From Inflation to Horizon Entry

- Inflation provides a source of nearly scale invariant comoving curvature fluctuations $\mathcal{R}$ or equivalently gravitational potential fluctuations $\Psi$ as well as gravitational waves $h_{+,\times}$.

- Fluctuations are frozen outside while the mode is outside the horizon.

- Upon horizon (re)entry, causal microphysics of interaction and particle propagation alters the initial spectrum.

- Initial fluctuations transferred to observable fluctuations through transfer functions that encode these processes.

- For the CMB, Thomson scattering is the dominant process and converts a scale free spectrum in $k$ to one with 3 fundamental scales in multipole $\ell$: acoustic scale, equality scale, damping scale.
CMB Temperature Fluctuations

- Angular Power Spectrum

Angular Scale

Multiplicity moment $(l)$

\[ l(l+1)C_l/2\pi \ (\mu K^2) \]

Model

- WMAP
- CBI
- ACBAR

Multipole moment $(l)$
- Take apart features in the power spectrum
Last Scattering

- Angular distribution of radiation is the 3D temperature field projected onto a shell - surface of last scattering
- Shell radius is distance from the observer to recombination: called the last scattering surface
- Take the radiation distribution at last scattering to also be described by an isotropic temperature fluctuation field $\Theta(x)$
Angular Power Spectrum

- Take recombination to be instantaneous

\[
\Theta(\hat{n}) = \int dD \, \Theta(\mathbf{x}) \delta(D - D_*)
\]

where \(D\) is the comoving distance and \(D_*\) denotes recombination.

- Describe the temperature field by its Fourier moments

\[
\Theta(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} \Theta(k) e^{i\mathbf{k} \cdot \mathbf{x}}
\]

- Power spectrum

\[
\langle \Theta(k)^* \Theta(k') \rangle = (2\pi)^3 \delta(k - k') P_T(k)
\]

\[
\Delta^2_T = k^3 P_T / 2\pi^2
\]
Angular Power Spectrum

- Temperature field

\[ \Theta(\hat{n}) = \int \frac{d^3k}{(2\pi)^3} \Theta(k) e^{i k \cdot D_* \hat{n}} \]

- Multipole moments \( \Theta(\hat{n}) = \sum_{\ell m} \Theta_{\ell m} Y_{\ell m} \)

- Expand out plane wave in spherical coordinates

\[ e^{i k D_* \hat{n}} = 4\pi \sum_{\ell m} i^\ell j_\ell(k D_*) Y_{\ell m}^*(k) Y_{\ell m}(\hat{n}) \]
Angular Power Spectrum

- Power spectrum

\[
\Theta_{\ell m} = \int \frac{d^3 k}{(2\pi)^3} \Theta(k) 4\pi i^{\ell} j_\ell(k D_*) Y^*_\ell m(k)
\]

\[
\langle \Theta^*_{\ell m} \Theta_{\ell' m'} \rangle = \int \frac{d^3 k}{(2\pi)^3} (4\pi)^2 i^{\ell - \ell'} j_\ell(k D_*) j_{\ell'}(k D_*) Y_{\ell m}(k) Y^*_{\ell' m'}(k) P_T(k)
\]

\[
= \delta_{\ell \ell'} \delta_{m m'} 4\pi \int d \ln k \, j^2_\ell(k D_*) \Delta^2_T(k)
\]

with \( \int_0^\infty j^2_\ell(x) d \ln x = 1/(2\ell(\ell + 1)) \), slowly varying \( \Delta^2_T \)

- Angular power spectrum:

\[
C_\ell = \frac{4\pi \Delta^2_T(\ell/D_*)}{2\ell(\ell + 1)} = \frac{2\pi}{\ell(\ell + 1)} \Delta^2_T(\ell/D_*)
\]
Thomson Scattering

- Thomson scattering of photons off of free electrons is the most important CMB process with a cross section (averaged over polarization states) of

\[ \sigma_T = \frac{8\pi\alpha^2}{3m_e^2} = 6.65 \times 10^{-25}\text{cm}^2 \]

- Density of free electrons in a fully ionized $x_e = 1$ universe

\[ n_e = (1 - Y_p/2)x_e n_b \approx 10^{-5}\Omega_b h^2(1 + z)^3\text{cm}^{-3}, \]

where $Y_p \approx 0.24$ is the Helium mass fraction, creates a high (comoving) Thomson opacity

\[ \dot{\tau} \equiv n_e\sigma_T a \]

where dots are conformal time $\eta \equiv \int dt/a$ derivatives and $\tau$ is the optical depth.
Tight Coupling Approximation

- Near recombination $z \approx 10^3$ and $\Omega_b h^2 \approx 0.02$, the (comoving) mean free path of a photon

$$\lambda_C \equiv \frac{1}{\tau} \sim 2.5\text{Mpc}$$

small by cosmological standards!

