Astro 321

Set 2: Thermal History

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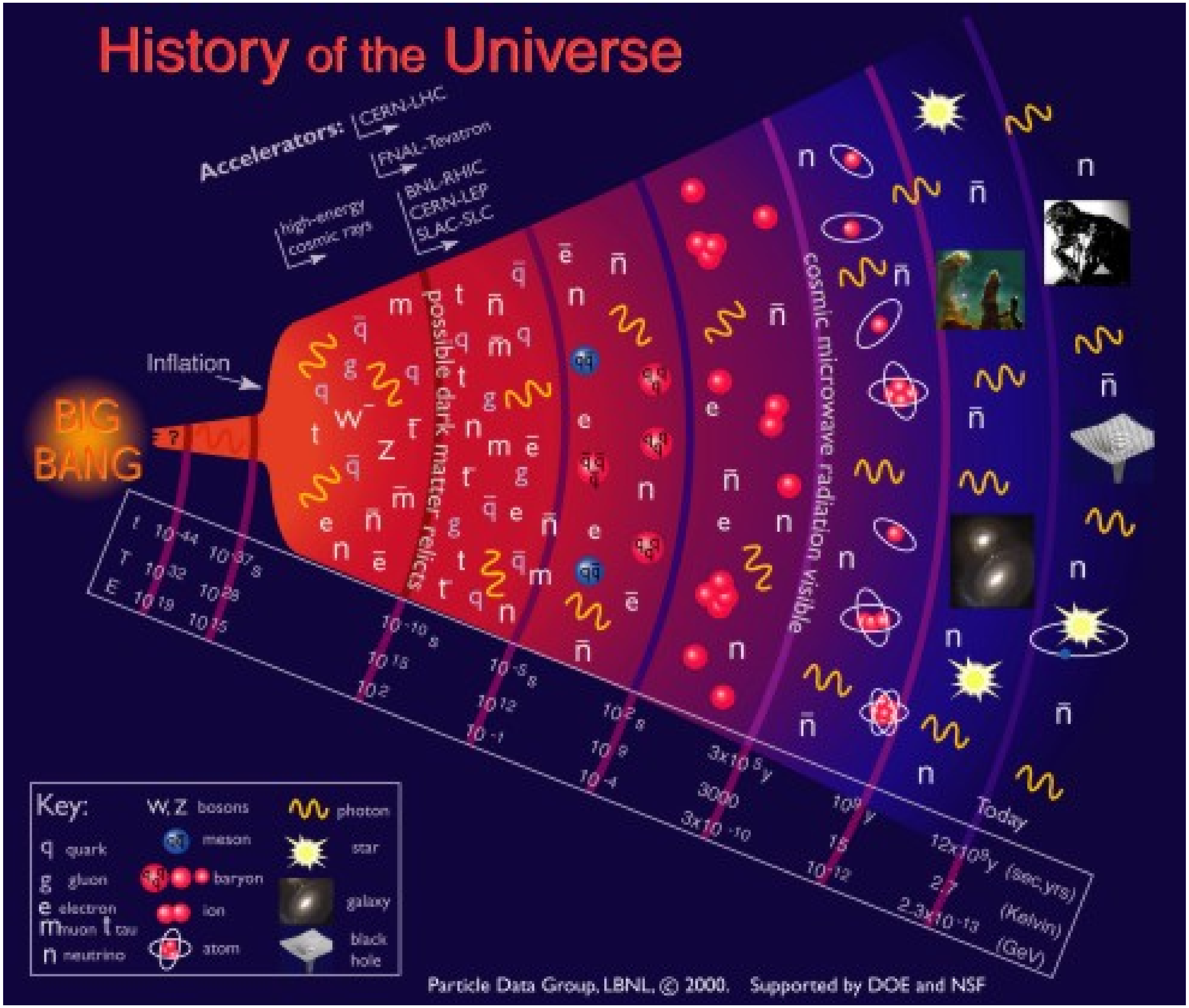
Macro vs Micro Description

- In the first set of notes, we used a macroscopic description.
- Gravity only cares about bulk properties: energy density, momentum density, pressure, anisotropic stress – stress tensor
- Matter and radiation is composed of particles whose properties can be described by their phase space distribution or occupation function
- Macroscopic properties are integrals or moments of the phase space distribution
- Particle interactions involve the evolution of the phase space distribution
- Rapid interactions drive distribution to thermal equilibrium but must compete with the expansion rate of universe
- Freeze out, the origin of species

