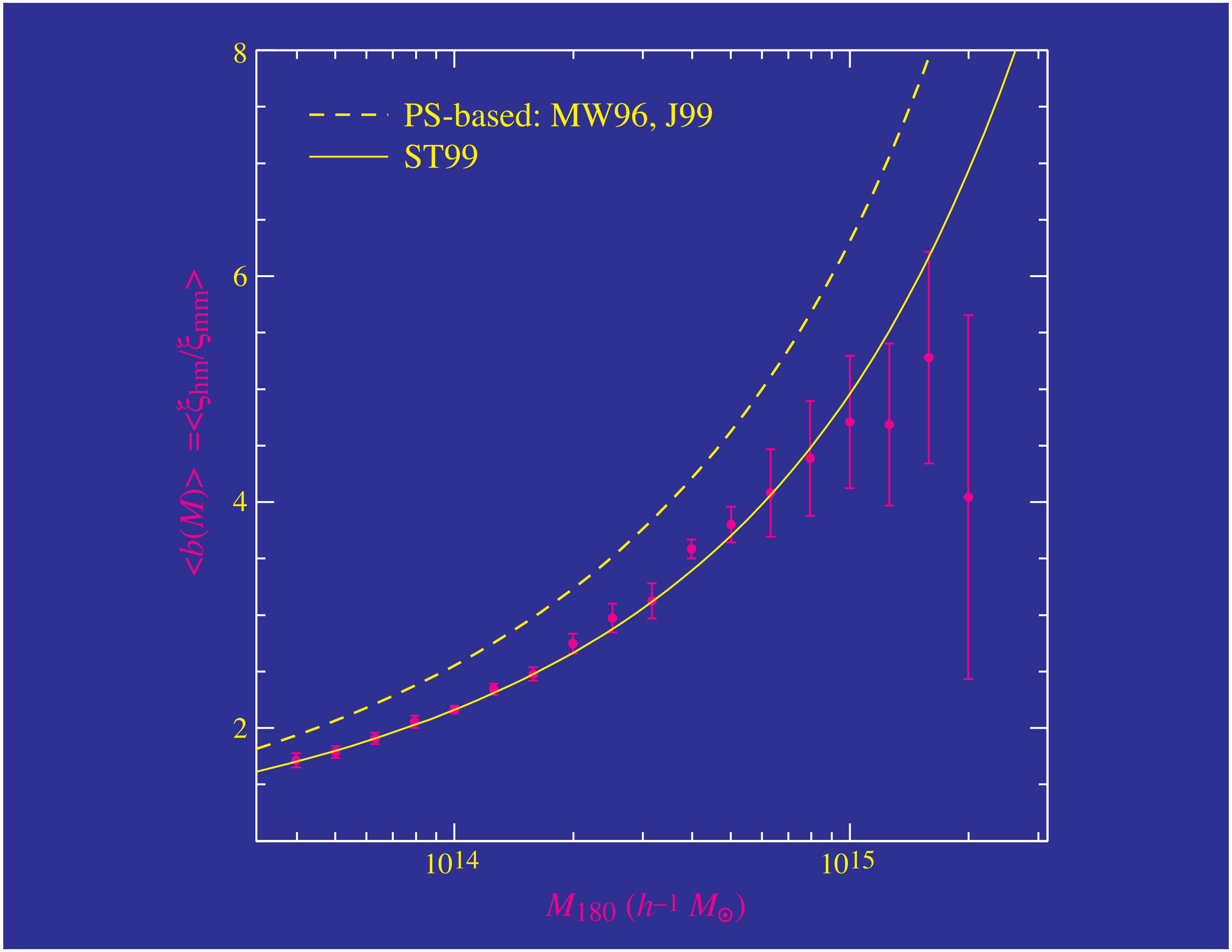
- Improved by the Sheth-Torman mass function

$$b(M) = 1 + \frac{a\nu^2 - 1}{\delta_c} + \frac{2p}{\delta_c[1 + (a\nu^2)^p]}$$

with $a = 0.75$ and $p = 0.3$ to match simulations.