- On scales $\lambda \gg \lambda_C$ photons are tightly coupled to the electrons by Thomson scattering which in turn are tightly coupled to the baryons by Coulomb interactions

- Specifically, their bulk velocities are defined by a single fluid velocity $v_\gamma = v_b$ and the photons carry no anisotropy in the rest frame of the baryons

- $\rightarrow$ No heat conduction or viscosity (anisotropic stress) in fluid
Zeroth Order Approximation

- Momentum density of a fluid is \((\rho + p)v\), where \(p\) is the pressure

- Neglect the momentum density of the baryons

\[
R \equiv \frac{(\rho_b + p_b)v_b}{(\rho_\gamma + p_\gamma)v_\gamma} = \frac{\rho_b + p_b}{\rho_\gamma + p_\gamma} = \frac{3\rho_b}{4\rho_\gamma}
\]

\[
\approx 0.6 \left( \frac{\Omega_b h^2}{0.02} \right) \left( \frac{a}{10^{-3}} \right)
\]

since \(\rho_\gamma \propto T^4\) is fixed by the CMB temperature \(T = 2.73(1 + z)K\) – OK substantially before recombination

- Neglect radiation in the expansion

\[
\frac{\rho_m}{\rho_r} = 3.6 \left( \frac{\Omega_m h^2}{0.15} \right) \left( \frac{a}{10^{-3}} \right)
\]

- Neglect gravity
Fluid Equations

- Density $\rho_{\gamma} \propto T^4$ so define temperature fluctuation $\Theta$
  \[ \delta_{\gamma} = 4 \frac{\delta T}{T} \equiv 4\Theta \]

- Real space continuity equation
  \[ \dot{\delta}_{\gamma} = -(1 + w_{\gamma}) k\nu_{\gamma} \]
  \[ \dot{\Theta} = -\frac{1}{3} k\nu_{\gamma} \]

- Euler equation (neglecting gravity)
  \[ \dot{v}_{\gamma} = -(1 - 3w_{\gamma}) \frac{\dot{a}}{a} v + \frac{kc_s^2}{1 + w_{\gamma}} \delta_{\gamma} \]
  \[ \dot{v}_{\gamma} = kc_s^2 \frac{3}{4} \delta_{\gamma} = 3c_s^2 k\Theta \]
Oscillator: Take One

- Combine these to form the simple harmonic oscillator equation

\[ \ddot{\Theta} + c_s^2 k^2 \Theta = 0 \]

where the sound speed is adiabatic

\[ c_s^2 = \frac{\delta p}{\delta \rho} = \frac{\dot{p}_\gamma}{\dot{\rho}_\gamma} \]

here \( c_s^2 = 1/3 \) since we are photon-dominated

- General solution:

\[ \Theta(\eta) = \Theta(0) \cos(ks) + \frac{\dot{\Theta}(0)}{kc_s} \sin(ks) \]

where the sound horizon is defined as \( s \equiv \int c_s d\eta \)
Harmonic Extrema

- All modes are \textbf{frozen} in at recombination (denoted with a subscript \(*\))
- Temperature perturbations of \textbf{different amplitude} for different modes.
- For the \textbf{adiabatic} (curvature mode) initial conditions

\[
\dot{\Theta}(0) = 0
\]

- So solution

\[
\Theta(\eta_*) = \Theta(0) \cos(ks_*)
\]
Harmonic Extrema

- Modes caught in the **extrema** of their oscillation will have enhanced fluctuations

\[ k_n s_* = n\pi \]

yielding a **fundamental scale** or frequency, related to the inverse sound horizon

\[ k_A = \pi / s_* \]

and a **harmonic relationship** to the other extrema as \( 1 : 2 : 3 \ldots \)
Peak Location

- The fundamental **physical scale** is translated into a fundamental **angular scale** by simple projection according to the angular diameter distance $D_A$

$$\theta_A = \lambda_A / D_A$$

$$\ell_A = k_A D_A$$

- In a flat universe, the distance is simply $D_A = D \equiv \eta_0 - \eta_* \approx \eta_0$, the horizon distance, and $k_A = \pi / s_* = \sqrt{3\pi} / \eta_*$ so

$$\theta_A \approx \frac{\eta_*}{\eta_0}$$

- In a **matter-dominated** universe $\eta \propto a^{1/2}$ so $\theta_A \approx 1/30 \approx 2^\circ$ or

$$\ell_A \approx 200$$
In a curved universe, the apparent or angular diameter distance is no longer the conformal distance $D_A = R \sin(D/R) \neq D$.

Objects in a closed universe are further than they appear! gravitational lensing of the background...

Curvature scale of the universe must be substantially larger than current horizon.
• Flat universe indicates critical density and implies missing energy given local measures of the matter density “dark energy”

• $D$ also depends on dark energy density $\Omega_{DE}$ and equation of state $w = p_{DE}/\rho_{DE}$.