Brief Thermal History



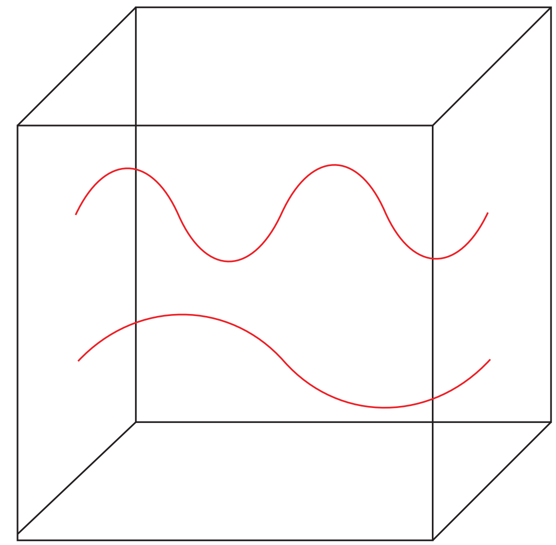
Astro-Particle Dictionary

Astro and physics literature use different words to describe same thing:

- Specific intensity $I_\nu \leftrightarrow$ phase space distribution f
- Surface brightness conservation $\leftrightarrow$ Liouville equation
- Absorption, emission, scattering $\leftrightarrow$ Collision term
- Einstein relations for absorption, stimulated and spontaneous emission $\leftrightarrow$ matrix element determines strength of all interactions of a given type
- Radiative transfer equation $\leftrightarrow$ Boltzmann equation
- Optically thin conditions $\leftrightarrow$ freezeout of interactions

We take physics notation but the content is the same as in other astro courses but placed in an expanding universe context.

Allowed Particle States



- Counting momentum states with momentum q and de Broglie wavelength

$$\lambda = \frac{h}{q} = \frac{2\pi\hbar}{q}$$

- In a discrete volume L^3 there is a discrete set of states that satisfy periodic boundary conditions
- We will hereafter set $\hbar = c = 1$
- As in Fourier analysis

$$e^{2\pi i x/\lambda} = e^{iqx} = e^{iq(x+L)} \to e^{iqL} = 1$$

Fitting in a Box

- Periodicity yields a discrete set of allowed states

$$Lq = 2\pi m_i, \quad m_i = 1, 2, 3...$$
$$q_i = \frac{2\pi}{L} m_i$$

- In each of 3 directions

$$\sum_{m_{xi} m_{yj} m_{zk}} \to \int d^3 m$$

- The differential number of allowed momenta in the volume

$$d^3 m = \left(\frac{L}{2\pi}\right)^3 d^3 q$$

Density of States

- The total number of states allows for a number of internal degrees of freedom, e.g. spin, quantified by the degeneracy factor g
- Total density of states:

$$\frac{dN_s}{V} = \frac{g}{V} d^3 m = \frac{g}{(2\pi)^3} d^3 q$$

- If all states were occupied by a single particle, then particle density

$$n_s = \frac{N_s}{V} = \frac{1}{V} \int dN_s = \int \frac{g}{(2\pi)^3} d^3 q$$

Distribution Function

- The distribution function f quantifies the occupation of the allowed momentum states

$$n = \frac{N}{V} = \frac{1}{V}\int f dN_s = \int \frac{g}{(2\pi)^3} f d^3q$$

- f, aka phase space occupation number, also quantifies the density of particles per unit phase space $dN/(\Delta x)^3(\Delta q)^3$
- For photons, the spin degeneracy $g = 2$ accounting for the 2 polarization states
- Energy $E(q) = (q^2 + m^2)^{1/2}$
- Momentum $\rightarrow$ frequency $q = 2\pi/\lambda = 2\pi\nu = \omega = E$ (where $m = 0$ and $\lambda\nu = c = 1$)

Bulk Properties

- Integrals over the distribution function define the bulk properties of the collection of particles
- Number density

$$n(\mathbf{x}, t) \equiv \frac{N}{V} = g \int \frac{d^3q}{(2\pi)^3} f$$

- Energy density

$$\rho(\mathbf{x}, t) = g \int \frac{d^3q}{(2\pi)^3} E(q) f$$

where $E^2 = q^2 + m^2$

- Momentum density

$$(\rho + p)\mathbf{v}(\mathbf{x}, t) = g \int \frac{d^3q}{(2\pi)^3} \mathbf{q} f$$

