Astro 321

Lecture Notes Set 1

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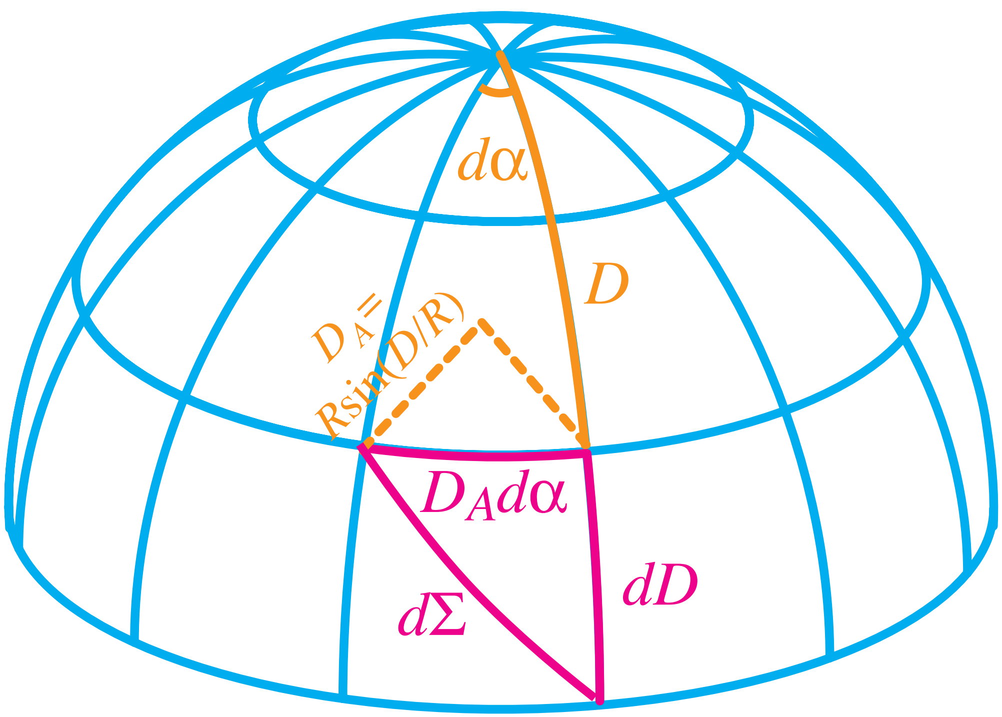
FRW Cosmology

- FRW cosmology = homogeneous and isotropic on large scales
- Universe observed to be nearly isotropic (e.g. CMB, radio point sources, galaxy surveys)
- Copernican principle: must be isotropic to all observers (all locations)
- Implies homogeneity; also galaxy redshift surveys (LCRS, 2dF, SDSS) have seen the "end of greatness", large scale homogeneity directly
- FRW cosmology (homogeneity, isotropy & Einstein equations) generically implies the expansion of the universe, except for special unstable cases

FRW Geometry

- Spatial geometry is that of a constant curvature (positive, negative, zero) surface
- Metric tells us how to measure distances on this surface
- Consider the closed geometry of a sphere of radius R and suppress one dimension



Angular Diameter Distance

- Spatial distance: restore 3rd dimension with the usual spherical polar angles

$$
\begin{aligned}
d\Sigma^2 &= dD^2 + D_A^2 d\alpha^2 \\
&= dD^2 + D_A^2 (d\theta^2 + \sin^2\theta d\phi^2)
\end{aligned}
$$

- D_A is called the angular diameter distance since $D_A d\alpha$ corresponds to the transverse separation or size as opposed to the Euclidean $D d\alpha$, i.e. is the apparent distance to an object through the gravitational lens of the background geometry
- In a positively curved geometry $D_A < D$ and objects are further than they appear
- In a negatively curved universe R is imaginary and $R \sin(D/R) = i|R| \sin(D/i|R|) = |R| \sinh(D/|R|)$ – and $D_A > D$ objects are closer than they appear

Volume Element

- Metric also defines the volume element

$$
\begin{aligned}
dV &= (dD)(D_A d\theta)(D_A \sin\theta d\phi) \\
&= D_A^2 dD d\Omega
\end{aligned}
$$

- Most of classical cosmology boils down to these three quantities, (comoving) distance, (comoving) angular diameter distance, and volume element
- For example, distance to a high redshift supernova, angular size of the horizon at last scattering, number density of clusters...