• Expansion rate at recombination or matter-radiation ratio enters into calculation of $k_A$. 
Doppler Effect

- **Bulk motion** of fluid changes the observed temperature via Doppler shifts

\[
\left( \frac{\Delta T}{T} \right)_{\text{dop}} = \hat{n} \cdot \mathbf{v}_\gamma
\]

- Averaged over directions

\[
\left( \frac{\Delta T}{T} \right)_{\text{rms}} = \frac{\mathbf{v}_\gamma}{\sqrt{3}}
\]

- Acoustic solution

\[
\frac{\mathbf{v}_\gamma}{\sqrt{3}} = -\frac{\sqrt{3}}{k} \dot{\Theta} = \frac{\sqrt{3}}{k} k c_s \Theta(0) \sin(ks) = \Theta(0) \sin(ks)
\]
Doppler Peaks?

- **Doppler effect** for the photon dominated system is of equal amplitude and $\pi/2$ out of phase: extrema of temperature are turning points of velocity.

- Effects add in quadrature:

  \[
  \left( \frac{\Delta T}{T} \right)^2 = \Theta^2(0) \left[ \cos^2(ks) + \sin^2(ks) \right] = \Theta^2(0)
  \]

- **No peaks** in $k$ spectrum! However the Doppler effect carries an angular dependence that changes its projection on the sky:

  \[
  \hat{n} \cdot v_\gamma \propto \hat{n} \cdot \hat{k}
  \]
Doppler Peaks?

- Coordinates where $\hat{z} \parallel \hat{k}$

\[
Y_{10} Y_{\ell 0} \rightarrow Y_{\ell \pm 1 0}
\]

recoupling $j'_\ell Y_{\ell 0}$: no peaks in Doppler effect
Restoring Gravity

- Take a simple photon dominated system with gravity
- Continuity altered since a gravitational potential represents a stretching of the spatial fabric that dilutes number densities – formally a spatial curvature perturbation
- Think of this as a perturbation to the scale factor $a \rightarrow a(1 + \Phi)$ so that the cosmogical redshift is generalized to

$$\frac{\dot{a}}{a} \rightarrow \frac{\dot{a}}{a} + \dot{\Phi}$$

so that the continuity equation becomes

$$\dot{\Theta} = -\frac{1}{3} k \nu_\gamma - \dot{\Phi}$$
Restoring Gravity

- **Gravitational force** in momentum conservation $\mathbf{F} = -m\nabla\Psi$ generalized to momentum density modifies the Euler equation to

$$\dot{v}_\gamma = k(\Theta + \Psi)$$

- General relativity says that $\Phi$ and $\Psi$ are the relativistic analogues of the Newtonian potential and that $\Phi \approx -\Psi$.

- In our matter-dominated approximation, $\Phi$ represents matter density fluctuations through the cosmological Poisson equation

$$k^2\Phi = 4\pi G a^2 \rho_m \Delta_m$$

where the difference comes from the use of **comoving coordinates** for $k$ ($a^2$ factor), the removal of the **background density** into the background expansion ($\rho \Delta_m$) and finally a **coordinate subtlety** that enters into the definition of $\Delta_m$. 
Constant Potentials

- In the matter dominated epoch potentials are constant because infall generates velocities as $v_m \sim k\eta \Psi$

- Velocity divergence generates density perturbations as $\Delta_m \sim -k\eta v_m \sim -(k\eta)^2 \Psi$

- And density perturbations generate potential fluctuations as $\Phi \sim \Delta_m/(k\eta)^2 \sim -\Psi$, keeping them constant. Note that because of the expansion, density perturbations must grow to keep potentials constant.

- Here we have used the Friedman equation $H^2 = \frac{8\pi G \rho_m}{3}$ and $\eta = \int d\ln a/(aH) \sim 1/(aH)$

- More generally, if stress perturbations are negligible compared with density perturbations ($\delta p \ll \delta \rho$) then potential will remain roughly constant – more specifically a variant called the Bardeen or comoving curvature $\mathcal{R}$ is constant
Oscillator: Take Two

- Combine these to form the simple harmonic oscillator equation

\[ \ddot{\Theta} + c_s^2 k^2 \Theta = -\frac{k^2}{3} \Psi - \dot{\Phi} \]

- In a CDM dominated expansion \( \dot{\Phi} = \dot{\Psi} = 0 \). Also for photon domination \( c_s^2 = 1/3 \) so the oscillator equation becomes

\[ \ddot{\Theta} + \ddot{\Psi} + c_s^2 k^2 (\Theta + \Psi) = 0 \]

- Solution is just an offset version of the original

\[ (\Theta + \Psi)(\eta) = (\Theta + \Psi)(0) \cos(ks) \]

- \( \Theta + \Psi \) is also the observed temperature fluctuation since photons lose energy climbing out of gravitational potentials at recombination
Effective Temperature

- Photons climb out of potential wells at last scattering
- Lose energy to gravitational redshifts
- Observed or effective temperature

\[ \Theta + \Psi \]

- Effective temperature oscillates around zero with amplitude given by the initial conditions
- Note: initial conditions are set when the perturbation is outside of horizon, need inflation or other modification to matter-radiation FRW universe.
- GR says that initial temperature is given by initial potential
Sachs-Wolfe Effect and the Magic 1/3