Vacuum Energy

- We have assumed here that the state of zero particles has zero energy
- In QFT, like the simple harmonic oscillator in ordinary quantum mechanics, there is a zero point energy to the ground state
- For bosons, $\hbar\omega/2 = E(q)/2$, so the most naive version of the cosmological constant problem is that $\rho \propto M^4$ where $M = M_{\rm Pl} = 1/\sqrt{8\pi G}$ if the theory applies out to the Planck scale
- The critical energy density $\rho_c = 3H_0^2/8\pi G \approx 8 \times 10^{-47} h^2 \mathrm{GeV}^4$ is more than 10^{120} off $M_{\rm Pl}^4 \approx 2 \times 10^{76}$ GeV^4.
- Note that $p_{\rm vac} \approx \rho_{\rm vac}/3$ so this fixed momentum cutoff calculation is a bit too naive since we know that $p_{\rm vac} = -\rho_{\rm vac}$

Vacuum Energy

- A Lorentz invariant renormalization scheme corrects this to

$$\rho_{\text{vac}} = \frac{m^4}{64\pi^2} \ln(m^2/\mu^2)$$

  where μ is some renormalization scale

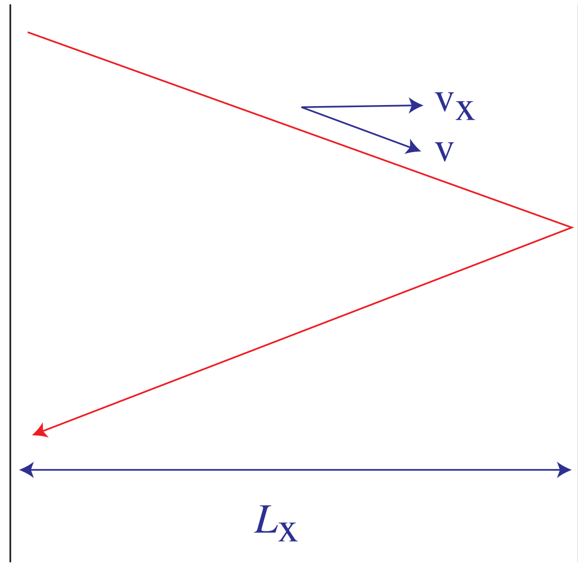
- But even if there are no mass states above the known standard model bosons, e.g. Higgs boson of $m \approx 125\text{GeV}$, this is way off, even though it helps by some 68 orders of magnitude!

- Caveat: fermions contribute negatively to the vacuum energy so if supersymmetry is unbroken would cancel

- But supersymmetry is clearly broken at low energies and has yet to be seen at LHC - so taking this as a lower limit on the scale of supersymmetry breaking the vacuum energy $m > 1\text{TeV}$, m^4 is still 60 orders of magnitude off.

Bulk Properties

- Pressure: particles bouncing off a surface of area A in a volume spanned by L_x: per momentum state



$$p_q = \frac{F}{A} = \frac{N_{\rm part}}{A}\frac{\Delta q}{\Delta t}$$

$$(\Delta q = 2|q_x|, \quad \Delta t = 2L_x/v_x,)$$

$$= \frac{N_{\rm part}}{V}|q_x||v_x| = \frac{N_{\rm part}}{V}\frac{|q||v|}{3}$$

$$(v = \gamma m v/\gamma m = q/E)$$

$$= \frac{N_{\rm part}}{V}\frac{q^2}{3E}$$

Bulk Properties

- So that summed over occupied momenta states

$$p(\mathbf{x},t) = g\int \frac{d^3q}{(2\pi)^3}\frac{|q|^2}{3E(q)}f$$

- Pressure is just one of the quadratic in q moments, in particular the isotropic one
- The remaining 5 components are the anisotropic stress (vanishes in the background)

$$\pi^i{}_j(\mathbf{x},t) = g\int \frac{d^3q}{(2\pi)^3}\frac{3q^iq_j - q^2\delta^i{}_j}{3E(q)}f$$