Numerical Bias

- Example of halo bias from a simulation (from Hu & Kravstov 2002)



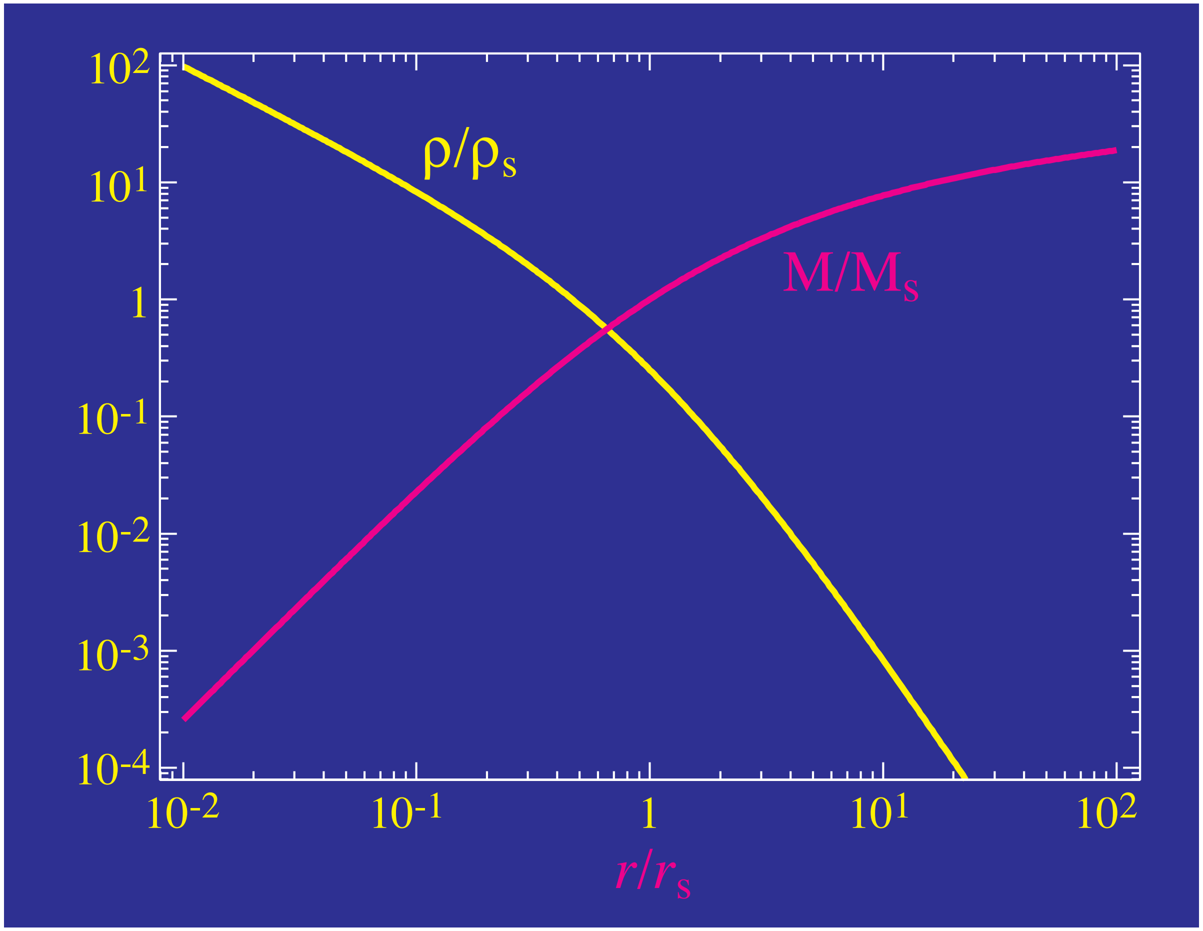
What is a Halo?

- Mass function and halo bias depend on the definition of mass of a halo
- Agreement with simulations depend on how halos are identified
- Other observables (associated galaxies, X-ray, SZ) depend on the details of the density profile
- Fortunately, simulations have shown that halos take on a near universal form in their density profile at least on large scales.