• A gravitational potential is a perturbation to the temporal coordinate [formally a gauge transformation]

\[
\frac{\delta t}{t} = \Psi
\]

• Convert this to a perturbation in the scale factor,

\[
t = \int \frac{da}{aH} \propto \int \frac{da}{a\rho^{1/2}} \propto a^{3(1+w)/2}
\]

where \( w \equiv p/\rho \) so that during matter domination

\[
\frac{\delta a}{a} = \frac{2}{3} \frac{\delta t}{t}
\]

• CMB temperature is cooling as \( T \propto a^{-1} \) so

\[
\Theta + \Psi \equiv \frac{\delta T}{T} + \Psi = -\frac{\delta a}{a} + \Psi = \frac{1}{3} \Psi
\]
**Sachs-Wolfe Normalization**

- Use measurements of $\Delta T/T \approx 10^{-5}$ in the Sachs-Wolfe effect to infer $\Delta_R^2$.

- Recall in matter domination $\Psi = -3\mathcal{R}/5$

\[
\frac{\ell(\ell+1)C_\ell}{2\pi} \approx \Delta_T^2 \approx \frac{1}{25} \Delta_R^2
\]

- So that the amplitude of initial curvature fluctuations is $\Delta_R \approx 5 \times 10^{-5}$.

- Modern usage: WMAP’s measurement of 1st peak plus known radiation transfer function is used to convert $\Delta T/T$ to $\Delta_R$.  


Baryon Loading

- Baryons add extra **mass** to the photon-baryon fluid
- Controlling parameter is the **momentum density ratio**:

\[ R \equiv \frac{p_b + \rho_b}{p_\gamma + \rho_\gamma} \approx 30\Omega_b h^2 \left( \frac{a}{10^{-3}} \right) \]

of order **unity** at recombination

- Momentum density of the **joint system** is conserved

\[ (\rho_\gamma + p_\gamma)v_\gamma + (\rho_b + p_b)v_b \approx (p_\gamma + p_\gamma + \rho_b + \rho_\gamma)v_\gamma \]

\[ = (1 + R)(\rho_\gamma + p_\gamma)v_\gamma b \]

where the controlling parameter is the **momentum density ratio**:

\[ R \equiv \frac{p_b + \rho_b}{p_\gamma + \rho_\gamma} \approx 30\Omega_b h^2 \left( \frac{a}{10^{-3}} \right) \]

of order **unity** at recombination
New Euler Equation

- Momentum density ratio enters as

\[ (1 + R)v_{\gamma b} \cdot = k\dot{\Theta} + (1 + R)k\Psi \]

- Photon continuity remains the same

\[ \dot{\Theta} = -\frac{k}{3}v_{\gamma b} - \dot{\Phi} \]

- Modification of oscillator equation

\[ (1 + R)\dot{\Theta} \cdot + \frac{1}{3}k^2\dot{\Theta} = -\frac{1}{3}k^2(1 + R)\Psi - [(1 + R)\dot{\Phi}] \cdot \]
Oscillator: Take Three

- Combine these to form the not-quite-so simple harmonic oscillator equation

\[ c_s^2 \frac{d}{d\eta} \left( c_s^{-2} \dot{\Theta} \right) + c_s^2 k_s^2 \Theta = -\frac{k^2}{3} \Psi - c_s^2 \frac{d}{d\eta} \left( c_s^{-2} \dot{\Phi} \right) \]

where \( c_s^2 \equiv \frac{\dot{p}_{\gamma b}}{\dot{\rho}_{\gamma b}} \)

\[ c_s^2 = \frac{1}{3} \frac{1}{1 + R} \]

- In a CDM dominated expansion \( \dot{\Phi} = \dot{\Psi} = 0 \) and the adiabatic approximation \( \dot{R}/R \ll \omega = k c_s \)

\[ [\Theta + (1 + R)\Psi](\eta) = [\Theta + (1 + R)\Psi](0) \cos(k_s) \]
• Photon-baryon ratio enters in three ways
• Overall larger amplitude:
\[
[\Theta + (1 + R)\Psi](0) = \frac{1}{3}(1 + 3R)\Psi(0)
\]
• Even-odd peak modulation of effective temperature
\[
[\Theta + \Psi]_{\text{peaks}} = [\pm(1 + 3R) - 3R] \frac{1}{3}\Psi(0)
\]
\[
[\Theta + \Psi]_1 - [\Theta + \Psi]_2 = [-6R] \frac{1}{3}\Psi(0)
\]
• Shifting of the sound horizon down or \(\ell_A\) up
\[
\ell_A \propto \sqrt{1 + R}
\]
Photon Baryon Ratio Evolution