- We shall see that these are related to the 5 quadrupole moments of the angular distribution

Bulk Properties

- These are more generally the components of the stress-energy tensor

$$T^{\mu}{}_{\nu} = g \int \frac{d^3 q}{(2\pi)^3} \frac{q^{\mu} q_{\nu}}{E(q)} f$$

- 0-0: energy density
- 0-i: momentum density
- $i - i$: pressure
- $i \neq j$: anisotropic stress
- In the FRW background cosmology, isotropy requires that there be only a net energy density and pressure

Observable Properties

- Only get to measure luminous properties of the universe. For photons mass $m = 0$, $g = 2$ (units: $J\ m^{-3}$)

$$\rho(\mathbf{x}, t) = 2\int \frac{d^3q}{(2\pi)^3} qf = 2\int dq d\Omega \left(\frac{q}{2\pi}\right)^3 f$$

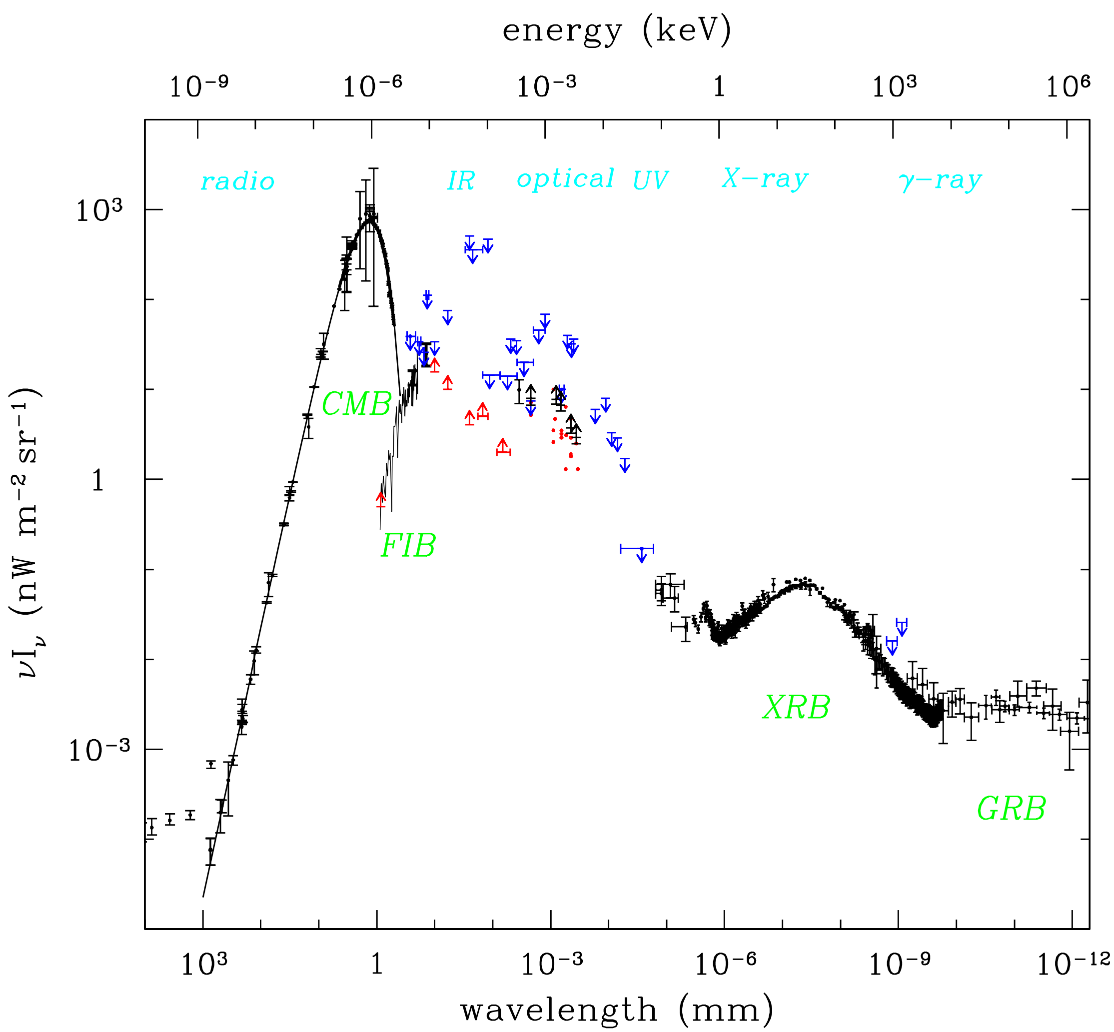
- Spectral energy density (per unit frequency $q = h\nu = \hbar 2\pi\nu = 2\pi\nu$, solid angle)

