Astro 321

Set 1: FRW Cosmology

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FRW Cosmology

- The Friedmann-Robertson-Walker (FRW sometimes Lemaitre, FLRW) cosmology has two elements
  - The FRW geometry or metric
  - The FRW dynamics or Einstein/Friedmann equation(s)
- Same as the two pieces of General Relativity (GR)
  - A metric theory: geometry tells matter how to move
  - Field equations: matter tells geometry how to curve
- Useful to separate out these two pieces both conceptually and for understanding alternate cosmologies, e.g.
  - Modifying gravity while remaining a metric theory
  - Breaking the homogeneity or isotropy assumption under GR

FRW Geometry

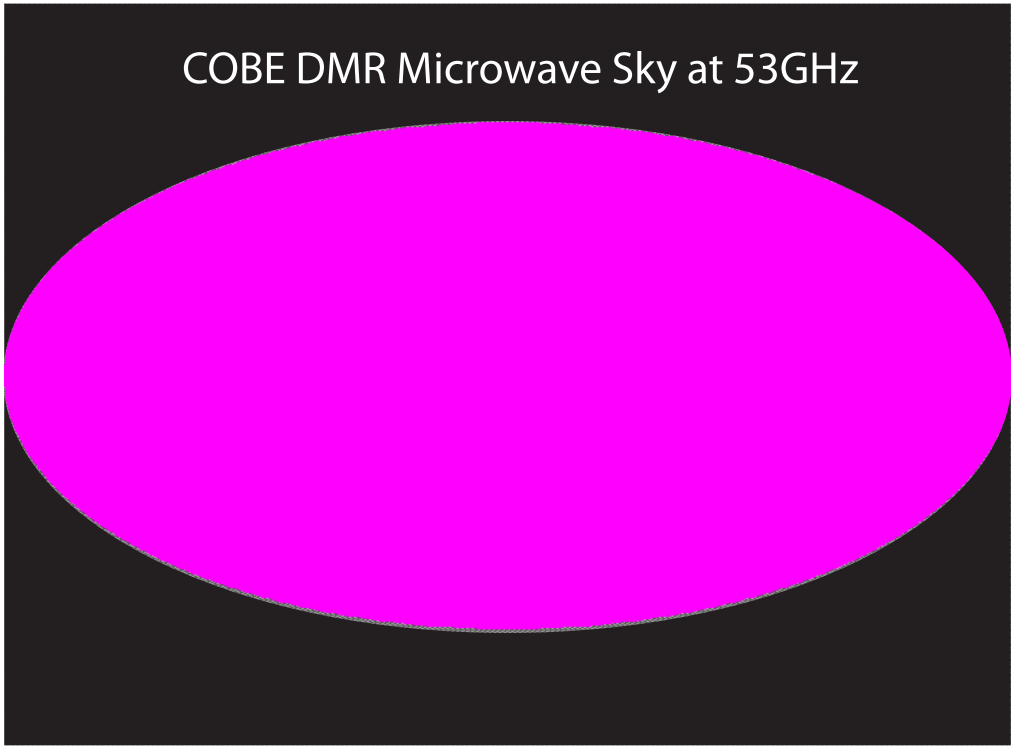
- FRW geometry = homogeneous and isotropic on large scales
- Universe observed to be nearly isotropic (e.g. CMB, radio point sources, galaxy surveys)
- Copernican principle: we're not special, must be isotropic to all observers (all locations)

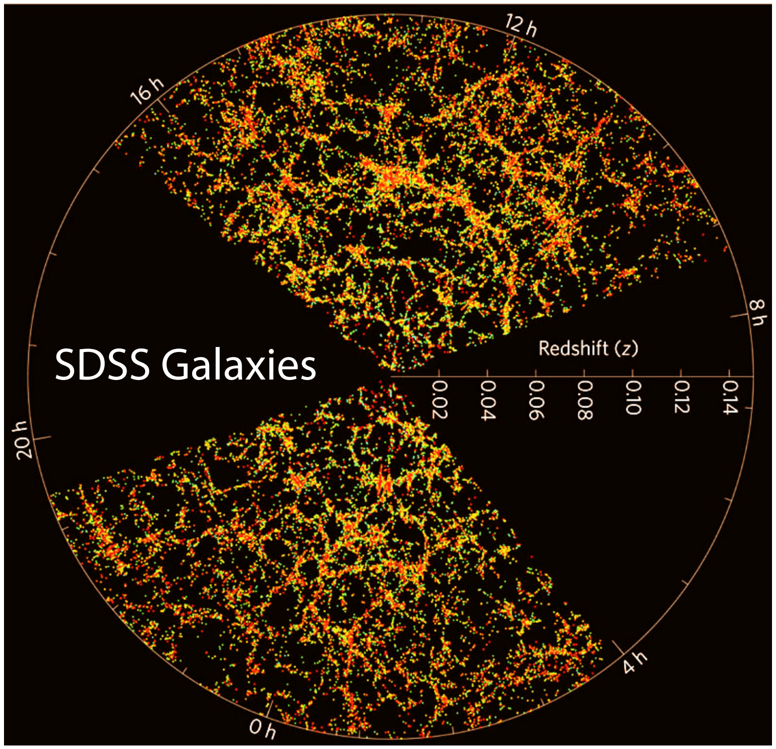
  Implies homogeneity

  Verified through galaxy redshift surveys
- FRW cosmology (homogeneity, isotropy & field equations) generically implies the expansion of the universe, except for special unstable cases

Isotropy & Homogeneity

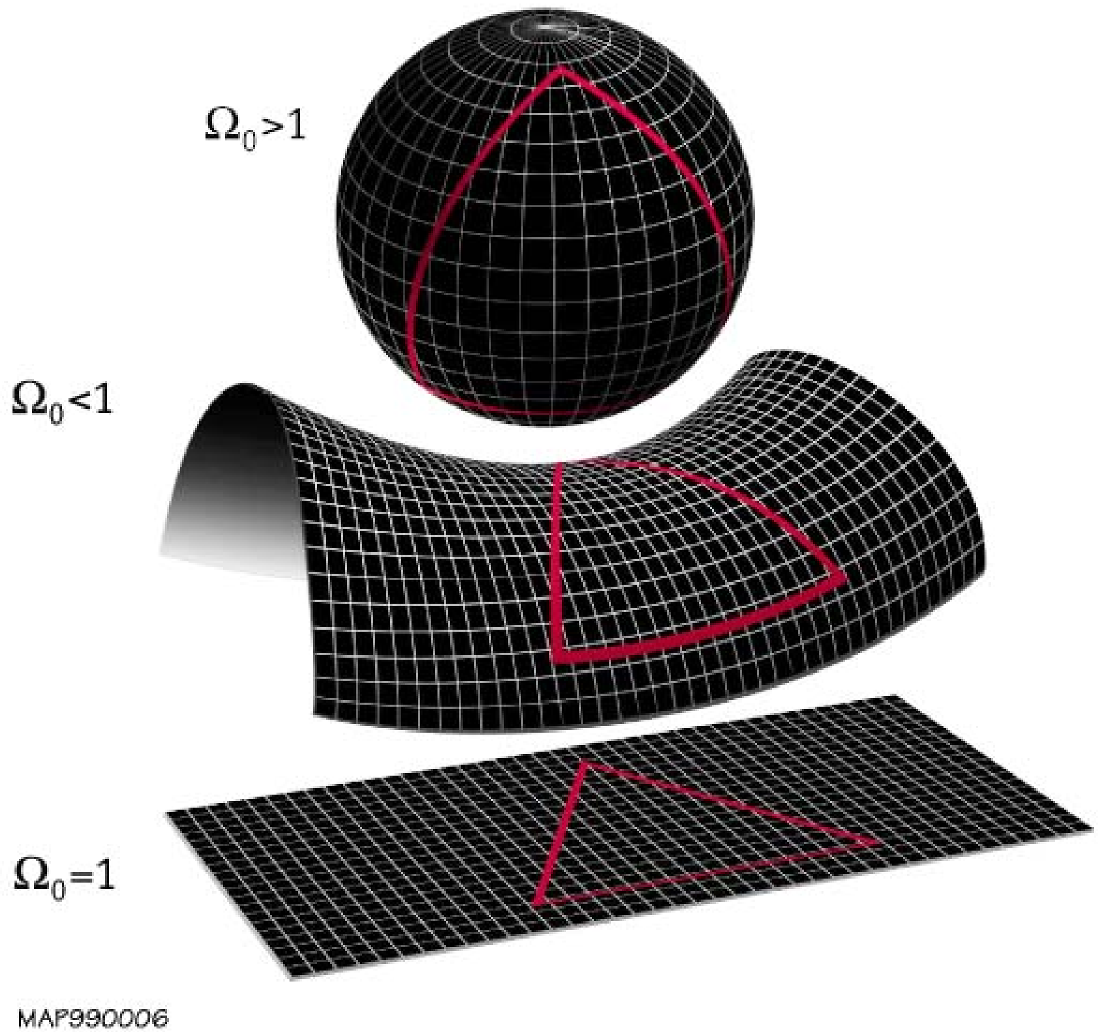
- Isotropy: CMB isotropic to 10^{-3}, 10^{-5} if dipole subtracted
- Redshift surveys show return to homogeneity on the >100Mpc scale