- Actual effects smaller since $R$ evolves
- Oscillator equation has time evolving mass

$$c_s^2 \frac{d}{d\eta} (c_s^{-2} \dot{\Theta}) + c_s^2 k^2 \Theta = 0$$

- Effective mass is $m_{\text{eff}} = 3c_s^{-2} = (1 + R)$
- Adiabatic invariant

$$\frac{E}{\omega} = \frac{1}{2} m_{\text{eff}} \omega A^2 = \frac{1}{2} 3c_s^{-2} k c_s A^2 \propto A^2 (1 + R)^{1/2} = \text{const.}$$

- Amplitude of oscillation $A \propto (1 + R)^{-1/4}$ decays adiabatically as the photon-baryon ratio changes
Baryons in the Power Spectrum

- Relative heights of peaks
Oscillator: Take Three and a Half

• The not-quite-so simple harmonic oscillator equation is a forced harmonic oscillator

\[ c_s^2 \frac{d}{d\eta} \left( c_s^{-2} \dot{\Theta} \right) + c_s^2 k^2 \Theta = -\frac{k^2}{3} \Psi - c_s^2 \frac{d}{d\eta}(c_s^{-2} \Phi) \]

changes in the gravitational potentials alter the form of the acoustic oscillations

• If the forcing term has a temporal structure that is related to the frequency of the oscillation, this becomes a driven harmonic oscillator

• Term involving \( \Psi \) is the ordinary gravitational force

• Term involving \( \Phi \) involves the \( \dot{\Phi} \) term in the continuity equation as a (curvature) perturbation to the scale factor
Potential Decay

- Matter-to-radiation ratio

\[ \frac{\rho_m}{\rho_r} \approx 24 \Omega_m h^2 \left(\frac{a}{10^{-3}}\right) \]

of order unity at recombination in a low \( \Omega_m \) universe

- Radiation is not stress free and so impedes the growth of structure

\[ k^2 \Phi = 4\pi G a^2 \rho_r \Delta_r \]

\( \Delta_r \sim 4\Theta \) oscillates around a constant value, \( \rho_r \propto a^{-4} \) so the Newtonian curvature decays.

- General rule: potential decays if the dominant energy component has substantial stress fluctuations, i.e. below the generalized sound horizon or Jeans scale
Radiation Driving

- Decay is timed precisely to drive the oscillator - close to fully coherent

\[
|\Theta + \Psi(\eta)| = |\Theta + \Psi(0) + \Delta \Psi - \Delta \Phi| \\
= \left| \frac{1}{3} \Psi(0) - 2\Psi(0) \right| = \left| \frac{5}{3} \Psi(0) \right|
\]

- \(5\times\) the amplitude of the Sachs-Wolfe effect!
Matter-Radiation in the Power Spectrum

- Coherent approximation is **exact** for a photon-baryon fluid but reality is reduced to $\sim 4\times$ because of neutrino contribution to radiation

- Actual **initial conditions** are $\Theta + \Psi = \Psi/2$ for radiation domination but comparison to matter dominated SW correct
Damping

- Tight coupling equations assume a **perfect fluid**: no viscosity, no heat conduction
- Fluid imperfections are related to the **mean free path of the photons in the baryons**

\[ \lambda_C = \frac{1}{\dot{\tau}} \quad \text{where} \quad \dot{\tau} = n_e \sigma_T a \]

\( \lambda_C \) is the conformal opacity to Thomson scattering
- Dissipation is related to the **diffusion length**: random walk approximation

\[ \lambda_D = \sqrt{N} \lambda_C = \sqrt{\frac{\eta}{\lambda_C}} \lambda_C = \sqrt{\eta \lambda_C} \]

the geometric mean between the horizon and mean free path
- \( \lambda_D / \eta_* \sim \text{few \%} \), so expect the peaks \( \gg 3 \) to be affected by dissipation
Equations of Motion

- **Continuity**

\[
\begin{align*}
\dot{\Theta} &= -\frac{k}{3}v_\gamma - \dot{\Phi}, \\
\dot{\delta}_b &= -kv_b - 3\dot{\Phi}
\end{align*}
\]

where the photon equation remains unchanged and the baryons follow number conservation with \( \rho_b = m_b n_b \)

- **Euler**

\[
\begin{align*}
\dot{v}_\gamma &= k(\Theta + \Psi) - \frac{k}{6}\pi_\gamma - \dot{\tau}(v_\gamma - v_b) \\
\dot{v}_b &= -\frac{\dot{a}}{a}v_b + k\Psi + \dot{\tau}(v_\gamma - v_b)/R
\end{align*}
\]

where the photons gain an anisotropic stress term \( \pi_\gamma \) from radiation viscosity and a momentum exchange term with the baryons and are compensated by the opposite term in the baryon Euler equation
Viscosity

- **Viscosity** is generated from radiation streaming from hot to cold regions.
- Expect

\[
\pi_\gamma \sim \nu_\gamma \frac{k}{\tau}
\]

generated by streaming, suppressed by **scattering** in a wavelength of the fluctuation. **Radiative transfer** says

\[
\pi_\gamma \approx 2A_v \nu_\gamma \frac{k}{\tau}
\]

where \(A_v = 16/15\)

\[
\dot{\nu}_\gamma = k(\Theta + \Psi) - \frac{k}{3} A_v \frac{k}{\tau} \nu_\gamma
\]
Oscillator: Penultimate Take