Comoving Coordinates

- Remaining degree of freedom (preserving homogeneity and isotropy) is an overall scale factor that relates the geometry (fixed by the radius of curvature R) to physical coordinates – a function of time only

$$d\sigma^2 = a^2(t)d\Sigma^2$$

  our conventions are that the scale factor today $a(t_0) \equiv 1$

- Similarly physical distances are given by $d(t) = a(t)D$, $d_A(t) = a(t)D_A$.
- Distances in capital case are *comoving* i.e. they comove with the expansion and do not change with time – simplest coordinates to work out geometrical effects

Time and Conformal Time

- Proper time

$$
\begin{aligned}
d\tau^2 &= dt^2 - d\sigma^2 \\
&= dt^2 - a^2(t) d\Sigma^2 \\
&\equiv a^2(t) \left(d\eta^2 - d\Sigma^2 \right)
\end{aligned}
$$

- Taking out the scale factor in the time coordinate $d\eta = dt/a$ defines conformal time – useful in that photons travelling radially from observer then obey

$$\Delta D = \Delta \eta = \int \frac{dt}{a}$$

so that time and distance may be interchanged

Horizon

- Distance travelled by a photon in the whole lifetime of the universe defines the horizon
- By $d\tau = 0$, the horizon is simply the conformal time elapsed

$$D_{\rm horizon}(t) = \int_0^t \frac{dt'}{a} = \eta(t)$$

- Since the horizon always grows with time, there is always a point in time before which two observers separated by a distance D could not have been in causal contact
- Horizon problem: why is the universe homogeneous and isotropic on large scales, near the current horizon – problem deepens for objects seen at early times, e.g. CMB

FRW Metric

- Proper time defines the metric $g_{\mu\nu}$

$$d\tau^2 \equiv g_{\mu\nu} dx^\mu dx^\nu$$

  signature follows Peacock's convention. Caveat reader: this is opposite to what I'm used to so I *will* occasionally mess up the sign
- Usually we will use comoving coordinates and conformal time as the "x" 's unless otherwise specified – metric for other choices are related by $a(t)$
- We will generally skirt around real General Relativity but rudimentary knowledge will be useful

Hubble Parameter

- Useful to define the expansion rate or Hubble parameter

$$H(t) \equiv \frac{1}{a}\frac{da}{dt}$$

  since dynamics (Einstein equations) will give this directly as $H(a) \equiv H(t(a))$

- Time becomes

$$t = \int dt = \int \frac{da}{aH(a)}$$

- Conformal time becomes

$$\eta = \int \frac{dt}{a} = \int \frac{da}{a^2 H(a)}$$

Redshift

- Wavelength of light "stretches" with the scale factor, so that it is convenient to define a shift-to-the-red or redshift as the scale factor increases

$$\lambda(a) = a(t)\Lambda$$
$$\frac{\lambda(1)}{\lambda(a)} = \frac{1}{a} \equiv (1+z)$$
$$\frac{\delta\lambda}{\lambda} = -\frac{\delta\nu}{\nu} = z$$

- Given known frequency of emission $\nu(a)$, redshift can be precisely measured (modulo Doppler shifts from peculiar velocities) – interpreting the redshift as a Doppler shift, objects receed in an expanding universe
- Given a measure of distance, $D(z) \equiv D(z(a))$ can be measured

Distance-Redshift Relation

- All distance redshift relations are based on the comoving distance $D(z)$

$$D(a) = \int dD = \int_a^1 \frac{da'}{a'^2 H(a')}$$

$$(da = -(1+z)^{-2} dz = -a^2 dz)$$

$$D(z) = -\int_z^0 \frac{dz'}{H(z')} = \int_0^z \frac{dz'}{H(z')}$$

- Note limiting case is the Hubble law

$$\lim_{z \to 0} D(z) = z/H(z=0) \equiv z/H_0$$

- Hubble constant usually quoted as $H_0 = 100h$ km s^{-1} Mpc^{-1}, observationally $h \sim 0.7$; in natural units $H_0 = (2997.9)^{-1} h$ Mpc^{-1} defines an inverse length scale