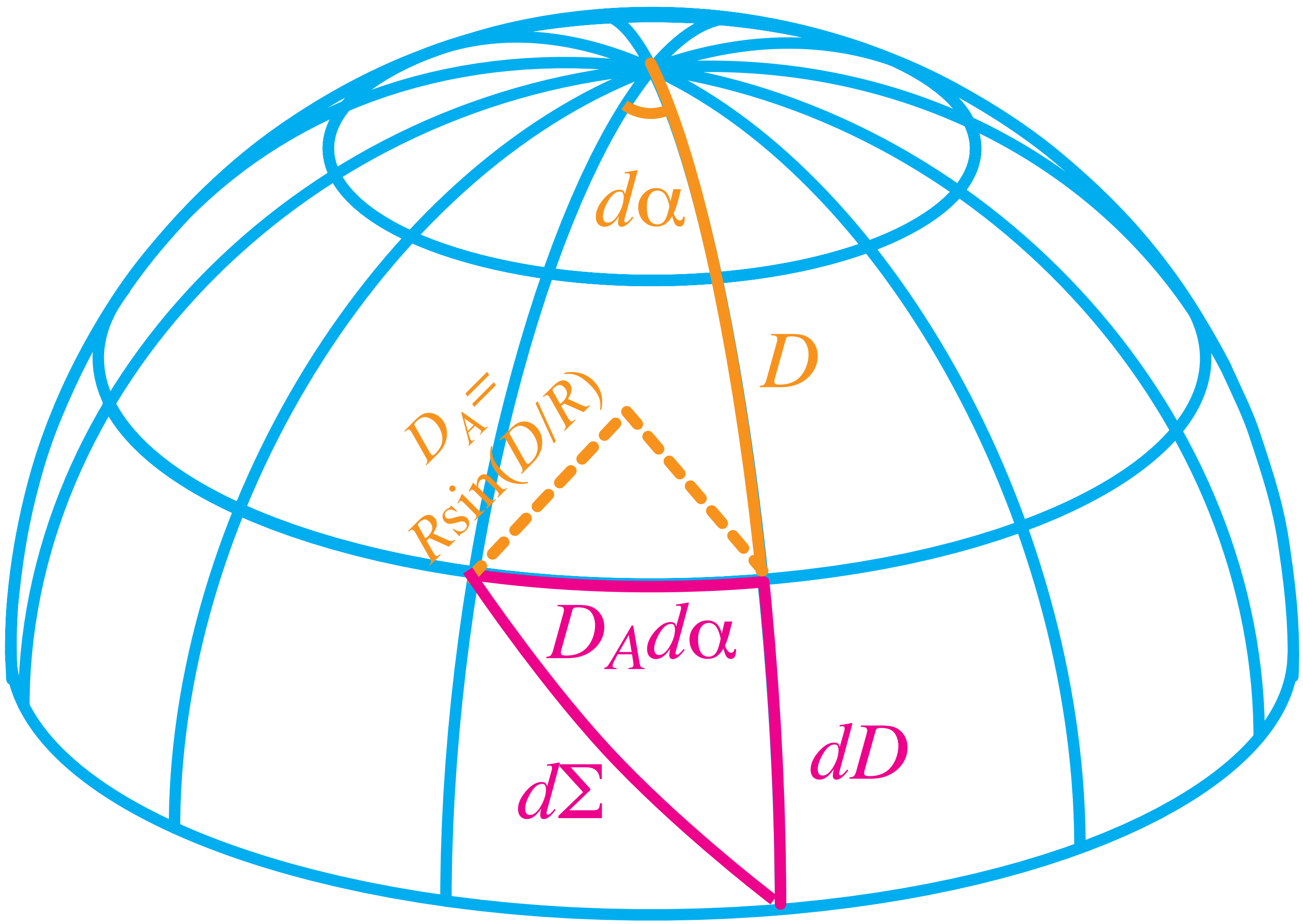
FRW Geometry

- Spatial geometry is that of a constant curvature $K = 1/R^2$
- Positive: sphere
  Negative: saddle
  Flat: plane
- Metric tells us how to measure distances on this surface



FRW Geometry

- Closed geometry of a sphere of radius R
- Suppress 1 dimension α represents total angular separation (θ, ϕ)



FRW Geometry

- Two types of distances:
  - Radial distance on the arc D

    Distance (for e.g. photon) traveling along the arc
  - Angular diameter distance D_A

  Distance inferred by the angular separation $d\alpha$ for a known transverse separation (on a constant latitude) $D_A d\alpha$

  Relationship $D_A = R\sin(D/R)$
- As if background geometry (gravitationally) lenses image
- Positively curved geometry $D_A < D$ and objects are further than they appear
- Negatively curved universe R is imaginary and

  $$R\sin(D/R) = i|R|\sin(D/i|R|) = |R|\sinh(D/|R|)$$

  and $D_A > D$ objects are closer than they appear

Angular Diameter Distance

- 3D distances restore usual spherical polar angles

$$
\begin{aligned}
d\Sigma^2 &= dD^2 + D_A^2 d\alpha^2 \\
&= dD^2 + D_A^2 (d\theta^2 + \sin^2\theta d\phi^2)
\end{aligned}
$$

- GR allows arbitrary choice of coordinates, alternate notation is to use D_A as radial coordinate
- D_A useful for describing observables (flux, angular positions)
- D useful for theoretical constructs (causality, relationship to temporal evolution)

Angular Diameter Distance

- The line element is often also written using D_A as the coordinate distance

$$dD_A^2 = \left(\frac{dD_A}{dD}\right)^2 dD^2$$

$$\left(\frac{dD_A}{dD}\right)^2 = \cos^2(D/R) = 1 - \sin^2(D/R) = 1 - (D_A/R)^2$$

$$dD^2 = \frac{1}{1 - D_A^2/R^2} dD_A^2$$

and defining the curvature $K = 1/R^2$ the line element becomes

$$d\Sigma^2 = \frac{1}{1 - D_A^2 K} dD_A^2 + D_A^2(d\theta^2 + \sin^2\theta d\phi^2)$$

where $K < 0$ for a negatively curved space

Volume Element

- Metric also defines the volume element

$$
\begin{aligned}
dV &= (dD)(D_A d\theta)(D_A \sin\theta d\phi) \\
&= D_A^2 dD d\Omega
\end{aligned}
$$

  where $d\Omega = \sin\theta d\theta d\phi$ is solid angle

- Most of classical cosmology boils down to these three quantities, (comoving) radial distance, (comoving) angular diameter distance, and volume element
- For example, distance to a high redshift supernova, angular size of the horizon at last scattering and BAO feature, number density of clusters...

Comoving Coordinates

- Remaining degree of freedom (preserving homogeneity and isotropy) is the temporal evolution of overall scale factor
- Relates the geometry (fixed by the radius of curvature R) to physical coordinates – a function of time only

$$d\sigma^2 = a^2(t) d\Sigma^2$$

  our conventions are that the scale factor today $a(t_0) \equiv 1$

- Similarly physical distances are given by $d(t) = a(t)D$, $d_A(t) = a(t)D_A$.
- Distances in upper case are comoving; lower, physical

  Do not change with time

  Simplest coordinates to work out geometrical effects

Time and Conformal Time

- Proper time (with $c = 1$)

$$d\tau^2 = dt^2 - d\sigma^2$$
$$= dt^2 - a^2(t) d\Sigma^2$$

- Taking out the scale factor in the time coordinate

$$d\tau^2 \equiv a^2(t) \left(d\eta^2 - d\Sigma^2 \right)$$

$d\eta = dt/a$ defines conformal time – useful in that photons travelling radially from observer then obey

$$\Delta D = \Delta\eta = \int \frac{dt}{a}$$

so that time and distance may be interchanged

FRW Metric

- Relationship between coordinate differentials and space-time separation defines the metric $g_{\mu\nu}$
- Mostly plus convention $ds^2 = -d\tau^2$

$$ds^2 \equiv g_{\mu\nu} dx^\mu dx^\nu = a^2(\eta)(-d\eta^2 + d\Sigma^2)$$

Einstein summation - repeated lower-upper pairs summed

- Usually we will use comoving coordinates and conformal time as the x^μ unless otherwise specified – metric for other choices are related by $a(t)$

Horizon

- Distance travelled by a photon in the whole lifetime of the universe defines the horizon
- Since $d\tau = 0$, the horizon is simply the elapsed conformal time

$$D_{\rm horizon}(t) = \int_0^t \frac{dt'}{a} = \eta(t)$$

- Horizon always grows with time
- Always a point in time before which two observers separated by a distance D could not have been in causal contact

Horizon

- Horizon problem: why is the universe homogeneous and isotropic on large scales especially for objects seen at early times, e.g. CMB, when horizon small
- Intuition: in each doubling (or efolding) of the scale factor, photons travel larger and larger distances

  Consequence: horizon is approximately the distance travelled in the last efolding
- To avoid the horizon problem, we want the distance to get smaller and smaller with each efolding
- Quantify by transforming time to efolds through the Hubble parameter

Hubble Parameter

- Useful to define the expansion rate or Hubble parameter

$$H(t) \equiv \frac{1}{a}\frac{da}{dt} = \frac{d\ln a}{dt}$$

fractional change in the scale factor per unit time - $\ln a = N$ is also known as the e-folds of the expansion

- Cosmic time becomes

$$t = \int dt = \int \frac{d\ln a}{H(a)}$$

- Conformal time becomes

$$\eta = \int \frac{dt}{a} = \int \frac{d\ln a}{aH(a)}$$

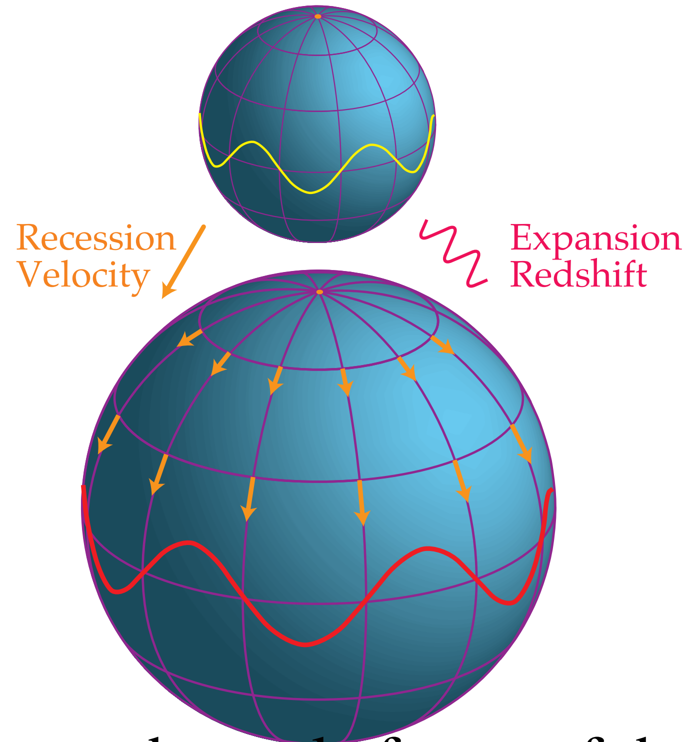
Horizon Problem Redux

- Does aH increase or decrease with a?
- If aH decreases then for each successive $\Delta \ln a$, a photon travels a larger ΔD, total distance dominated by last efold
- If aH increases then for each successive $\Delta \ln a$, a photon travels a smaller ΔD, total distance dominated by first efold
- Critical point is when the acceleration of the expansion switches sign

$$\frac{d(aH)}{dt} = \frac{d^2 a}{dt^2}$$

Redshift

- Wavelength of light "stretches" with the scale factor
- The physical wavelength $\lambda_{\rm emit}$ associated with an observed wavelength today $\lambda_{\rm obs}(a=1)$ (or comoving=physical units today) is



$$\lambda_{\rm emit} = a(t)\lambda_{\rm obs}$$

so that the redshift of spectral lines measures the scale factor of the universe at t, $1+z = 1/a$.