• Adiabatic approximation \((\omega \gg \dot{a}/a)\)

\[ \dot{\Theta} \approx -\frac{k}{3} \nu_\gamma \]

• Oscillator equation contains a \(\dot{\Theta}\) damping term

\[ c_s^2 \frac{d}{d\eta} (c_s^{-2} \dot{\Theta}) + \frac{k^2 c_s^2}{\dot{\tau}} A_v \dot{\Theta} + k^2 c_s^2 \Theta = -\frac{k^2}{3} \Psi - c_s^2 \frac{d}{d\eta} (c_s^{-2} \dot{\Phi}) \]

• Heat conduction term similar in that it is proportional to \(\nu_\gamma\) and is suppressed by scattering \(k/\dot{\tau}\). Expansion of Euler equations to leading order in \(k/\dot{\tau}\) gives

\[ A_h = \frac{R^2}{1 + R} \]

since the effects are only significant if the baryons are dynamically important
Oscillator: Final Take

- Final oscillator equation

\[ c_s^2 \frac{d}{d\eta} (c_s^{-2} \dot{\Theta}) + \frac{k^2 c_s^2}{\dot{\tau}} [A_v + A_h] \dot{\Theta} + k^2 c_s^2 \Theta = -\frac{k^2}{3} \Psi - c_s^2 \frac{d}{d\eta} (c_s^{-2} \dot{\Phi}) \]

- Solve in the adiabatic approximation

\[ \Theta \propto \exp(i \int \omega d\eta) \]

\[-\omega^2 + \frac{k^2 c_s^2}{\dot{\tau}} (A_v + A_h)i\omega + k^2 c_s^2 = 0 \]  

(1)
Dispersion Relation

- Solve

\[
\omega^2 = k^2 c_s^2 \left[ 1 + i \frac{\omega}{\dot{\tau}} (A_v + A_h) \right]
\]

\[
\omega = \pm kc_s \left[ 1 + \frac{i}{2} \frac{\omega}{\dot{\tau}} (A_v + A_h) \right]
\]

\[
= \pm kc_s \left[ 1 \pm \frac{i}{2} \frac{k c_s}{\dot{\tau}} (A_v + A_h) \right]
\]

- Exponentiate

\[
\exp(i \int \omega d\eta) = e^{\pm i ks} \exp[-k^2 \int d\eta \frac{1}{2} \frac{c_s^2}{\dot{\tau}} (A_v + A_h)]
\]

\[
= e^{\pm i ks} \exp[-(k/k_D)^2]
\]

- Damping is exponential under the scale \( k_D \)
Diffusion Scale

• Diffusion wavenumber

\[
 k_D^{-2} = \int d\eta \frac{1}{\frac{\eta}{\tau}} \frac{1}{6(1 + R)} \left( \frac{16}{15} + \frac{R^2}{(1 + R)} \right)
\]

• Limiting forms

\[
 \lim_{R \to 0} k_D^{-2} = \frac{1}{6} \frac{16}{15} \int d\eta \frac{1}{\frac{\eta}{\tau}}
\]

\[
 \lim_{R \to \infty} k_D^{-2} = \frac{1}{6} \int d\eta \frac{1}{\frac{\eta}{\tau}}
\]

• Geometric mean between horizon and mean free path as expected from a random walk

\[
 \lambda_D = \frac{2\pi}{k_D} \sim \frac{2\pi}{\sqrt{6}} (\eta \frac{\dot{\tau}}{-1})^{1/2}
\]
Thomson Scattering

- Polarization state of radiation in direction $\hat{n}$ described by the intensity matrix $\langle E_i(\hat{n})E_j^*(\hat{n}) \rangle$, where $\mathbf{E}$ is the electric field vector and the brackets denote time averaging.

- Differential cross section

$$\frac{d\sigma}{d\Omega} = \frac{3}{8\pi} |\hat{\mathbf{E}}' \cdot \hat{\mathbf{E}}|^2 \sigma_T,$$

where $\sigma_T = \frac{8\pi \alpha^2}{3m_e}$ is the Thomson cross section, $\hat{\mathbf{E}}'$ and $\hat{\mathbf{E}}$ denote the incoming and outgoing directions of the electric field or polarization vector.