$$u_\nu = \frac{d\rho}{d\nu d\Omega} = 2(2\pi)\nu^3 f$$

- Photons travelling at speed of light so that $u_\nu = I_\nu = 4\pi\nu^3 f$ the specific intensity or brightness, energy flux across a surface, units of W m^{-2} Hz^{-1} sr^{-1} (SI); ergs s^{-1} cm^{-2} Hz^{-1} sr^{-1} (cgs)

Diffuse Extragalactic Light

- νI_ν peaks in the microwave mm-cm region: CMB black body $T = 2.725 \pm 0.002K$ or $n_\gamma = 410$ cm^{-3}, $\Omega_\gamma = 2.47 \times 10^{-5} h^{-2}$.



Observable Properties

- Integrate over frequencies for total intensity

$$I = \int d\nu I_\nu = \int d\ln\nu\, \nu I_\nu$$

νI_ν often plotted since it shows peak under a log plot; I and νI_ν have units of W m^{-2} sr^{-1} and is independent of choice of frequency unit

- Flux density (specific flux): integrate over the solid angle of a radiation source, units of W m^{-2} Hz^{-1} or Jansky = 10^{-26} W m^{-2} Hz^{-1}

$$F_\nu = \int_{\rm source} I_\nu d\Omega$$

a.k.a. spectral energy distribution

Observable Properties

- Flux integrate over frequency, units of W m^{-2}

$$F = \int d\ln\nu\, \nu F_\nu$$

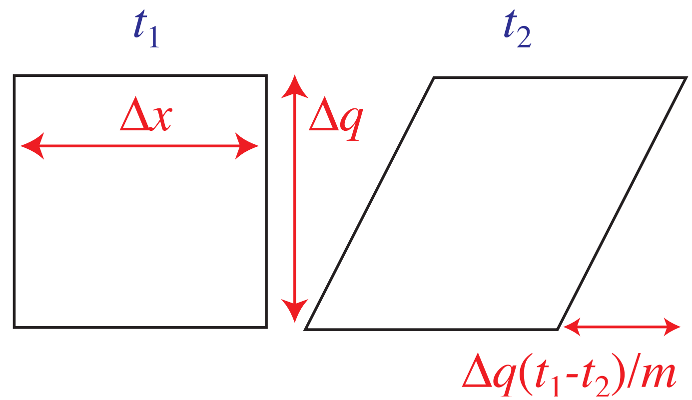
- Flux in a frequency band S_b measured in terms of magnitudes (optical), set to some standard zero point per band

$$m_b - m_{\rm norm} = 2.5\log_{10}(F_{\rm norm}/F_b) \approx \ln(F_{\rm norm}/F_b)$$

- Luminosity: integrate over area assuming isotropic emission or beaming factor, units of W

$$L = 4\pi d_L^2 F$$

Liouville Equation



- In absence of interactions and changes to the momentum, particle conservation implies that the phase space distribution is invariant along the trajectory of the particles
- Follow an element in Δx with spread Δq. For example for non relativistic particles a spread in velocity of $\Delta v = \Delta q/m$.
- After a time δt the low velocity tail will lag the high velocity tail by $\delta x = \Delta v \delta t = \Delta q \delta t/m$
- For ultrarelativistic particles $v = c = 1$ and $\Delta v = 0$, so obviously true

Liouville Equation

- The phase space element can shear but preserves area $\Delta x \Delta q$
- This remains true under Lorentz and even a general coordinate transform
- Therefore $df/dt = 0$ or f is conserved when evaluated along the path of the particles
- Liouville Equation: $f \propto I_\nu/\nu^3$ and $ds = cdt$

$$\frac{df}{dt} = 0 \rightarrow \frac{dI}{ds} = 0$$

if frequency is also conserved on the path

Liouville Equation

- In general, expand out the total derivative

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \sum_i \left(\frac{dx_i}{dt} \frac{\partial f}{\partial x_i} + \frac{dq_i}{dt} \frac{\partial f}{\partial q_i} \right) = 0$$