Distance-Redshift Relation

- Example: object of known physical size $\lambda = a(t)\Lambda$ ("standard ruler") subtending an (observed) angle on the sky α

$$\alpha = \frac{\Lambda}{D_A(z)} = \frac{\lambda}{aR\sin(D(z)/R)}$$
$$= \frac{\lambda}{R\sin(D(z)/R)}(1+z) \equiv \frac{\lambda}{d_A(z)}$$

- Example: object of known luminosity L ("standard candle") with a measured flux F. Comoving surface area $4\pi D_A^2$, frequency/energy $(1+z)$, time-dilation or arrival rate of photons (crests) $(1+z)$:

$$F = \frac{L}{4\pi D_A^2}\frac{1}{(1+z)^2}$$
$$\equiv \frac{L}{4\pi d_L^2} \quad (d_L = (1+z)D_A = (1+z)^2 d_A)$$

Absolute calibration

- If absolute calibration of standards unknown, then Hubble constant not measured
- Still measures evolution of Hubble parameter $H(z)/H_0$:

$$\frac{d_{A,L}(z)}{d_{A,L}(\delta z)} = \frac{H_0}{\delta z} d_{A,L}(z)$$

- Alternately, distances & curvature are measured in units of h^{-1} Mpc.
- Fundamental dependence (aside from $(1+z)$ factors)

$$\begin{aligned} H_0 D_A(z) &= H_0 R \sin(D(z)/R) \\ &= \tilde{R} \sin(H_0 D(z)/\tilde{R}), \quad \tilde{R} = H_0 R \end{aligned}$$

$$H_0 D(z) = \int \frac{da}{a^2} \frac{H_0}{H(a)}$$

Evolution of Scale Factor

- FRW cosmology is fully specified if the function $a(t)$ is given
- General relativity relates the scale factor with the matter content of universe.
- Build the Einstein tensor $G_{\mu\nu}$ out of the metric and use Einstein equation

$$G_{\mu\nu} = -8\pi G T_{\mu\nu}$$

$$G^0{}_0 = -\frac{3}{a^2}\left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{1}{R^2}\right]$$

$$G^i{}_j = -\frac{1}{a^2}\left[2\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 + \frac{1}{R^2}\right]\delta^i{}_j$$

Einstein Equations

- Isotropy demands that the stress-energy tensor take the form

$$T^0{}_0 = \rho$$
$$T^i{}_j = -p\delta^i{}_j$$

where ρ is the energy density and p is the pressure

- So Einstein equations become

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{1}{R^2} = \frac{8\pi G}{3}a^2\rho$$
$$2\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 + \frac{1}{R^2} = -8\pi G a^2 p$$
$$\text{or} \quad \frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 = -\frac{4\pi G}{3}a^2(\rho + 3p)$$

Friedman Equations

- More usual to see Einstein equations expressed in time not conformal time

$$\frac{\dot a}{a} = \frac{da}{d\eta}\frac{1}{a} = \frac{da}{dt} = aH(a)$$

$$\frac{\ddot a}{a} - \left(\frac{\dot a}{a}\right)^2 = \frac{d}{d\eta}\left(\frac{\dot a}{a}\right) = a\frac{d}{dt}\left(\frac{da}{dt}\right) = a\frac{d^2a}{dt^2}$$

- Friedmann equations:

$$H^2(a) + \frac{1}{a^2R^2} = \frac{8\pi G}{3}\rho$$

$$\frac{1}{a}\frac{d^2a}{dt^2} = -\frac{4\pi G}{3}(\rho + 3p)$$

- Convenient fiction to describe curvature as an energy density component $\rho_K = -3/(8\pi G a^2 R^2) \propto a^{-2}$

Critical Density

- Friedmann equation for H then reads

$$H^2(a) = \frac{8\pi G}{3}(\rho + \rho_K) \equiv \frac{8\pi G}{3}\rho_c$$

  defining a critical density today ρ_c in terms of the expansion rate

- In particular, its value today is given by the Hubble constant as

$$\rho_{\rm c}(z=0) = 3H_0^2/8\pi G = 1.8788 \times 10^{-29} h^2 {\rm g\,cm}^{-3}$$