- Interpreting the redshift as a Doppler shift, objects recede in an expanding universe
- More generally the de Broglie wavelength of any particle redshifts in this way

Distance-Redshift Relation

- Given atomically known rest wavelength $\lambda_{\rm emit}$, redshift can be precisely measured from spectra
- Combined with a measure of distance, distance-redshift $D(z) \equiv D(z(a))$ can be inferred - given that photons travel $D = \Delta\eta$ this tells us how the scale factor of the universe evolves with time.
- Related to the expansion history as

$$D(a) = \int dD = \int_a^1 \frac{d\ln a'}{a' H(a')}$$

$$[d\ln a' = -d\ln(1+z) = -a' dz]$$

$$D(z) = -\int_z^0 \frac{dz'}{H(z')} = \int_0^z \frac{dz'}{H(z')}$$

Hubble Law

- Note limiting case is the Hubble law

$$\lim_{z \to 0} D(z) = z/H(z=0) \equiv z/H_0$$

independently of the geometry and expansion dynamics

- Hubble constant usually quoted as as dimensionless h

$$H_0 = 100h \, \mathrm{km\, s^{-1} Mpc^{-1}}$$

- Observationally $h \sim 0.7$ (see below)

Scale of the Universe

- In natural units of $\hbar = c = 1$ used here, H_0 sets an length, time, energy, mass scale
- $H_0^{-1} = 9.7778\,(h^{-1}\ \mathrm{Gyr})$

  e-folding time scale of the expansion (Hubble time), age of (decelerating) universe
- $H_0^{-1} = 2997.9\,(h^{-1}\ \mathrm{Mpc})$

  Observable length scale (Hubble scale), horizon scale of (decelerating) universe
- $H_0 = 2.1332h \times 10^{-33}\mathrm{eV} = m_{\rm de}$

  Mass scale of explanations of dark energy
- $H_0 = 10^{-6}h \times \ (2.9979\,\mathrm{kpc})^{-1} = (GM/r) \times r^{-1}$

  Acceleration/MOND scale - order of magnitude at which dark matter in galaxies flatten rotation curve ($\sim 10^{-10}\mathrm{m\,s^{-2}}$)

Scale of the Universe

- Since GM/r is dimensionless and r has inverse M dimensions, gravity sets a natural mass scale in the reduced Planck mass $M_{\rm Pl} = 1/\sqrt{8\pi G} = 1.22 \times 10^{19}$ GeV

$$
\begin{aligned}
M^4 &\equiv \rho_c = 3H_0^2/8\pi G \\
&= (3.000 \times 10^{-12}\mathrm{GeV})^4 h^2 = 8.098 \times 10^{-47} h^2 \mathrm{Gev}^4
\end{aligned}
$$

  Density scale of the expansion, critical energy density (see below)

- $M/M_{\rm Pl} = 2.46 h^{1/2} \times 10^{-31}$ – seems highly unnatural in natural units! (famous 120 orders of magnitude in density, see below)
- $M = 3^{1/4}\sqrt{m_{\rm de} M_{\rm Pl}}$, geometric mean
- $m_{\rm de}$ as far from any standard model particle – what protects such a hierarchy? (note that M is comparable to neutrino masses)

Measuring $D(z)$

- Standard Ruler: object of known physical size

$$\lambda = a(t)\Lambda$$

subtending an observed angle α on the sky α

$$\alpha = \frac{\Lambda}{D_A(z)} \equiv \frac{\lambda}{d_A(z)}$$

$$d_A(z) = aD_A(a) = \frac{D_A(z)}{1+z}$$

where, by analogy to D_A, d_A is the physical angular diameter distance

- Since $D_A \to D_{\rm horizon}$ whereas $(1+z)$ unbounded, angular size of a fixed physical scale at high redshift actually increases with radial distance

Measuring $D(z)$

- Standard Candle: object of known luminosity L with a measured flux F (energy/time/area)
  - Comoving surface area $4\pi D_A^2$
  - Frequency/energy redshifts as $(1+z)$
  - Time-dilation or arrival rate of photons (crests) $dt = a d\eta$ lowered as $(1+z)$ vs emission rate:

$$F = \frac{L}{4\pi D_A^2}\frac{1}{(1+z)^2} \equiv \frac{L}{4\pi d_L^2}$$

- So luminosity distance

$$d_L = (1+z)D_A = (1+z)^2 d_A$$

- As $z \to 0$, $d_L = d_A = D_A$

Olber's Paradox

- Surface brightness

$$S = \frac{F}{\Delta\Omega} = \frac{L}{4\pi d_L^2}\frac{d_A^2}{\lambda^2}$$

- In a non-expanding geometry (regardless of curvature), surface brightness is conserved $d_A = d_L$

$$S = \text{const.}$$

- So since each site line in universe full of stars will eventually end on surface of star, night sky should be as bright as sun (not infinite)
- In an expanding universe

$$S \propto (1+z)^{-4}$$

Olber's Paradox

- Second piece: age finite so even if stars exist in the early universe, not all site lines end on stars
- But even as age goes to infinity and the number of site lines goes to 100%, surface brightness of distant objects (of fixed physical size) goes to zero
  - Angular size increases
  - Redshift of energy and arrival time

Measuring $D(z)$

- Ratio of fluxes or difference in log flux (magnitude) measurable independent of knowing luminosity

$$m_1 - m_2 = -2.5 \log_{10}(F_1/F_2)$$

related to d_L by definition by inverse square law

$$m_1 - m_2 = 5 \log_{10}[d_L(z_1)/d_L(z_2)]$$

- If absolute magnitude is known

$$m - M = 5 \log_{10}[d_L(z)/10\mathrm{pc}]$$

absolute distances measured, e.g. at low $z = z_0$ Hubble constant

$$d_L \approx z_0/H_0 \rightarrow H_0 = z_0/d_L$$

- Also standard ruler whose length, calibrated in physical units

Measuring $D(z)$

- If absolute calibration of standards unknown, then both standard candles and standard rulers measure relative sizes and fluxes

  For standard candle, e.g. compare magnitudes low z_0 to a high z object involves

$$\Delta m = m_z - m_{z_0} = 5\log_{10}\frac{d_L(z)}{d_L(z_0)} = 5\log_{10}\frac{H_0 d_L(z)}{z_0}$$

  Likewise for a standard ruler comparison at the two redshifts

$$\frac{d_A(z)}{d_A(z_0)} = \frac{H_0 d_A(z)}{z_0}$$

- Distances are measured in units of h^{-1} Mpc.
- Change in expansion rate measured as $H(z)/H_0$

Hubble Constant

- Hubble in 1929 used the Cepheid period luminosity relation to infer distances to nearby galaxies thereby discovering the expansion of the universe

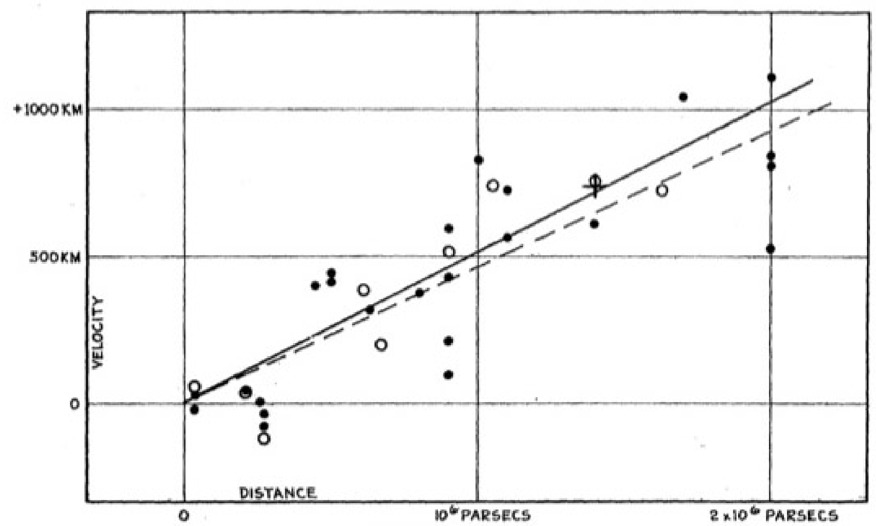


FIGURE 1

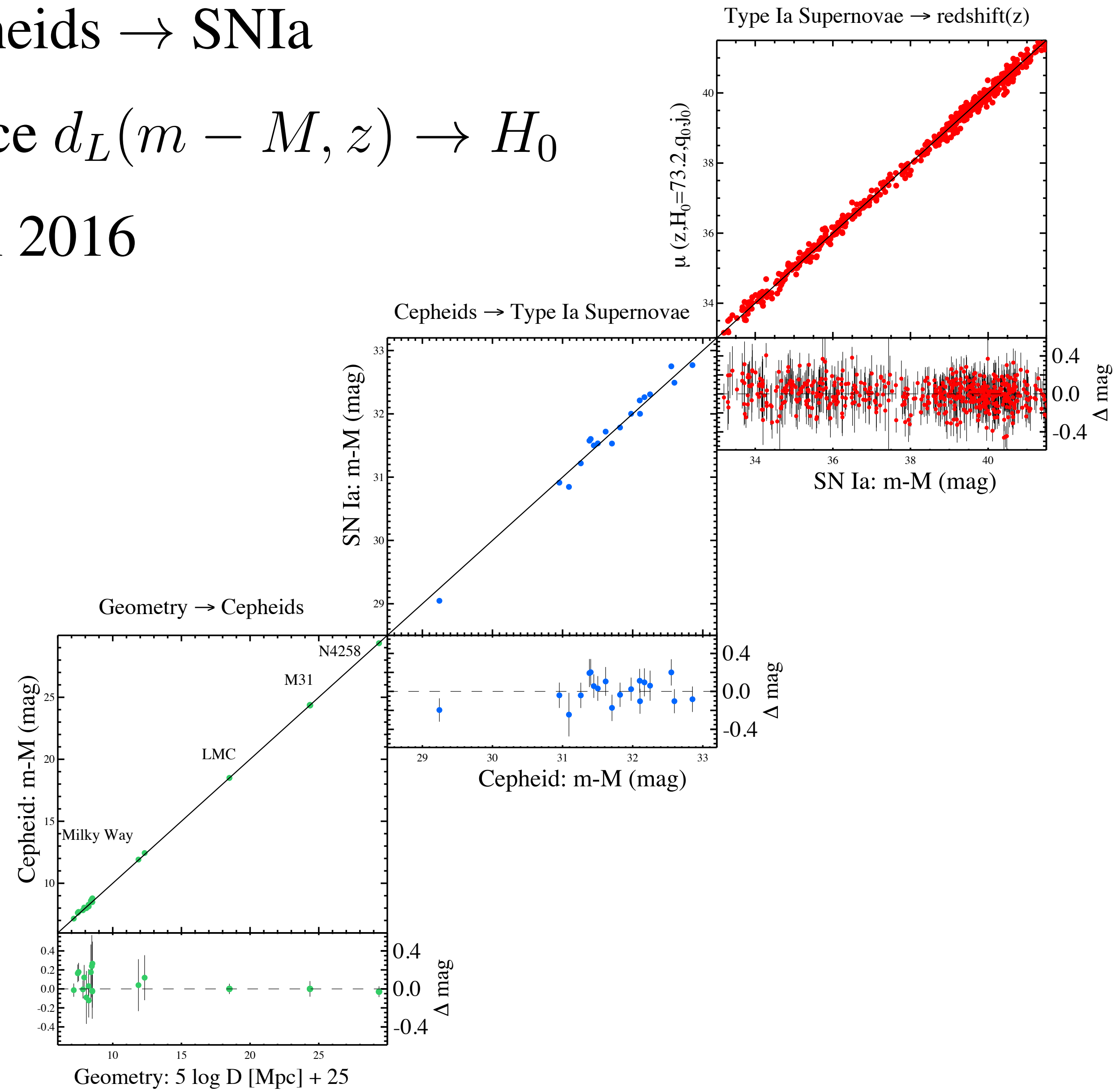
- Hubble actually inferred too large a Hubble constant of $H_0 \sim 500\,\mathrm{km/s/Mpc}$
- Miscalibration of the Cepheid distance scale - absolute measurement hard, checkered history