NFW Halo

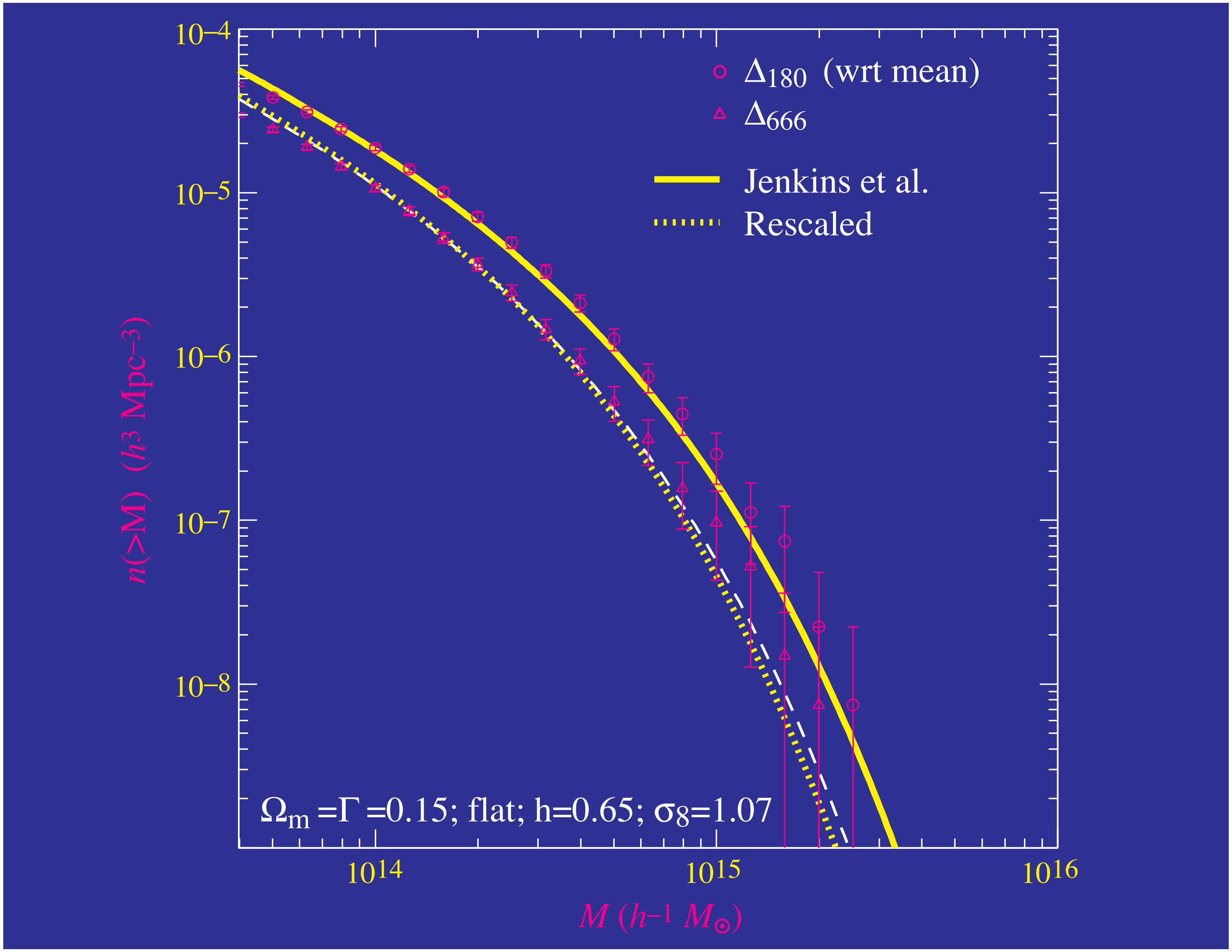
- Density profile well-described by (Navarro, Frenk & White 1997)

$$\rho(r) = \frac{\rho_s}{(r/r_s)(1 + r/r_s)^2}$$



Transforming the Masses

- NFW profile gives a way of transforming different mass definitions



Lack of Concentration?