- Summed over angle and incoming polarization

$$\sum_{i=1,2} \int d\hat{n}' \frac{d\sigma}{d\Omega} = \sigma_T$$
Polarization Generation

- Heuristic: incoming radiation shakes an electron in direction of electric field vector $\mathbf{E}'$
- Radiates photon with polarization also in direction $\mathbf{E}'$
- But photon cannot be longitudinally polarized so that scattering into $90^\circ$ can only pass one polarization
- Linearly polarized radiation like polarization by reflection
- Unlike reflection of sunlight, incoming radiation is nearly isotropic
- Missing from direction orthogonal to original incoming direction
- Only quadrupole anisotropy generates polarization by Thomson scattering
Acoustic Polarization

- Break down of tight-coupling leads to quadrupole anisotropy of

\[ \pi_\gamma \approx \frac{k}{\dot{\tau}} v_\gamma \]

- Scaling \( k_D = (\dot{\tau}/\eta_*)^{1/2} \rightarrow \dot{\tau} = k_D^2 \eta_* \)

- Know: \( k_D s_* \approx k_D \eta_* \approx 10 \)

- So:

\[ \pi_\gamma \approx \frac{k}{k_D} \frac{1}{10} v_\gamma \]

\[ \Delta_P \approx \frac{\ell}{\ell_D} \frac{1}{10} \Delta_T \]
Acoustic Polarization

- Gradient of velocity is along direction of wavevector, so polarization is pure $E$-mode
- Velocity is $90^\circ$ out of phase with temperature – turning points of oscillator are zero points of velocity:
  \[ \Theta + \Psi \propto \cos(ks); \quad v_\gamma \propto \sin(ks) \]
- Polarization peaks are at troughs of temperature power
Cross Correlation

- Cross correlation of temperature and polarization

\[(\Theta + \Psi)(v_\gamma) \propto \cos(ks) \sin(ks) \propto \sin(2ks)\]

- Oscillation at twice the frequency

- Correlation: radial or tangential around hot spots

- Partial correlation: easier to measure if polarization data is noisy, harder to measure if polarization data is high $S/N$ or if bands do not resolve oscillations

- Good check for systematics and foregrounds

- Comparison of temperature and polarization is proof against features in initial conditions mimicking acoustic features
Transfer Function

- **Transfer function** transfers the initial Newtonian curvature to its value today (linear response theory)

\[ T(k) = \frac{\Phi(k, a = 1)}{\Phi(k_\text{norm}, a_\text{init})} \frac{\Phi(k_\text{norm}, a_\text{init})}{\Phi(k_\text{norm}, a = 1)} \]

- Conservation of Bardeen curvature: Newtonian curvature is a constant when stress perturbations are negligible: above the horizon during radiation and dark energy domination, on all scales during matter domination

- When stress fluctuations dominate, perturbations are stabilized by the Jeans mechanism

- **Hybrid Poisson equation**: Newtonian curvature, comoving density perturbation \( \Delta \equiv (\delta \rho/\rho)_\text{com} \) implies \( \Phi \) decays

\[ (k^2 - 3K)\Phi = 4\pi G \rho \Delta \sim \eta^{-2} \Delta \]
Transfer Function

- Freezing of \( \Delta \) stops at \( \eta_{eq} \)

\[ \Phi \sim (k \eta_{eq})^{-2} \Delta H \sim (k \eta_{eq})^{-2} \Phi_{init} \]

- Transfer function has a \( k^{-2} \) fall-off beyond \( k_{eq} \sim \eta_{eq}^{-1} \)

- Small correction since growth with a smooth radiation component is logarithmic not frozen

- Transfer function is a direct output of an Einstein-Boltzmann code
Fitting Function

- Alternately accurate fitting formula exist, e.g. pure CDM form:

\[ T(k(q)) = \frac{L(q)}{L(q) + C(q)q^2} \]

\[ L(q) = \ln(e + 1.84q) \]

\[ C(q) = 14.4 + \frac{325}{1 + 60.5q^{1.11}} \]

\[ q = k/\Omega_m h^2 \text{Mpc}^{-1} (T_{\text{CMB}}/2.7K)^2 \]

- In $h \text{ Mpc}^{-1}$, the critical scale depends on $\Gamma \equiv \Omega_m h$ also known as the shape parameter
Transfer Function

- Numerical calculation

\[ T(k) \]

\[ k (h^{-1} \text{ Mpc}) \]

BAO

\[ k^{-2} \]
Dark Matter and the Transfer Function

- Baryons caught up in the acoustic oscillations of the CMB and impart acoustic wiggles to the transfer function. Density enhancements are produced kinematically through the continuity equation \( \delta_b \sim (k \eta) \nu_b \) and hence are out of phase with CMB temperature peaks.

- Dissipation of the acoustic oscillations eliminates both the CMB and baryon perturbations – known as Silk damping for the baryons. This suppression and the general fact that baryons are caught up with photons was one of the main arguments for CDM.

- Neutrino dark matter suffers similar effects and hence cannot be the main component of dark matter in the universe.
Massive Neutrinos

- Relativistic stresses of a light neutrino slow the growth of structure.

- Neutrino species with cosmological abundance contribute to matter as $\Omega_\nu h^2 = \sum m_\nu / 94\text{eV}$, suppressing power as $\Delta P/P \approx -8\Omega_\nu / \Omega_m$.