Hubble Constant History

- Took 70 years to settle on this value with a factor of 2 discrepancy persisting until late 1990's
- Difficult measurement since local galaxies where individual Cepheids can be measured have peculiar motions and so their velocity is not entirely due to the "Hubble flow"
- A "distance ladder" of cross calibrated measurements
- Primary distance indicators cepheids, novae planetary nebula, tip of red giant branch, AGN water maser
- GAIA will soon improve geometric calibration of galactic cepheids with parallax measurements
- More luminous secondary distance indicators into the Hubble flow: Tully-Fisher, fundamental plane, surface brightness fluctuations, Type 1A supernova

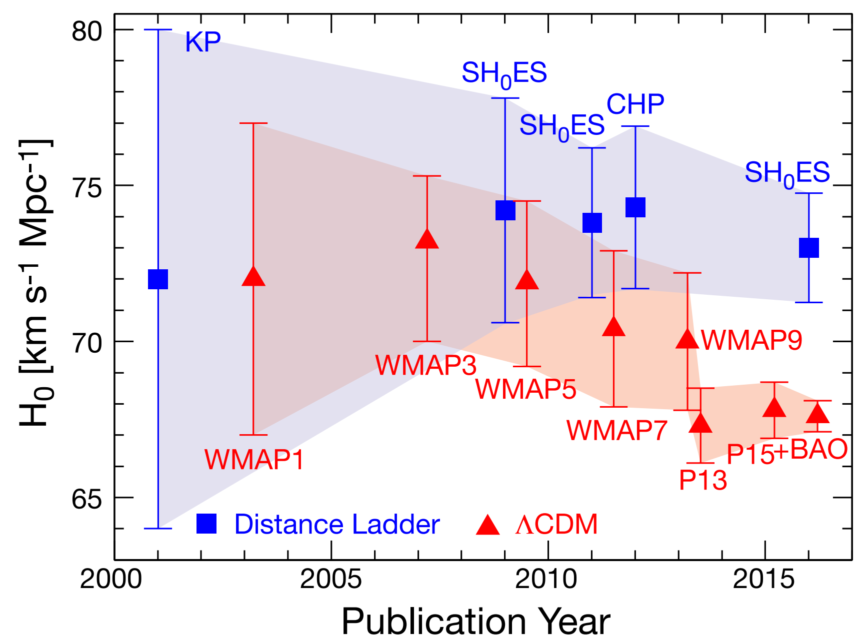
Modern Distance Ladder

- Geometry → Cepheids → SNIa
- Luminosity distance $d_L(m - M, z) \rightarrow H_0$
- SH0ES, Riess et al 2016



Hubble Constant

- H_0 now measured as $73.24 \pm 1.74\,\text{km/s/Mpc}$ by SH0ES calibrating SNIa off cepheids off AGN water maser as well as the local distance ladder.



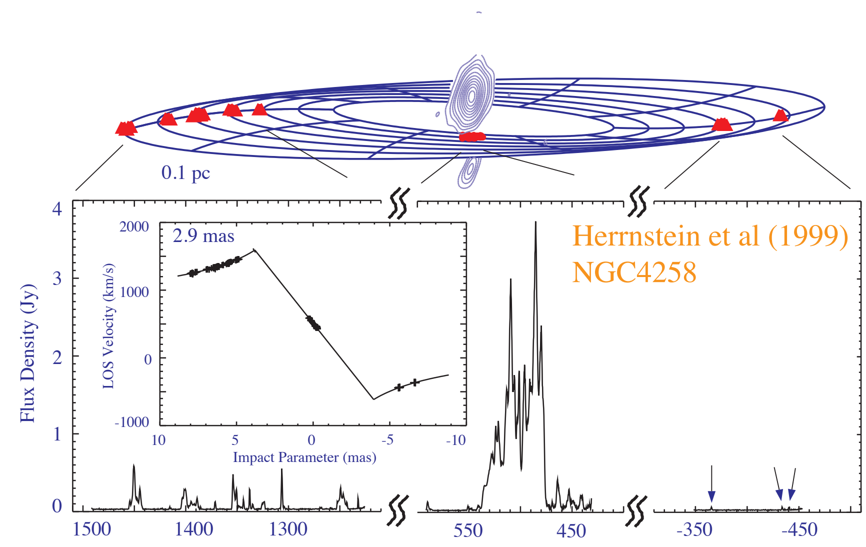
- Comparable precision from Carnegie-Chicago Hubble Program
- Inverse distance ladder: standard ruler CMB calibration at $z \sim 10^3$ to BAO to SNIa
- Assuming the ΛCDM model the inverse distance ladder gives: $H_0 = 67.6 \pm 0.5\,\text{km/s/Mpc}$

Hubble Constant

- Given the history and difficulty of connecting these ladders, this agreement is actually quite impressive – but not within the quoted errors
- Resolution remains to be seen: must ensure that both of these precise measurements are accurate in the presence of systematics.

Maser-Cepheid-SN Distance Ladder

- Water maser around AGN, gas in Keplerian orbit
- Measure proper motion, radial velocity, acceleration of orbit



- Method 1: radial velocity plus orbit infer tangential velocity = distance $\times$ angular proper motion

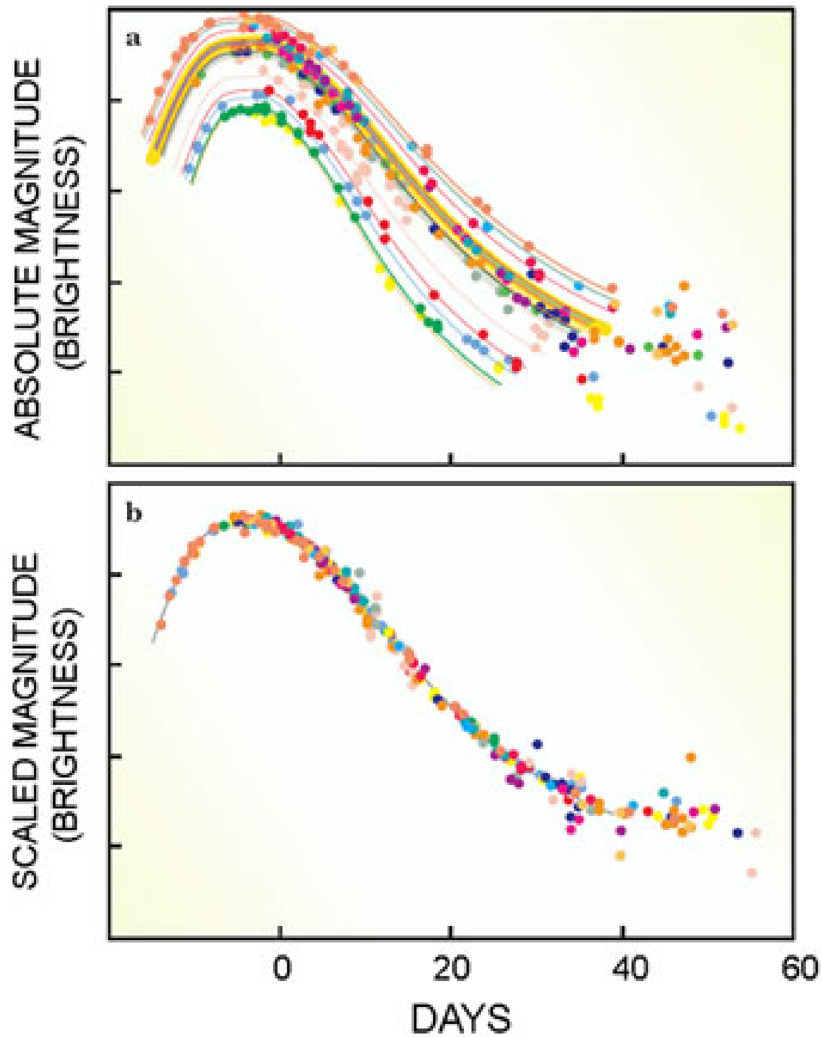
$$v_t = d_A(d\alpha/dt)$$

- Method 2: centripetal acceleration and radial velocity from line infer physical size

