Astro 321: Problem Set 4
Due Feb. 5

This problem set will help prepare you for the discussion of inflation and the CMB final project.

1 Problem 1: "Gauge Transformations" and the Bardeen-ζ

Recall from class the general equations of motion in relativistic linear perturbation theory. [A flat universe is assumed throughout. Dots are conformal time derivatives and $w \equiv p/\rho$.]

The continuity/energy equation:

$$\dot\delta = -3\frac{\dot a}{a}\left(c_s^2 - w\right)\delta - (1+w)(kv + 3\dot\Phi)\,, \tag{1}$$

with $c_s^2 \equiv \delta p/\delta\rho$, where the fluctuation δp is not to be confused with $\delta \times p$. The Euler equation:

$$\dot v = -(1-3w)\frac{\dot a}{a}v - \frac{\dot w}{1+w}v + \frac{kc_s^2}{1+w}\delta - \frac{2}{3}\frac{w}{1+w}k\pi + k\Psi\,. \tag{2}$$

The Poisson equation:

$$\begin{aligned} k^2\Phi &= 4\pi G a^2\rho[\delta + 3\frac{\dot a}{a}(1+w)v/k]\,, \\ k^2(\Psi+\Phi) &= -8\pi G a^2 p\pi\,, \end{aligned} \tag{3}$$

and an redunant combo of these equations that you will find useful:

$$\left(\frac{\dot a}{a}\right)\Psi - \dot\Phi = 4\pi G a^2(\rho+p)v/k\,. \tag{4}$$

Although the representation of the system in these variables [called the "Newtonian gauge" or "longitudinal guage" system] is complete and best corresponds to our Newtonian intuition, it is inconvenient for both numerical and analytic work on certain problems. In particular the gravitational potentials Φ and Ψ and their time evolution are not simply related to the matter fields.

Let look for a more convenient representation. You may think of this operation as purely a change of variables but for the GR cognescenti, this operation is a gauge transformation (of a time shift v/k) and the transformed matter fluctuation fields are simply the matter fluctuation fields in a "comoving" set of coordinates.

Noting the form of the Poisson equation, define a new density perturbation

$$\Delta\rho = \delta\rho - \dot\rho v/k \tag{5}$$

- Show

$$\Delta \equiv \delta + 3\frac{\dot a}{a}(1+w)v/k \tag{6}$$

- Rewrite the continuity equation and show that

$$\dot\Delta = -3\frac{\dot a}{a}\left(C_s^2 - w\right)\Delta - (1+w)(kv + 3\dot\zeta)\,, \tag{7}$$

  where the transformed sound speed

$$\begin{aligned} C_s^2 &\equiv \frac{\Delta p}{\Delta\rho} \quad (8)\\ \Delta p &\equiv \delta p - \dot p v/k \quad (9) \end{aligned}$$

  (again don't confuse Δp with $\Delta \times p$), and the Bardeen curvature ζ

$$\zeta \equiv \Phi - \frac{\dot a}{a}v/k\,. \tag{10}$$

  Now the potential is defined simply in terms of the matter fields $k^2\Phi = 4\pi G a^2\rho\Delta$ as you would expect from Newtonian gravity but at the price of introducing $\dot\zeta$ into its evolution equation.

  The introduction of $\dot\zeta$ is in fact also useful in that it is also simply related to the matter fields.

- Show that

$$\begin{aligned} \dot{\zeta} &= \frac{\dot{a}}{a}\xi \\ \xi &= -\frac{C_s^2}{1+w}\Delta + \frac{2}{3}\frac{w}{1+w}\pi \,. \end{aligned} \tag{11}$$

Hint: differentiate the definition of ζ and use the Euler equation to eliminate terms. You may find the auxiliary equation (4) and the acceleration equation for $\ddot{a}$ useful.

Since ξ is then directly related to the stress fluctuations, what this says is that if stress fluctuations can be ignored, as they can always be outside the horizon, the variable ζ is a constant, *independently* of the nature of the matter fields. This enormously useful fact proven first by Bardeen (1980) allows us to ignore the details of many processes since once ζ is calculated you are done with perturbation theory on large scales!

For the CMB final project: this representation of the linear perturbation equations is numerically stable, unlike the original form.