- Current data from 2dF galaxy survey and CMB indicate $\sum m_\nu < 0.9\text{eV}$ assuming a $\Lambda$CDM model with constant tilt based on the shape of the transfer function.
Growth Function

- Same physics applies to the dark energy dominated universe
- Under the dark energy sound horizon or Jeans scale, dark energy density frozen. Potential decays at the same rate for all scales

\[ G'(a) = \frac{\Phi(k_{\text{norm}}, a)}{\Phi(k_{\text{norm}}, a_{\text{init}})} \quad \equiv \quad \frac{d}{d \ln a} \]

- Continuity + Euler + Poisson

\[ G'' + \left( 1 - \frac{\rho''}{\rho'} + \frac{1}{2} \frac{\rho'_c}{\rho_c} \right) G' + \left( \frac{1}{2} \frac{\rho'_c}{\rho_c} - \frac{\rho''}{\rho'} \right) G = 0 \]

where \( \rho \) is the Jeans unstable matter and \( \rho_c \) is the critical density
**Dark Energy Growth Suppression**

- **Pressure growth suppression:** \( \delta \equiv \delta \rho_m / \rho_m \propto aG \)

\[
\frac{d^2 G}{d \ln a^2} + \left[ \frac{5}{2} - \frac{3}{2} w(z) \Omega_{DE}(z) \right] \frac{dG}{d \ln a} + \frac{3}{2} [1 - w(z)] \Omega_{DE}(z) G = 0,
\]

where \( w \equiv p_{DE}/\rho_{DE} \) and \( \Omega_{DE} \equiv \rho_{DE}/(\rho_m + \rho_{DE}) \) with initial conditions \( G = 1, \frac{dg}{d \ln a} = 0 \)

- As \( \Omega_{DE} \rightarrow 0 \) \( G = \text{const.} \) is a solution. The other solution is the decaying mode, eliminated by initial conditions.

- As \( \Omega_{DE} \rightarrow 1 \) \( G \propto a^{-1} \) is a solution. Corresponds to a frozen density field.
Velocity field

- Continuity gives the velocity from the density field as

\[
v = -\frac{\dot{\Delta}}{k} = -\frac{aH}{k} \frac{d\Delta}{d \ln a}
\]

\[
= -\frac{aH}{k} \Delta \frac{d \ln (ag)}{d \ln a}
\]

- In a \( \Lambda \) CDM model or open model \( \frac{d \ln (ag)}{d \ln a} \approx \Omega_m^{0.6} \)

- Measuring both the density field and the velocity field (through distance determination and redshift) allows a measurement of \( \Omega_m \)

- Practically one measures \( \beta = \Omega_m^{0.6}/b \) where \( b \) is a bias factor for the tracer of the density field, i.e. with galaxy numbers \( \delta n/n = b \Delta \)

- Can also measure this factor from the redshift space power spectrum - the Kaiser effect where clustering in the radial direction is apparently enhanced by gravitational infall
Lyman-α Forest

- QSO spectra absorbed by neutral hydrogen through the Lyα transition.

- Lack of complete absorption, known as the lack of a Gunn-Peterson trough indicates that the universe is nearly fully ionized out to the highest redshift quasar \( z \sim 6 \); recently SDSS QSO implies \( z \sim 6 \) is the end of the reionization epoch.

- In ionization equilibrium, the Gunn-Peterson optical depth is a tracer of the underlying baryon density which itself is a tracer of the dark matter \( \tau_{GP} \propto \rho_b T^{-0.7} \) with \( T(\rho_b) \).

- Clustering in the Lyα forest reflects the underlying linear power spectrum as calibrated through simulations.
Gravitational Lensing

- Gravitational potentials along the line of sight $\hat{n}$ to some source at comoving distance $D_s$ lens the images according to (flat universe)

$$\phi(\hat{n}) = 2 \int dD \frac{D_s - D}{DD_s} \Phi(D\hat{n}, \eta(D))$$

remapping image positions as

$$\hat{n}^I = \hat{n}^S + \nabla_{\hat{n}} \phi(\hat{n})$$

- Since absolute source position is unknown, use image distortion defined by the Jacobian matrix

$$\frac{\partial n^I_i}{\partial n^S_j} = \delta_{ij} + \psi_{ij}$$
Weak Lensing

- Small image distortions described by the convergence $\kappa$ and shear components $(\gamma_1, \gamma_2)$

$$\psi_{ij} = \begin{pmatrix} \kappa - \gamma_1 & -\gamma_2 \\ -\gamma_2 & \kappa + \gamma_1 \end{pmatrix}$$

where $\nabla \hat{n} = D \nabla$ and

$$\psi_{ij} = 2 \int dD \frac{D(D_s - D)}{D_s} \nabla_i \nabla_j \Phi(D \hat{n}, \eta(D))$$

- In particular, through the Poisson equation the convergence (measured from shear) is simply the projected mass

$$\kappa = \frac{3}{2} \Omega_m H_0^2 \int dD \frac{D(D_s - D)}{D_s} \frac{\Delta(D \hat{n}, \eta(D))}{a}$$