$$a = v^2/R, \qquad R = d_A\theta$$

Supernovae as Standard Candles

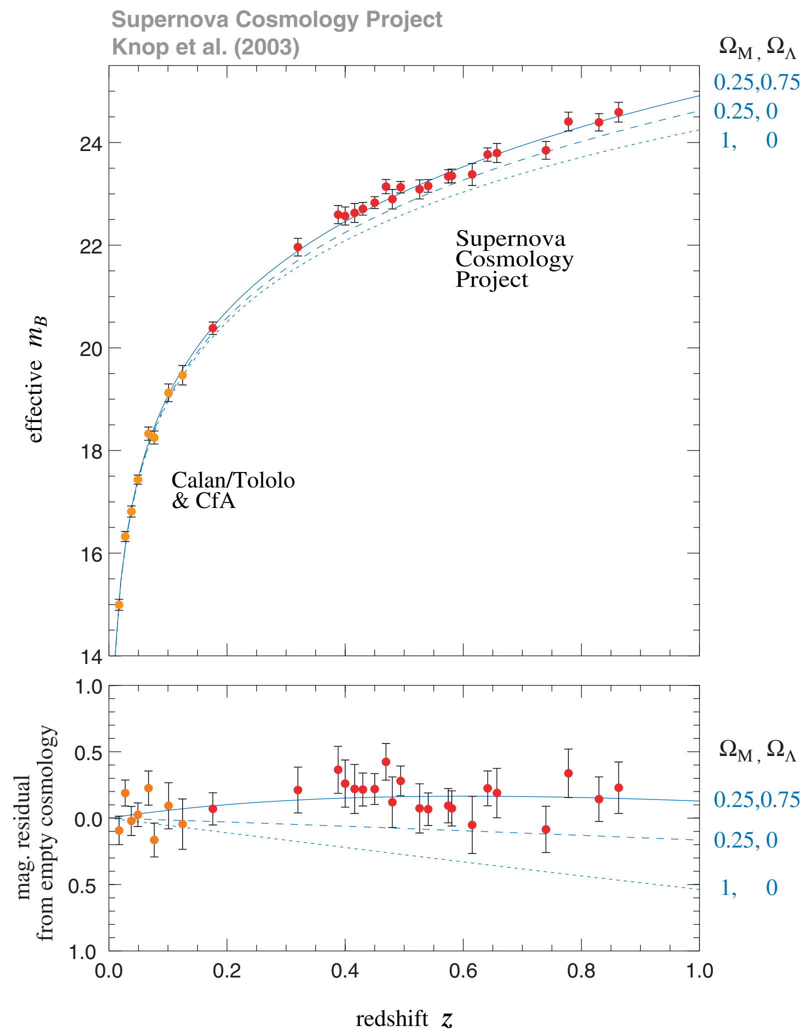
- Type 1A supernovae are white dwarfs that reach Chandrashekar mass where electron degeneracy pressure can no longer support the star, hence a very regular explosion
- Moreover, the scatter in absolute magnitude is correlated with the shape of the light curve - the rate of decline from peak light, empirical "Phillips relation"



- Higher ^{56}N, brighter SN, higher opacity, longer light curve duration

Beyond Hubble's Law

- Type 1A are therefore "standardizable" candles leading to a very low scatter $\delta m \sim 0.15$ and visible out to high redshift $z \sim 1$
- Two groups in 1999 found that SN more distant at a given redshift than expected
- Cosmic acceleration



Beyond Hubble's Law

- Using SN as a relative indicator (independent of absolute magnitude), comparison of low and high z gives

$$H_0 D(z) = \int dz \frac{H_0}{H}$$

more distant implies that $H(z)$ not increasing at expect rate, i.e. is more constant

- Take the limiting case where $H(z)$ is a constant (a.k.a. de Sitter expansion

$$H = \frac{1}{a}\frac{da}{dt} = \text{const}$$
$$\frac{dH}{dt} = \frac{1}{a}\frac{d^2a}{dt^2} - H^2 = 0$$
$$\frac{1}{a}\frac{d^2a}{dt^2} = H^2 > 0$$

Beyond Hubble's Law

- Indicates that the expansion of the universe is accelerating
- Intuition tells us (FRW dynamics shows) ordinary matter decelerates expansion since gravity is attractive
- Ordinary expectation is that

$$H(z > 0) > H_0$$

  so that the Hubble parameter is higher at high redshift
- Or equivalently that expansion rate decreases as it expands

FRW Dynamics

- This is as far as we can go on FRW geometry alone - we still need to know how the scale factor $a(t)$ evolves given matter-energy content
- General relativity: matter tells geometry how to curve, scale factor determined by content
- Build the Einstein tensor $G_{\mu\nu}$ out of the metric and use Einstein equation (overdots conformal time derivative)

$$G_{\mu\nu}\left(= R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) = 8\pi G T_{\mu\nu}$$

- Easier to work with mixed upper and lower indices since the metric $g^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu}$

Einstein Equations

- For the FRW metric

$$
\begin{aligned}
G^0{}_0 &= -3\left(H^2 + \frac{K}{a^2}\right) \\
G^i{}_j - G^0{}_0 \frac{\delta^i{}_j}{3} &= -\frac{2}{a^2}\left(\frac{\ddot{a}}{a} - a^2 H^2\right)\delta^i{}_j = -\frac{2}{a}\frac{d^2 a}{dt^2}\delta^i{}_j ,
\end{aligned}
$$

where recall the curvature $K = 1/R^2$ and overdots are $d/d\eta$

- Likewise isotropy demands that the stress-energy tensor take the form

$$
T^0{}_0 = -\rho, \quad T^i{}_j = p\delta^i{}_j \quad \to \quad T^i{}_j - T^0{}_0 \frac{\delta^i{}_j}{3} = p + \rho/3
$$

where ρ is the energy density and p is the pressure

- It is not necessary to assume that the content is a perfect fluid - consequence of FRW symmetry

Friedmann Equations

- Einstein equations given the FRW symmetries become the Friedmann equations

$$
\begin{aligned}
H^2 + \frac{K}{a^2} &= \frac{8\pi G}{3}\rho \\
\frac{1}{a}\frac{d^2 a}{dt^2} &= -\frac{4\pi G}{3}(\rho + 3p)
\end{aligned}
$$

- Acceleration source is $\rho + 3p$ requiring $p < -\rho/3$ for positive acceleration
- Curvature as an effective energy density component

$$
\rho_K = -\frac{3}{8\pi G}\frac{K}{a^2} \propto a^{-2}
$$

Positive curvature gives negative effective energy density

Critical Density

- Friedmann equation for H then reads

$$H^2(a) = \frac{8\pi G}{3}(\rho + \rho_K) \equiv \frac{8\pi G}{3}\rho_c$$

defining a critical density today ρ_c in terms of the expansion rate

- In particular, its value today is given by the Hubble constant as

$$\rho_{\rm c}(z=0) = \frac{3H_0^2}{8\pi G} = 1.8788 \times 10^{-29} h^2 {\rm g\,cm}^{-3}$$

- Energy density today is given as a fraction of critical

$$\Omega \equiv \frac{\rho}{\rho_c(z=0)}$$

- Note that physical energy density $\propto \Omega h^2$ (g cm^{-3})

Critical Density

- Likewise radius of curvature then given by

$$\Omega_K = (1-\Omega) = -\frac{1}{H_0^2 R^2} \to R = (H_0\sqrt{\Omega - 1})^{-1}$$

- If $\Omega \approx 1$, then true density is near critical $\rho \approx \rho_c$ and

$$\rho_K \ll \rho_c \leftrightarrow H_0 R \ll 1$$

Universe is flat across the Hubble distance

- $\Omega > 1$ positively curved

$$D_A = R\sin(D/R) = \frac{1}{H_0\sqrt{\Omega - 1}}\sin(H_0 D\sqrt{\Omega - 1})$$

- $\Omega < 1$ negatively curved

$$D_A = R\sin(D/R) = \frac{1}{H_0\sqrt{1-\Omega}}\sinh(H_0 D\sqrt{1-\Omega})$$

Newtonian Energy Interpretation

- Consider a test particle of mass m as part of expanding spherical shell of radius r and total mass M.

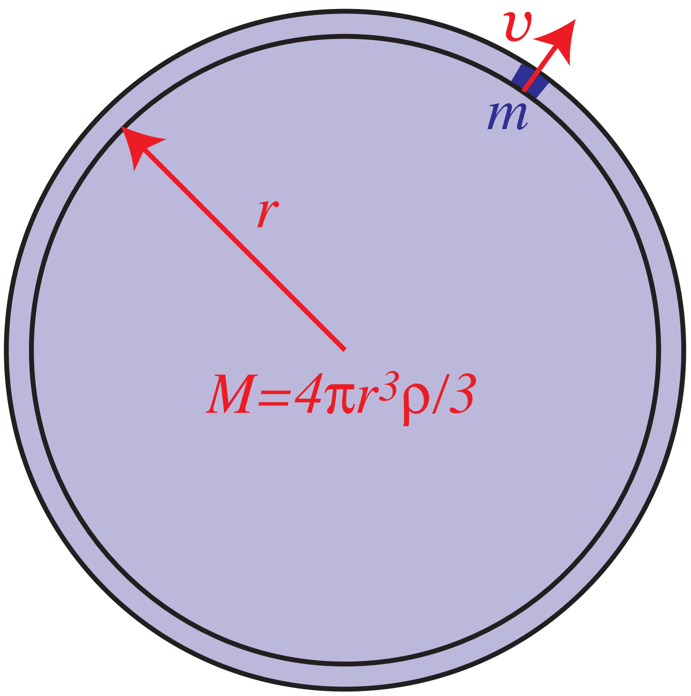
- Energy conservation

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r} = \text{const}$$

$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2 - \frac{GM}{r} = \text{const}$$

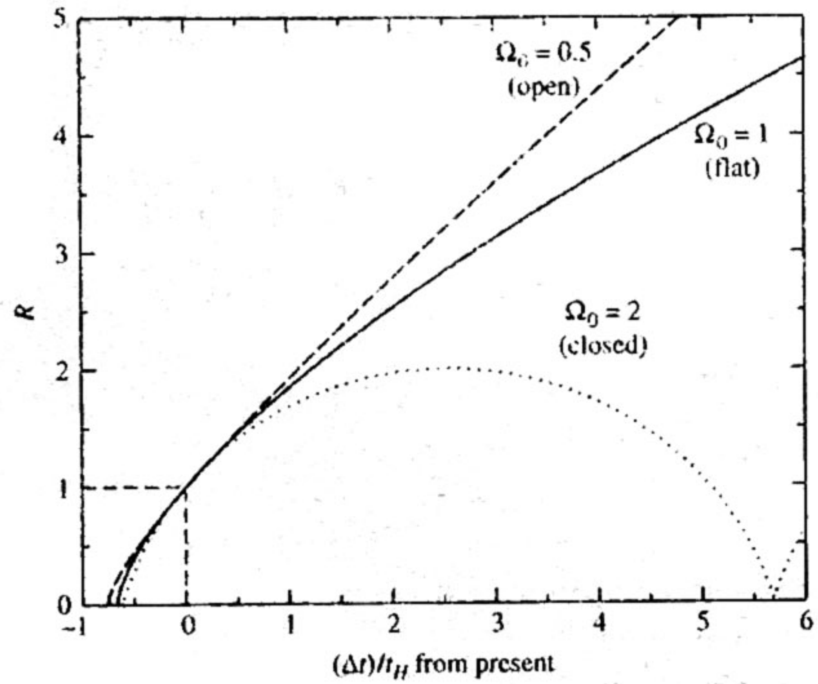
$$\frac{1}{2}\left(\frac{1}{r}\frac{dr}{dt}\right)^2 - \frac{GM}{r^3} = \frac{\text{const}}{r^2}$$

$$H^2 = \frac{8\pi G\rho}{3} - \frac{\text{const}}{a^2}$$



Newtonian Energy Interpretation

- Constant determines whether the system is bound and in the Friedmann equation is associated with curvature – not general since neglects pressure



- Nonetheless Friedmann equation is the same with pressure - but mass-energy within expanding shell is not constant
- Instead, rely on the fact that gravity in the weak field regime is Newtonian and forces unlike energies are unambiguously defined locally.