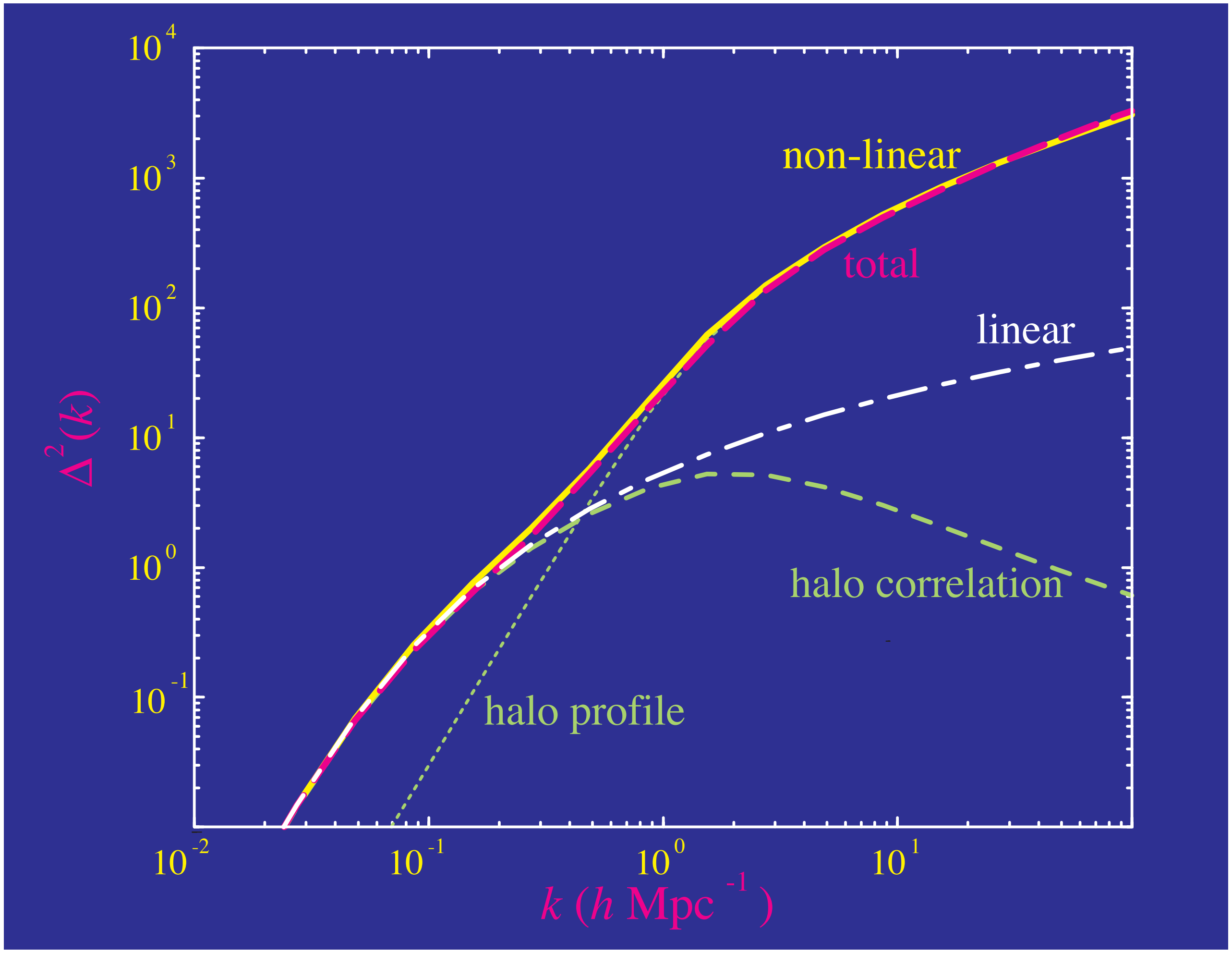
- NFW parameters may be recast into M_v, the mass of a halo out to the virial radius r_v where the overdensity wrt mean reaches $\Delta_v = 180$.
- Concentration parameter

$$c \equiv \frac{r_v}{r_s}$$

- CDM predicts $c \sim 10$ for M_* halos. Too centrally concentrated for galactic rotation curves?
- Possible discrepancy has lead to the exploration of dark matter alternatives: warm ($m \sim$keV) dark matter, self-interacting dark-matter, annihillating dark matter, ultra-light "fuzzy" dark matter, ...

The Halo Model

- NFW halos, of abundance n_M given by mass function, clustered according to the halo bias $b(M)$ and the linear theory $P(k)$
- Power spectrum example:



Incredible, Extensible Halo Model

- An industry developed to build semi-analytic models for wide variety of cosmological observables based on the halo model
- Idea: associate an observable (galaxies, gas, ...) with dark matter halos
- Let the halo model describe the statistics of the observable
- The overextended halo model?