- Energy density today is given as a fraction of critical $\Omega \equiv \rho/\rho_c|_{z=0}$. Radius of curvature then given by $R^{-2} = H_0^2(\Omega - 1)$

- If $\Omega \approx 1$, $\rho \approx \rho_c$, then $\rho_K \ll \rho_c$ or $H_0 R \ll 1$, universe is flat across the Hubble distance. $\Omega < 1$ negatively curved; $\Omega > 1$ positively curved

Newtonian Interpretation

- Consider a test particle of mass m in expanding spherical region of radius r and total mass M. Energy conservation

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r} = \mathrm{const}$$

$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2 - \frac{GM}{r} = \mathrm{const}$$

$$\frac{1}{2}\left(\frac{1}{r}\frac{dr}{dt}\right)^2 - \frac{GM}{r^3} = \frac{\mathrm{const}}{r^2}$$

$$H^2 = \frac{8\pi G\rho}{3} - \frac{\mathrm{const}}{a^2}$$

- Constant determines whether the system is bound and in the Friedmann equation is associated with curvature – not general since neglects pressure

Conservation Law

- Second Friedmann equation, or acceleration equation, simply expresses energy conservation (why: stress energy is automatically conserved in GR via Bianchi identity)

$$
\begin{aligned}
d\rho V + p dV &= 0 \\
d\rho a^3 + p d a^3 &= 0 \\
\dot{\rho} a^3 + 3\frac{\dot{a}}{a}\rho a^3 + 3\frac{\dot{a}}{a} p a^3 &= 0
\end{aligned}
$$

$$
\frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a} \qquad w \equiv p/\rho
$$

- If $w =$ const. then the energy density depends on the scale factor as $\rho \propto a^{-3(1+w)}$.

Multicomponent Universe

- The total energy density can be composed of a sum of components with differing equations of state

$$\rho(a) = \sum_i \rho_i(a) = \sum_i \rho_i(a=1) a^{-3(1+w_i)}, \quad \Omega_i \equiv \rho_i/\rho_c|_{a=1}$$

- Important cases: nonrelativistic matter $\rho_m = mn_m \propto a^{-3}$, $w_m = 0$; relativistic radiation $\rho_r = En_r = \nu n_r \propto a^{-4}$, $w_r = 1/3$; "curvature" $\rho_K \propto a^{-2}$, $w_K = -1/3$; constant energy density or cosmological constant $\rho_\Lambda \propto a^0$, $w_\Lambda = -1$
- Or generally with $w_c = p_c/\rho_c = (p + p_K)/(\rho + \rho_K)$

$$\rho_c(a) = \rho_c(a=1) e^{-\int d\ln a\, 3(1+w_c(a))}$$

$$H^2(a) = H_0^2 e^{-\int d\ln a\, 3(1+w_c(a))}$$

Acceleration Equation

- Time derivative of (first) Friedman equation

$$
\begin{aligned}
2\frac{1}{a}\frac{da}{dt}\left[\frac{1}{a}\frac{d^2a}{dt^2} - H^2(a)\right] &= \frac{8\pi G}{3}\frac{d\rho_c}{dt} \\
\left[\frac{1}{a}\frac{d^2a}{dt^2} - \frac{8\pi G}{3}\rho_c\right] &= \frac{4\pi G}{3}[-3(1+w_c)\rho_c] \\
\frac{1}{a}\frac{d^2a}{dt^2} &= -\frac{4\pi G}{3}[(1+3w_c)\rho_c] \\
&= -\frac{4\pi G}{3}(\rho + \rho_K + 3p + 3p_K) \\
&= -\frac{4\pi G}{3}(1+3w)\rho
\end{aligned}
$$

- Acceleration equation says that universe decelerates if $w > -1/3$

Expansion Required

- Friedmann equations "predict" the expansion of the universe. Non-expanding conditions $da/dt = 0$ and $d^2a/dt^2 = 0$ require

$$\rho = -\rho_K \qquad \rho = -3p$$

i.e. a positive curvature and a total equation of state $w \equiv p/\rho = -1/3$

- Since matter is known to exist, one can in principle achieve this with

$$\rho = \rho_m + \rho_\Lambda = -\rho_K = -3p = 3\rho_\Lambda$$

$$\rho_\Lambda = -\frac{1}{3}\rho_K \quad \rho_m = -\frac{2}{3}\rho_K$$

Einstein introduced ρ_Λ for exactly this reason – "biggest blunder"; but note that this balance is unstable: ρ_m can be perturbed but ρ_Λ, a true constant cannot