Newtonian Force Interpretation

- An alternate, more general Newtonian derivation, comes about by realizing that locally around an observer, gravity must look Newtonian.
- Given Newton's iron sphere theorem, the gravitational acceleration due to a spherically symmetric distribution of mass outside some radius r is zero (Birkhoff theorem in GR)
- We can determine the acceleration simply from the enclosed mass

$$\nabla^2 \Psi_N = 4\pi G(\rho + 3p)$$

$$\nabla \Psi_N = \frac{4\pi G}{3}(\rho + 3p) r = \frac{G M_N}{r^2}$$

where $\rho + 3p$ reflects the active gravitational mass provided by pressure.

Newtonian Force Interpretation

- Hence the gravitational acceleration

$$\frac{\ddot{r}}{r} = -\frac{1}{r}\nabla\Psi_N = -\frac{4\pi G}{3}(\rho + 3p)$$

- We'll come back to this way of viewing the effect of the expansion on spherical collapse including the dark energy.

Conservation Law

- The two Friedmann equation are redundant in that one can be derived from the other via energy conservation
  - (Consequence of Bianchi identities in GR: $\nabla^\mu G_{\mu\nu} = 0$)

$$
\begin{aligned}
d\rho V + p dV &= 0 \\
d\rho a^3 + p d a^3 &= 0 \\
\dot\rho a^3 + 3\frac{\dot a}{a}\rho a^3 + 3\frac{\dot a}{a} p a^3 &= 0 \\
\frac{\dot\rho}{\rho} = -3(1+w)\frac{\dot a}{a}
\end{aligned}
$$

  - Time evolution depends on "equation of state" $w(a) = p/\rho$
- If $w =$ const. then the energy density depends on the scale factor as $\rho \propto a^{-3(1+w)}$.

Multicomponent Universe

- Special cases:
  - nonrelativistic matter $\rho_m = m n_m \propto a^{-3}$, $w_m = 0$
  - ultrarelativistic radiation $\rho_r = E n_r \propto \nu n_r \propto a^{-4}$, $w_r = 1/3$
  - curvature $\rho_K \propto a^{-2}$, $w_K = -1/3$
  - (cosmological) constant energy density $\rho_\Lambda \propto a^0$, $w_\Lambda = -1$
  - total energy density summed over above

$$\rho(a) = \sum_i \rho_i(a) = \rho_c(a=1) \sum_i \Omega_i a^{-3(1+w_i)}$$

  - If constituent w also evolve (e.g. massive neutrinos)

$$\rho(a) = \rho_c(a=1) \sum_i \Omega_i e^{-\int d\ln a \, 3(1+w_i)}$$

Multicomponent Universe

- Friedmann equation gives Hubble parameter evolution in a

$$H^2(a) = H_0^2 \sum \Omega_i e^{-\int d\ln a\, 3(1+w_i)}$$

- In fact we can always define a critical equation of state

$$w_c = \frac{p_c}{\rho_c} = \frac{\sum w_i \rho_i - \rho_K/3}{\sum_i \rho_i + \rho_K}$$

- Critical energy density obeys energy conservation

$$\rho_c(a) = \rho_c(a=1) e^{-\int d\ln a\, 3(1+w_c(a))}$$

- And the Hubble parameter evolves as

$$H^2(a) = H_0^2 e^{-\int d\ln a\, 3(1+w_c(a))}$$

Acceleration Equation

- Time derivative of (first) Friedmann equation

$$
\begin{aligned}
\frac{dH^2}{dt} &= \frac{8\pi G}{3}\frac{d\rho_c}{dt} \\
2H\left[\frac{1}{a}\frac{d^2a}{dt^2} - H^2\right] &= \frac{8\pi G}{3}H[-3(1+w_c)\rho_c] \\
\left[\frac{1}{a}\frac{d^2a}{dt^2} - 2\frac{4\pi G}{3}\rho_c\right] &= -\frac{4\pi G}{3}[3(1+w_c)\rho_c] \\
\frac{1}{a}\frac{d^2a}{dt^2} &= -\frac{4\pi G}{3}[(1+3w_c)\rho_c] \\
&= -\frac{4\pi G}{3}(\rho + \rho_K + 3p + 3p_K) \\
&= -\frac{4\pi G}{3}(1+3w)\rho
\end{aligned}
$$

- Acceleration equation says that universe decelerates if $w > -1/3$

Expansion Required

- Friedmann equations "predict" the expansion of the universe. Non-expanding conditions $da/dt = 0$ and $d^2a/dt^2 = 0$ require

$$\rho = -\rho_K \qquad \rho = -3p$$

  i.e. a positive curvature and a total equation of state $w \equiv p/\rho = -1/3$

- Since matter is known to exist, one can in principle achieve this by adding a balancing cosmological constant

$$\rho = \rho_m + \rho_\Lambda = -\rho_K = -3p = 3\rho_\Lambda$$

$$\rho_\Lambda = -\frac{1}{3}\rho_K \,, \quad \rho_m = -\frac{2}{3}\rho_K$$

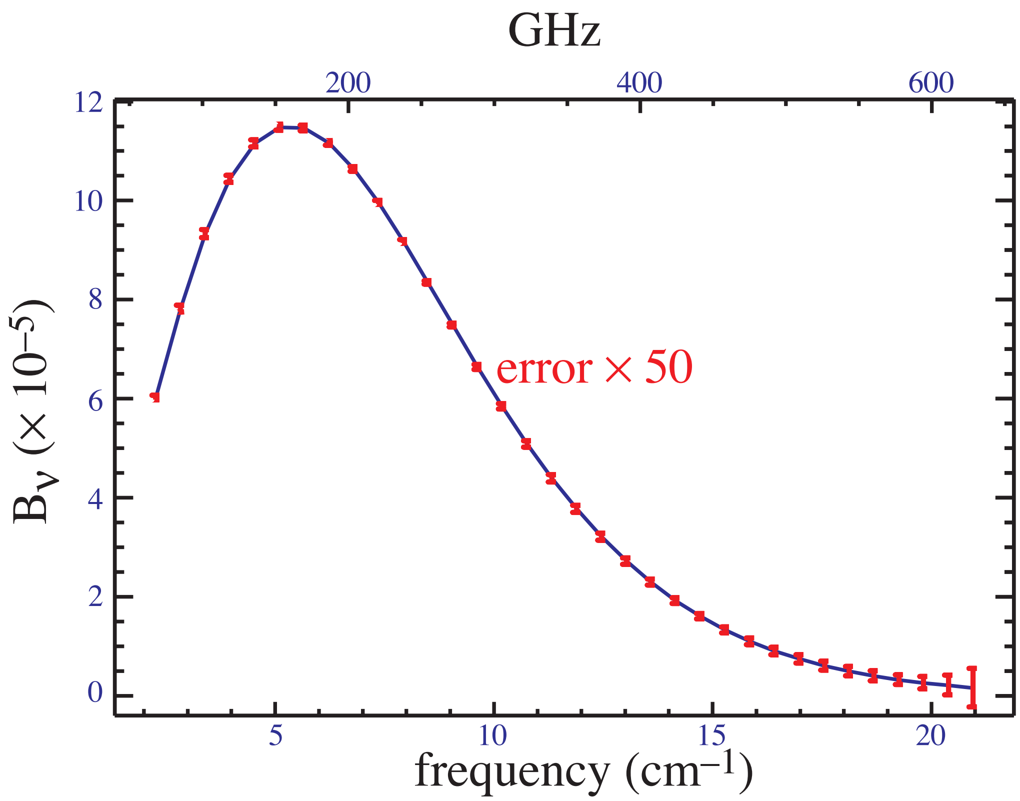
  Einstein introduced ρ_Λ for exactly this reason – "biggest blunder"; but note that this balance is unstable: ρ_m can be perturbed but ρ_Λ, a true constant cannot

Cosmic Microwave Radiation

- Existence of a $\sim$ 10K radiation background predicted by Gamow and Alpher in 1948 based on the formation of light elements in a hot big bang (BBN)
- Peebles, Dicke, Wilkinson & Roll in 1965 independently predicted this background and proceeded to build instrument to detect it
- Penzias & Wilson 1965 found unexplained excess isotropic noise in a communications antennae and learning of the Peebles et al calculation announced the discovery of the blackbody radiation
- Thermal radiation proves that the universe began in a hot dense state when matter and radiation was in equilibrium - ruling out a competing steady state theory

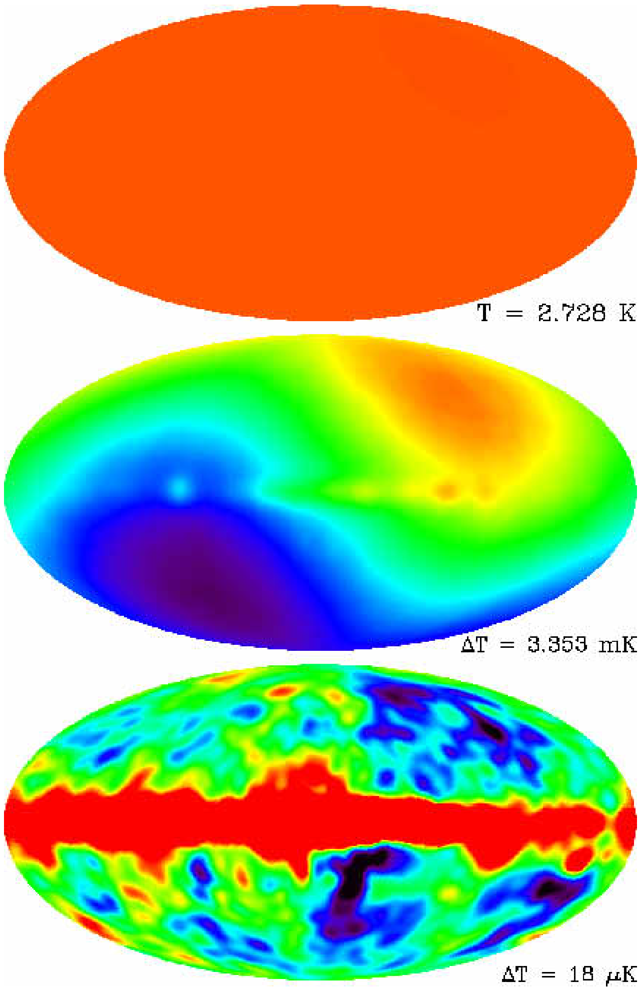
Cosmic Microwave Radiation

- Modern measurement from COBE satellite of blackbody spectrum.
  $T = 2.725\text{K}$, $\rho_\gamma = (\pi^2/15)T^4$ giving $\Omega_\gamma h^2 = 2.471 \times 10^{-5}$