Halo Temperature

- Motivate with isothermal distribution, correct from simulations

$$\rho(r) = \frac{\sigma^2}{2\pi G r^2}$$

- Express in terms of virial mass M_v enclosed at virial radius r_v

$$M_v = \frac{4\pi}{3} r_v \rho_m \Delta_v = \frac{2}{G} r_v \sigma^2$$

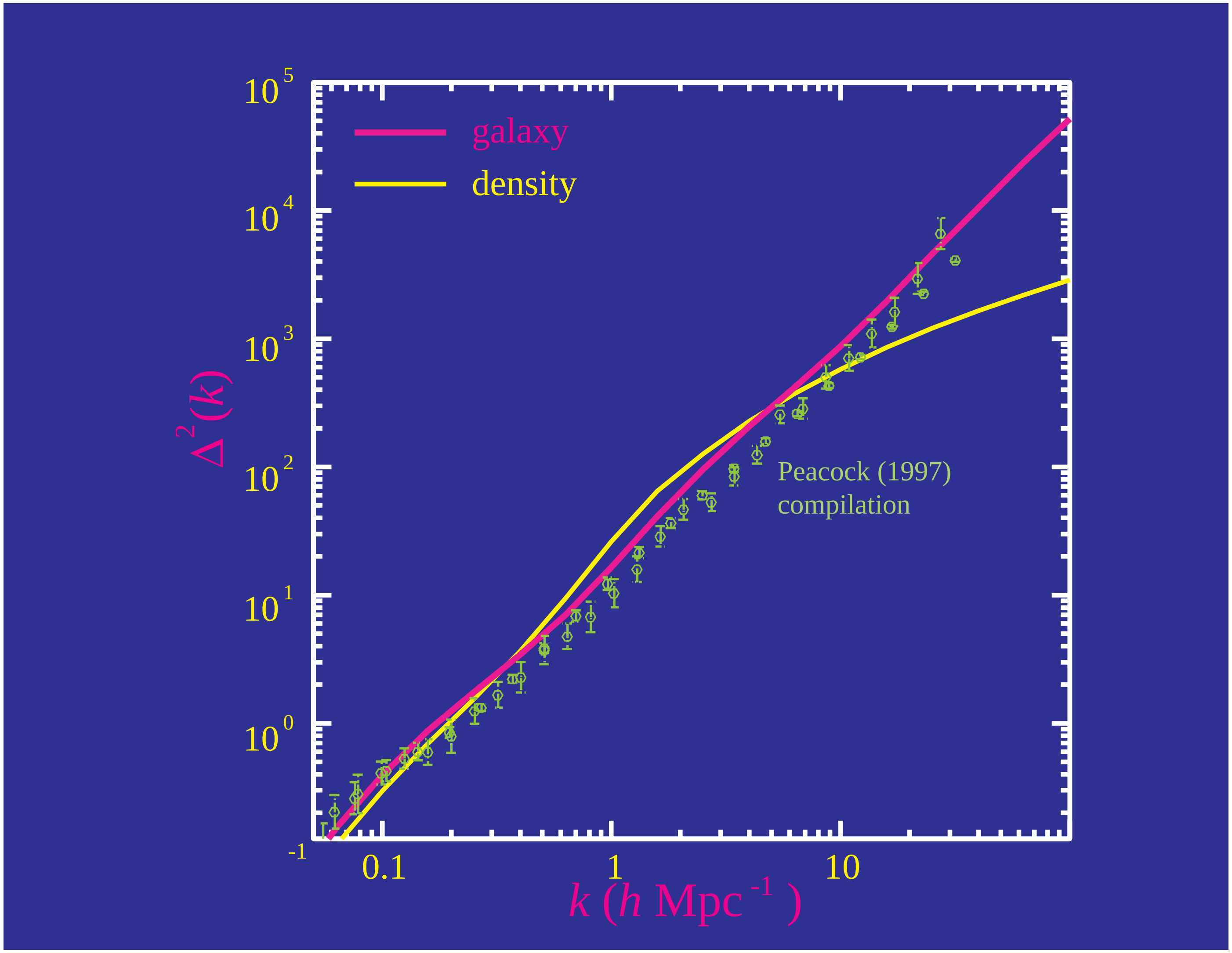
- Eliminate r_v, temperature $T \propto \sigma^2$ velocity dispersion^2
- Then $T \propto M_v^{2/3} (\rho_m \Delta_v)^{1/3}$ or

$$\left(\frac{M_v}{10^{15} h^{-1} M_\odot}\right) = \left[\frac{f}{(1+z)(\Omega_m \Delta_v)^{1/3}} \frac{T}{1\mathrm{keV}}\right]^{3/2}$$

- Theory (X-ray weighted): $f \sim 0.75$; observations $f \sim 0.54$. Difference is crucial in determining cosmology from cluster counts!

Galaxy Clustering

- Associate galaxies with halo of mass M: $N(M)$ (Seljak 2001)



- An explanation of the pure power law galaxy spectrum

Summary

- Dark matter simulations well-understood and can be modelled with dark matter halos
- Halo formation modelled by spherical collapse, two magic numbers $\delta_c = 1.686$ and $\Delta_v = 178$
- Halo abundance described by a mass function with exponential high mass cutoff – rare clusters extremely sensitive to power spectrum amplitude and growth rate $\rightarrow$ dark energy

  Possibly too many small halos or sub-structure?
- Halo clustering modelled with peak-background split leading to halo bias
- Halo profile described by NFW halos

  Possibly too high central concentration
- Associate an observable with a halo $\rightarrow$ a halo model