- The spatial gradient terms are responsible for flow of particles in and out of a fixed volume
- The momentum amplitude derivative terms are responsible for redshift effects
- The momentum direction derivative terms are responsible for gravitational lensing

Liouville Equation

- Liouville theorem states that the phase space distribution function is conserved along a trajectory in the absence of particle interactions

$$\frac{df}{dt} = \left[\frac{\partial}{\partial t} + \frac{d\mathbf{q}}{dt}\frac{\partial}{\partial \mathbf{q}} + \frac{d\mathbf{x}}{dt}\frac{\partial}{\partial \mathbf{x}} \right] f = 0$$

subtlety in expanding universe is that the de Broglie wavelength of particles changes with the expansion so that

$$q \propto a^{-1}$$

- Homogeneous and isotropic limit

$$\frac{\partial f}{\partial t} + \frac{dq}{dt}\frac{\partial f}{\partial q} = \frac{\partial f}{\partial t} - H(a)\frac{\partial f}{\partial \ln q} = 0$$

Energy Density Evolution

- Integrate Liouville equation over $g\int d^3q/(2\pi)^3 E$ to form

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} &= H(a) g \int \frac{d^3q}{(2\pi)^3} E q \frac{\partial}{\partial q} f \\
&= H(a) g \int \frac{d\Omega}{(2\pi)^3} \int dq q^3 E \frac{\partial}{\partial q} f \\
&= -H(a) g \int \frac{d\Omega}{(2\pi)^3} \int dq \frac{d(q^3 E)}{dq} f \\
&= -H(a) g \int \frac{d\Omega}{(2\pi)^3} \int dq (3q^2 E + q^3 \frac{dE}{dq}) f
\end{aligned}
$$

$$d(E^2 = q^2 + m^2) \to EdE = qdq$$

$$= -3H(a) g \int \frac{d^3q}{(2\pi)^3} (E + \frac{q^2}{3E}) f = -3H(a)(\rho + p)$$

as derived previously from energy conservation

Boltzmann Equation

- Boltzmann equation says that Liouville theorem must be modified to account for collisions

$$\frac{Df}{Dt} = C[f]$$

- Heuristically

$$C[f] = \text{particle sources - sinks}$$

- Collision term: integrate over phase space of incoming particles, connect to outgoing state with some interaction strength

Boltzmann Equation

- Form:

$$C[f] = \frac{1}{E} \int d(\text{phase space})[\text{energy-momentum conservation}]$$
$$\times |M|^2 [\text{emission} - \text{absorption}]$$

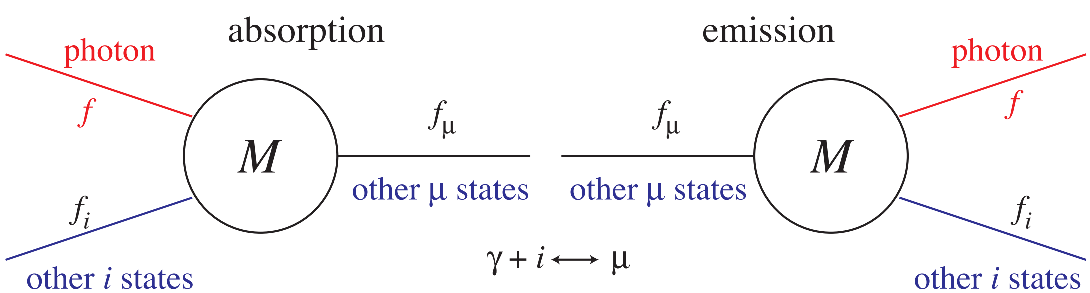
- Matrix element M, assumed T [or CP] invariant
- (Lorentz invariant) phase space element

$$\int d(\text{phase space}) = \Pi_i \frac{g_i}{(2\pi)^3} \int \frac{d^3 q_i}{2E_i}$$

- Energy conservation: $(2\pi)^4 \delta^{(4)}(q_1 + q_2 + ...)$

Boltzmann Equation

- Emission - absorption term involves the particle occupation of the various states
- For concreteness: take f to be the photon distribution function
- Interaction ($\gamma + \sum i \leftrightarrow \sum \mu$); sums are over all incoming and outgoing other particles