Cosmic Microwave Radiation

- Radiation is isotropic to 10^{-5} in temperature $\rightarrow$ horizon problem



Total Radiation

- Adding in neutrinos to the radiation gives the total radiation (next lecture set) content as $\Omega_r h^2 = 4.15 \times 10^{-5}$
- Since radiation redshifts faster than matter by one factor of $1 + z$ even this small radiation content will dominate the total energy density at sufficiently high redshift
- Matter-radiation equality

$$1 + z_{\rm eq} = \frac{\Omega_m h^2}{\Omega_r h^2}$$

$$1 + z_{\rm eq} = 3130 \frac{\Omega_m h^2}{0.13}$$

Dark Matter

- Since Zwicky in the 1930's non-luminous or dark matter has been known to dominate over luminous matter in stars (and hot gas)
- Arguments based on internal motion holding system up against gravitational force
- Equilibrium requires a balance pressure of internal motions

    rotation velocity of spiral galaxies

    velocity dispersion of galaxies in clusters

    gas pressure or thermal motion in clusters

    radiation pressure in CMB acoustic oscillations

Classical Argument

- Classical argument for measuring total amount of dark matter
- Assuming that the object is somehow typical in its non-luminous to luminous density: "mass-to-light ratio"
- Convert to dark matter density as $M/L\times$ luminosity density
- From galaxy surveys the luminosity density in solar units is

$$\rho_L = 2 \pm 0.7 \times 10^8 h\, L_\odot \mathrm{Mpc}^{-3}$$

  (h's: $L \propto F d^2$ so $\rho_L \propto L/d^3 \propto d^{-1}$ and d in h^{-1} Mpc
- Critical density in solar units is

$$\rho_c = 2.7754 \times 10^{11} h^2\, M_\odot \mathrm{Mpc}^{-3}$$

  so that the critical mass-to-light ratio in solar units is

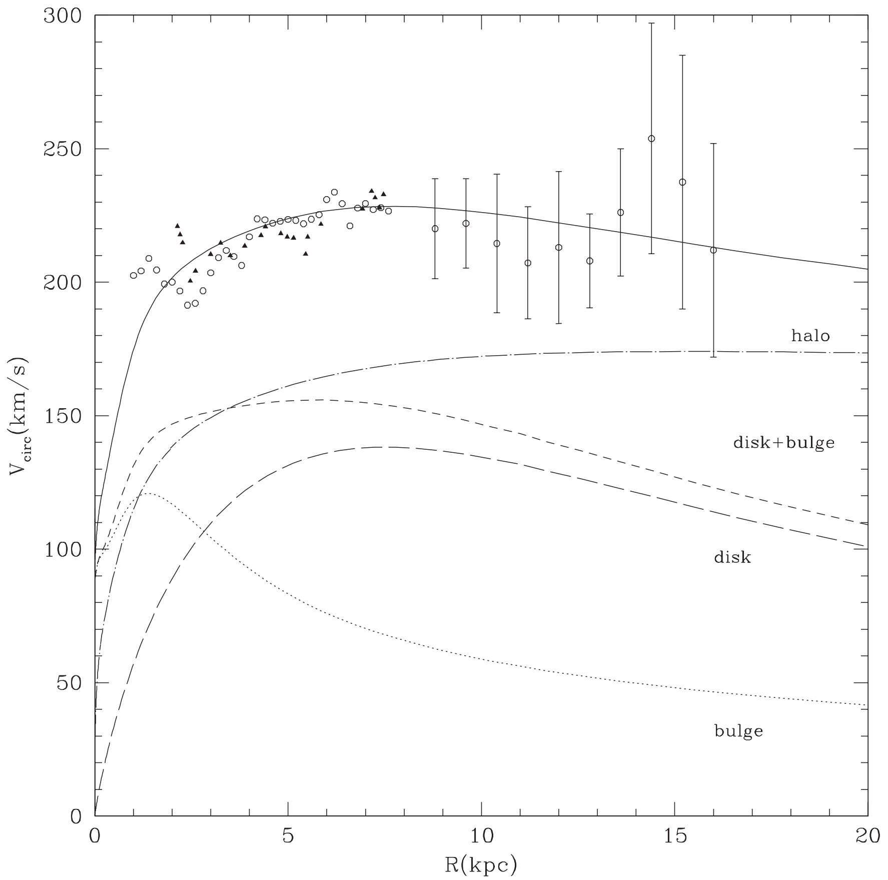
$$M/L \approx 1400h$$

Dark Matter: Rotation Curves

- Flat rotation curves:

$$GM/r^2 \approx v^2/r$$
$$M \approx v^2 r/G$$

  so $M \propto r$ out to tens of kpc



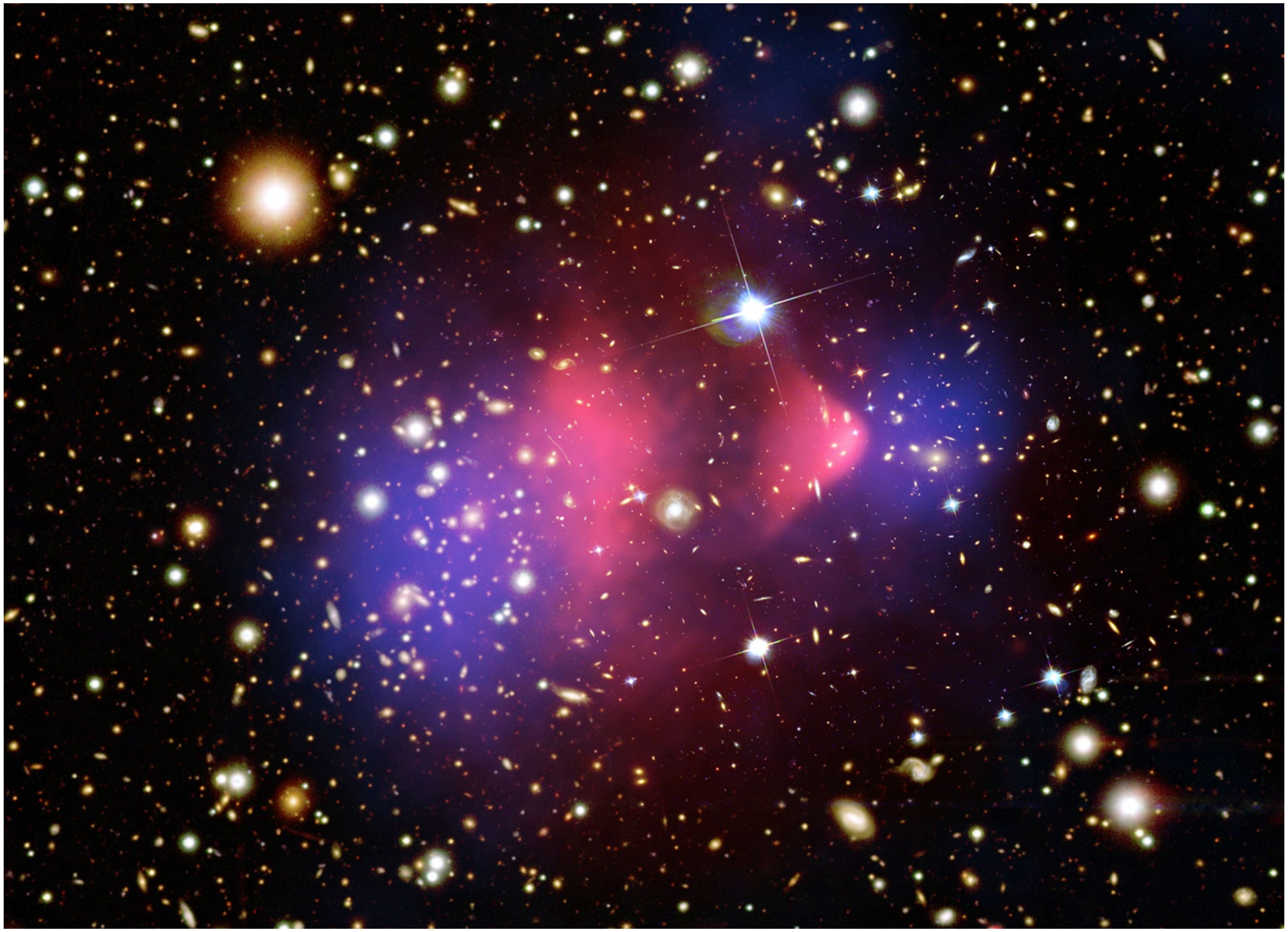
- Implies $M/L > 30h$
  and perhaps more –
  closure if flat out to ~ 1 Mpc.
- Mass required to keep rotation
  curves flat much larger than implied by stars and gas.
- Hence "dark" matter

Dark Matter: Clusters

- Similar argument holds in clusters of galaxies
- Velocity dispersion replaces circular velocity
- Centripetal force is replaced by a "pressure gradient" $T/m = \sigma^2$ and $p = \rho T/m = \rho\sigma^2$
- Zwicky got $M/L \approx 300h$.
- Generalization to the gas distribution also gives evidence for dark matter

Dark Matter: Bullet Cluster

- Merging clusters: gas (visible matter) collides and shocks (X-rays), dark matter measured by gravitational lensing passes through



Hydrostatic Equilibrium

- Evidence for dark matter in X-ray clusters also comes from direct hydrostatic equilibrium inference from the gas: balance radial pressure gradient with gravitational potential gradient
- Infinitesimal volume of area dA and thickness dr at radius r and interior mass $M(r)$: pressure difference supports the gas

$$[p_g(r) - p_g(r+dr)]dA = \frac{GmM}{r^2} = \frac{G\rho_g M}{r^2} dV$$
$$\frac{dp_g}{dr} = -\rho_g \frac{d\Phi}{dr}$$

with $p_g = \rho_g T_g / m$ becomes

$$\frac{GM}{r} = -\frac{T_g}{m}\left(\frac{d\ln\rho_g}{d\ln r} + \frac{d\ln T_g}{d\ln r}\right)$$