Dark Energy

- Distance redshift relation depends on energy density components

$$H_0 D(z) = \int \frac{da}{a^2} \frac{H_0}{H(a)}$$
$$= \int \frac{da}{a^2} e^{\int d\ln a \, \frac{3}{2}(1+w_c(a))}$$

- Distant supernova Ia as standard candles imply that $w_c < -1/3$ so that the expansion is accelerating
- Consistent with a cosmological constant that is $\Omega_\Lambda = \rho_\Lambda/\rho_{\rm crit} = 2/3$ of the total energy density
- Coincidence problem: different components of matter scale differently with a. Why are (at least) two components comparable today? – Anthropic?

Dark Matter

- Since Zwicky in the 1930's non-luminous or dark matter has been known to dominate over luminous matter in stars (and hot gas)
- Arguments are basically from a balance of gravitational force against "pressure" from internal motions: rotation velocity in galactic disks, velocity dispersion of galaxies in clusters, gas pressure in clusters, radiation pressure in CMB
- Assuming that the object is somehow typical in its non-luminous to luminous density, these measures are converted to an overall dark matter density through a "mass-to-light ratio"
- From galaxy surveys the luminosity density in solar units is

$$\rho_L = 2 \pm 0.7 \times 10^8 h \, L_\odot \mathrm{Mpc}^{-3}$$

(h's: distances in h^{-1} Mpc; luminosity inferred from flux $L \propto Sd^2 \propto h^{-2}$; inverse volume $\propto h^3$)

Dark Matter

- Critical density in solar units is $\rho_c = 2.7754 \times 10^{11} h^2\, M_\odot \mathrm{Mpc}^{-3}$ so that the critical mass-to-light ratio in solar units is

$$\left(\frac{M}{L}\right) \approx 1400h$$

- Flat rotation curves: $GM/r^2 \approx v^2/r \rightarrow M \approx v^2 r/G$, so the observed flat rotation curve implies $M \propto r$ out to $30h^{-1}$ kpc, beyond the light. Implies $M/L > 30h$ and perhaps more – closure if flat out to ~ 1 Mpc.

- Similar argument holds in clusters of galaxies where velocity dispersion replaces circular velocity and centripetal force is replaced by a "pressure gradient" $T/m = \sigma^2$ and $p = \rho T/m = \rho\sigma^2$– generalization of hydrostatic equilibrium: Zwicky got $M/L \approx 300$h.

Hydrostatic Equilibrium

- Evidence for dark matter in X-ray clusters also comes from direct hydrostatic equilibrium inference from the gas: balance radial pressure gradient with gravitational potential gradient
- Infinitesimal volume of area dA and thickness dr at radius r and interior mass $M(r)$: pressure difference supports the gas

$$[p_g(r) - p_g(r+dr)]dA = \frac{GmM}{r^2} = \frac{G\rho_g M}{r^2} dV$$
$$\frac{dp_g}{dr} = -\rho_g \frac{d\Phi}{dr}$$

with $p_g = \rho_g T_g / m$ becomes

$$\frac{GM}{r} = -\frac{T_g}{m}\left(\frac{d\ln\rho_g}{d\ln r} + \frac{d\ln T_g}{d\ln r}\right)$$

- ρ_g from X-ray luminosity; T_g sometimes taken as isothermal

Gravitational Lensing

- Mass can be directly measured in the gravitational lensing of sources behind the cluster
- Strong lensing (giant arcs) probes central region of clusters
- Weak lensing (1-10%) elliptical distortion to galaxy image probes outer regions of cluster and total mass
- All techniques agree on the necessity of dark matter and are roughly consistent with a dark matter density $\Omega_m \sim 0.2 - 0.4$
- $\Omega_m + \Omega_\Lambda \approx 1$ from matter density + dark energy
- CMB provides a test of $D_A \neq D$ through the standard rulers of the acoustic peaks and shows that the universe is close to flat $\Omega \approx 1$
- Consistency has lead to the standard model for the cosmological matter budget