- [emission-absorption] $+ =$ boson; $- =$ fermion

$$\Pi_i \Pi_\mu f_\mu (1 \pm f_i)(1 \pm f) - \Pi_i \Pi_\mu (1 \pm f_\mu) f_i f$$

Boltzmann Equation

- Photon Emission: $f_\mu(1 \pm f_i)(1 + f)$

  f_μ: proportional to number of emitters

  $(1 \pm f_i)$: if final state is occupied and a fermion, process blocked; if boson the process enhanced

  $(1 + f)$: final state factor for photons: "1": spontaneous emission (remains if $f = 0$); "$+f$": stimulated and proportional to the occupation of final photon

- Photon Absorption: $-(1 \pm f_\mu) f_i f$

  $(1 \pm f_\mu)$: if final state is occupied and fermion, process blocked; if boson the process enhanced

  f_i: proportional to number of absorbers

  f: proportional to incoming photons

Boltzmann Equation

- If interactions are rapid they will establish an equilibrium distribution where the distribution functions no longer change $C[f_{\rm eq}] = 0$
- Solve by inspection

$$\Pi_i \Pi_\mu f_\mu (1 \pm f_i)(1 \pm f) - \Pi_i \Pi_\mu (1 \pm f_\mu) f_i f = 0$$

- Try $f_a = (e^{E_a/T} \mp 1)^{-1}$ so that $(1 \pm f_a) = e^{-E_a/T}(e^{E_a/T} \mp 1)^{-1}$

$$e^{-\sum (E_i + E)/T} - e^{-\sum E_\mu/T} = 0$$

and energy conservation says $E + \sum E_i = \sum E_\mu$, so identity is satisfied if the constant T is the same for all species, i.e. are in thermal equilibrium

Boltzmann Equation

- If the interaction does not create or destroy particles then the distribution

$$f_{\rm eq} = (e^{(E-\mu)/T} \mp 1)^{-1}$$

also solves the equilibrium equation: e.g. a scattering type reaction

$$\gamma_E + i \to \gamma_{E'} + j$$

where i and j represent the same collection of particles but with different energies after the scattering

$$\sum (E_i - \mu_i) + (E - \mu) = \sum (E_j - \mu_j) + (E' - \mu)$$

since $\mu_i = \mu_j$ for each particle

- Not surprisingly, this is the Fermi-Dirac distribution for fermions and the Bose-Einstein distribution for bosons

Boltzmann Equation

- More generally, equilibrium is satisfied if the sum of the chemical potentials on both sides of the interaction are equal, $\gamma + i \leftrightarrow \nu$

$$\sum \mu_i + \mu = \sum \mu_\nu$$

i.e. the law of mass action is satisfied

- If interactions that create or destroy particles are in equilibrium then this law says that the chemical potential will vanish: e.g. $\gamma + e^- \to 2\gamma + e^-$

$$\mu_e + \mu = \mu_e + 2\mu \to \mu = 0$$

so that the chemical potential is driven to zero if particle number is not conserved in interaction

Maxwell Boltzmann Distribution

- For the nonrelativistic limit $E = m + \frac{1}{2}q^2/m$, nondegenerate limit $(E-\mu)/T \gg 1$ so both distributions go to the Maxwell-Boltzmann distribution

$$f_{\rm eq} = \exp[-(m-\mu)/T]\exp(-q^2/2mT)$$

- Here it is even clearer that the chemical potential μ is the normalization parameter for the number density of particles whose number is conserved.
- μ and n can be used interchangably

Poor Man's Boltzmann Equation

- Non expanding medium

$$\frac{\partial f}{\partial t} = \Gamma\,(f - f_{\rm eq})$$

where Γ is some rate for collisions

- Add in expansion in a homogeneous medium

$$\frac{\partial f}{\partial t} + \frac{dq}{dt}\frac{\partial f}{\partial q} = \Gamma\,(f - f_{\rm eq})$$

$$(q \propto a^{-1} \rightarrow \frac{1}{q}\frac{dq}{dt} = -\frac{1}{a}\frac{da}{dt} = H)$$

$$\frac{\partial f}{\partial t} - H\frac{\partial f}{\partial \ln q} = \Gamma\,(f - f_{\rm eq})$$