- ρ_g from X-ray luminosity; T_g sometimes taken as isothermal

CMB Hydrostatic Equilibrium

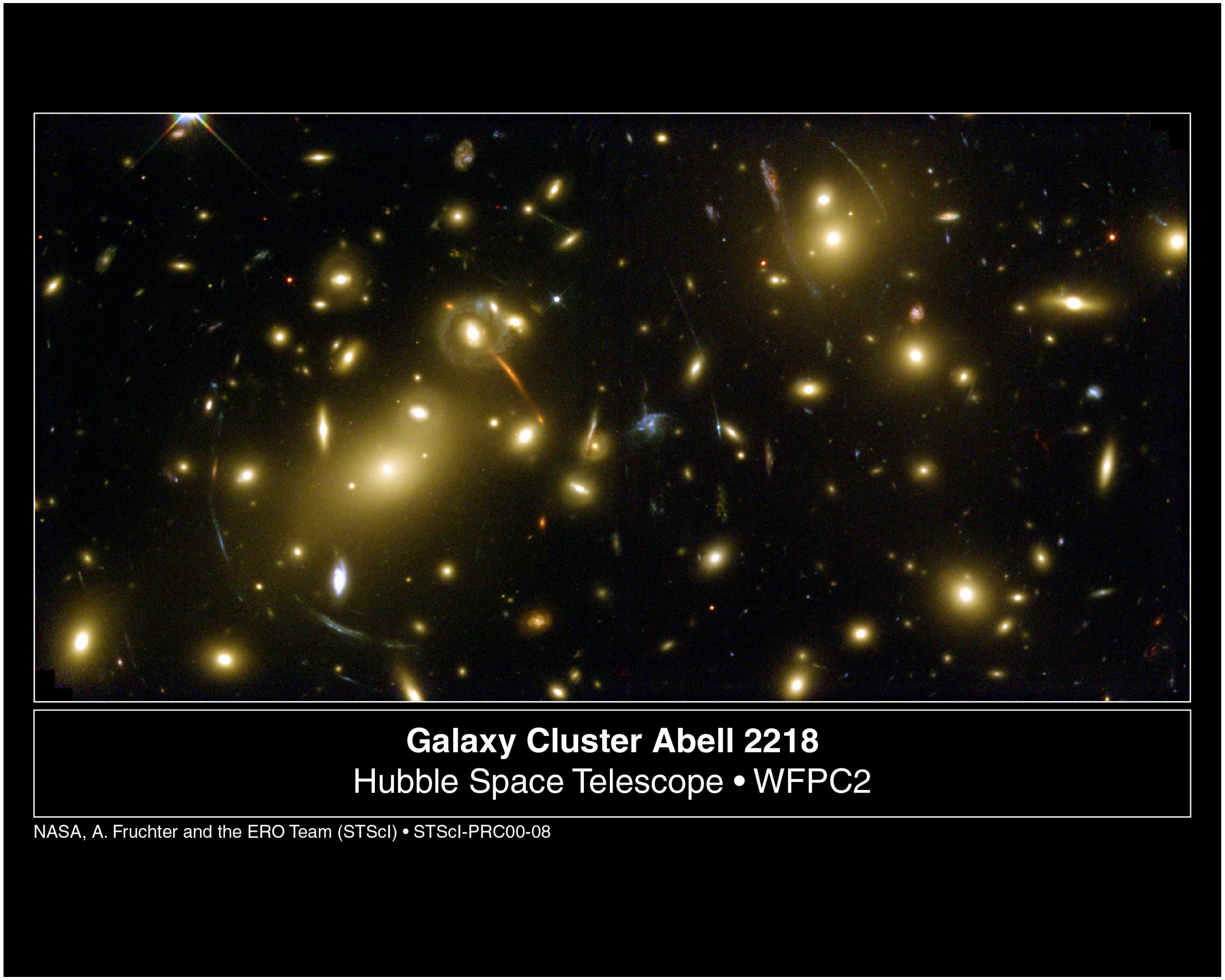
- Same argument in the CMB with radiation pressure in the gas balancing the gravitational potential gradients of linear fluctuations
- Best measurement of the dark matter density to date (Planck 2015): $\Omega_c h^2 = 0.1188 \pm 0.0010$, $\Omega_b h^2 = (2.23 \pm 0.014) \times 10^{-2}$.
- Unlike other techniques, measures the physical density of the dark matter rather than contribution to critical since the CMB temperature sets the physical density and pressure of the photons

Gravitational Lensing

- Mass can be directly measured in the gravitational lensing of sources behind the cluster
- Strong lensing (giant arcs) probes central region of clusters
- Weak lensing (1-10%) elliptical distortion to galaxy image probes outer regions of cluster and total mass

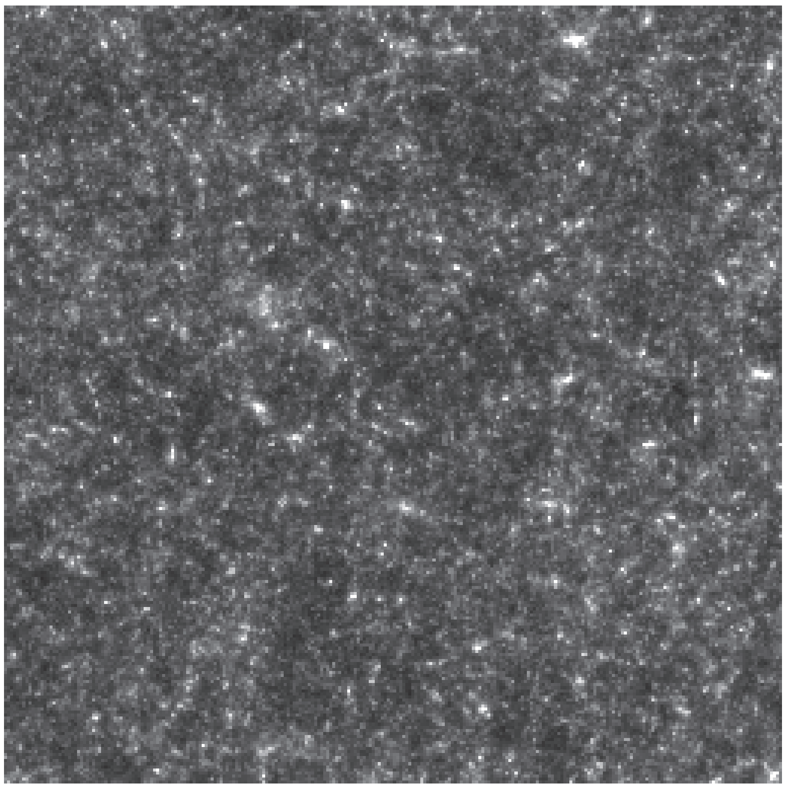
Giant Arcs

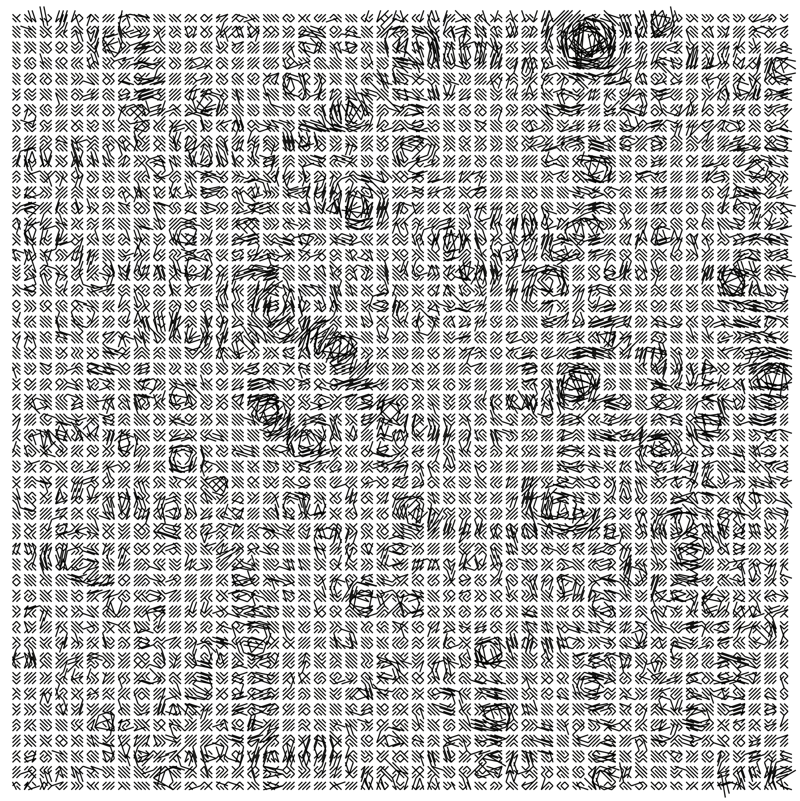
- Giant arcs in galaxy clusters: galaxies, source; cluster, lens



Cosmic Shear

- On even larger scales, the large-scale structure weakly shears background images: weak lensing





Dark Energy

- Distance redshift relation depends on energy density components

$$H_0 D(z) = \int dz \frac{H_0}{H(a)}$$

- SN dimmer, distance further than in a matter dominated epoch
- Hence $H(a)$ must be smaller than expected in a matter only $w_c = 0$ universe where it increases as $(1+z)^{3/2}$

$$H_0 D(z) = \int dz e^{\int d\ln a \, \frac{3}{2}(1+w_c(a))}$$

- Distant supernova Ia as standard candles imply that $w_c < -1/3$ so that the expansion is accelerating
- Consistent with a cosmological constant that is $\Omega_\Lambda \approx 0.70$
- Coincidence problem: different components of matter scale differently with a. Why are two components comparable today?

Cosmic Census

- With $h = 0.68$ and CMB $\Omega_m h^2 = 0.14$, $\Omega_m = 0.30$ - consistent with other, less precise, dark matter measures
- CMB provides a test of $D_A \neq D$ through the standard rulers of the acoustic peaks and shows that the universe is close to flat $\Omega \approx 1$
- Consistency has lead to the standard model for the cosmological matter budget:
  - 70% dark energy
  - 30% non-relativistic matter (with 84% of that in dark matter)
  - 0% spatial curvature