

## 1 Problem 1: “Gauge Transformations” and the Bardeen- $\zeta$

Recall from class the general equations of motion in relativistic linear perturbation theory. [A flat universe is assumed throughout. Dots are conformal time derivatives and  $w \equiv p/\rho$ .]

The continuity/energy equation:

$$\dot{\delta} = -3\frac{\dot{a}}{a}(c_s^2 - w)\delta - (1+w)(kv + 3\dot{\Phi}), \quad (1)$$

with  $c_s^2 \equiv \delta p/\delta\rho$ , where the fluctuation  $\delta p$  is not to be confused with  $\delta \times p$ . The Euler equation:

$$\dot{v} = -(1-3w)\frac{\dot{a}}{a}v - \frac{\dot{w}}{1+w}v + \frac{kc_s^2}{1+w}\delta - \frac{2}{3}\frac{w}{1+w}k\pi + k\Psi. \quad (2)$$

The Poisson equation:

$$\begin{aligned} k^2\Phi &= 4\pi G a^2 \rho [\delta + 3\frac{\dot{a}}{a}(1+w)v/k], \\ k^2(\Psi + \Phi) &= -8\pi G a^2 p \pi, \end{aligned} \quad (3)$$

and an redundant combo of these equations that you will find useful:

$$\left(\frac{\dot{a}}{a}\right)\Psi - \dot{\Phi} = 4\pi G a^2(\rho + p)v/k. \quad (4)$$

Although the representation of the system in these variables [called the “Newtonian gauge” or “longitudinal gauge” system] is complete and best corresponds to our Newtonian intuition, it is inconvenient for both numerical and analytic work on certain problems. In particular the gravitational potentials  $\Phi$  and  $\Psi$  and their time evolution are not simply related to the matter fields.

Let look for a more convenient representation. You may think of this operation as purely a change of variables but for the GR cognescenti, this operation is a gauge transformation (of a time shift  $v/k$ ) and the transformed matter fluctuation fields are simply the matter fluctuation fields in a “comoving” set of coordinates.

Noting the form of the Poisson equation, define a new density perturbation

$$\Delta\rho = \delta\rho - \dot{p}v/k \quad (5)$$

- Show

$$\Delta \equiv \delta + 3\frac{\dot{a}}{a}(1+w)v/k \quad (6)$$

- Rewrite the continuity equation and show that

$$\dot{\Delta} = -3\frac{\dot{a}}{a}(C_s^2 - w)\Delta - (1+w)(kv + 3\dot{\zeta}), \quad (7)$$

where the transformed sound speed

$$C_s^2 \equiv \frac{\Delta p}{\Delta\rho} \quad (8)$$

$$\Delta p \equiv \delta p - \dot{p}v/k \quad (9)$$

(again don’t confuse  $\Delta p$  with  $\Delta \times p$ ), and the Bardeen curvature  $\zeta$

$$\zeta \equiv \Phi - \frac{\dot{a}}{a}v/k. \quad (10)$$

Now the potential is defined simply in terms of the matter fields  $k^2\Phi = 4\pi G a^2 \rho \Delta$  as you would expect from Newtonian gravity but at the price of introducing  $\dot{\zeta}$  into its evolution equation.

The introduction of  $\dot{\zeta}$  is in fact also useful in that it is also simply related to the matter fields.

- Show that

$$\begin{aligned} \dot{\zeta} &= \frac{\dot{a}}{a}\xi \\ \xi &= -\frac{C_s^2}{1+w}\Delta + \frac{2}{3}\frac{w}{1+w}\pi. \end{aligned} \quad (11)$$

Hint: differentiate the definition of  $\zeta$  and use the Euler equation to eliminate terms. You may find the auxiliary equation (4) and the acceleration equation for  $\ddot{a}$  useful.

Since  $\xi$  is then directly related to the stress fluctuations, what this says is that if stress fluctuations can be ignored, as they can always be outside the horizon, the variable  $\zeta$  is a constant, *independently* of the nature of the matter fields. This enormously useful fact proven first by Bardeen (1980) allows us to ignore the details of many processes since once  $\zeta$  is calculated you are done with perturbation theory on large scales!

For the CMB final project: this representation of the linear perturbation equations is numerically stable, unlike the original form.