- So equilibrium will be maintained if collision rate exceeds expansion rate $\Gamma > H$

Non-Relativistic Bulk Properties

- Number density

$$
\begin{aligned}
n &= g e^{-(m-\mu)/T} \frac{4\pi}{(2\pi)^3} \int_0^\infty q^2 dq \exp(-q^2/2mT) \\
&= g e^{-(m-\mu)/T} \frac{2^{3/2}}{2\pi^2} (mT)^{3/2} \int_0^\infty x^2 dx \exp(-x^2) \\
&= g(\frac{mT}{2\pi})^{3/2} e^{-(m-\mu)/T}
\end{aligned}
$$

- Energy density $E = m \rightarrow \rho = mn$
- Pressure $q^2/3E = q^2/3m \rightarrow p = nT$, ideal gas law

Ultra-Relativistic Bulk Properties

- Chemical potential $\mu = 0$, $\zeta(3) \approx 1.202$
- Number density

$$n_{\rm boson} = gT^3 \frac{\zeta(3)}{\pi^2} \qquad \zeta(n+1) \equiv \frac{1}{n!}\int_0^\infty \frac{x^n}{e^x - 1}dx$$

$$n_{\rm fermion} = \frac{3}{4} gT^3 \frac{\zeta(3)}{\pi^2}$$

- Energy density

$$\rho_{\rm boson} = gT^4 \frac{3}{\pi^2}\zeta(4) = gT^4 \frac{\pi^2}{30}$$

$$\rho_{\rm fermion} = \frac{7}{8} gT^4 \frac{3}{\pi^2}\zeta(4) = \frac{7}{8} gT^4 \frac{\pi^2}{30}$$

- Pressure $q^2/3E = E/3 \rightarrow p = \rho/3$, $w_r = 1/3$

Entropy Density

- First law of thermodynamics

$$dS = \frac{1}{T}(d\rho(T)V + p(T)dV)$$

so that

$$\left.\frac{\partial S}{\partial V}\right|_T = \frac{1}{T}[\rho(T) + p(T)]$$

$$\left.\frac{\partial S}{\partial T}\right|_V = \frac{V}{T}\frac{d\rho}{dT}$$

- Since $S(V,T) \propto V$ is extensive

$$S = \frac{V}{T}[\rho(T) + p(T)] \qquad \sigma = S/V = \frac{1}{T}[\rho(T) + p(T)]$$

Entropy Density

- Integrability condition $dS/dV dT = dS/dT dV$ relates the evolution of entropy density

$$\frac{d\sigma}{dT} = \frac{1}{T}\frac{d\rho}{dT}$$
$$\frac{d\sigma}{dt} = \frac{1}{T}\frac{d\rho}{dt} = \frac{1}{T}[-3(\rho + p)]\frac{d\ln a}{dt}$$
$$\frac{d\ln\sigma}{dt} = -3\frac{d\ln a}{dt} \qquad \sigma \propto a^{-3}$$

comoving entropy density is conserved in thermal equilibrium

- For ultra relativisitic bosons $s_{\rm boson} = 3.602 n_{\rm boson}$; for fermions factor of $7/8$ from energy density.

$$g_* = \sum_{\rm bosons} g_b + \frac{7}{8}\sum g_f$$

Neutrino Freezeout

- Neutrino equilibrium maintained by weak interactions, e.g. $e^+ + e^- \leftrightarrow \nu + \bar{\nu}$
- Weak interaction cross section $T_{10} = T/10^{10}K \sim T/1\text{MeV}$

$$\sigma_w \sim G_F^2 E_\nu^2 \approx 4 \times 10^{-44}\, T_{10}^2 \text{cm}^2$$

- Rate $\Gamma = n_\nu \sigma_w = H$ at $T_{10} \sim 3$ or $t \sim 0.2\text{s}$
- After neutrino freezeout, electrons and positrons annihilate dumping their entropy into the photons
- Before g_*: $\gamma, e^+, e^- = 2 + \frac{7}{8}(2+2) = \frac{11}{2}$
- After g_*: γ = 2; so conservation of entropy gives

$$g_* T^3\Big|_{\text{initial}} = g_* T^3\Big|_{\text{final}} \qquad T_\nu = \left(\frac{4}{11}\right)^{1/3} T_\gamma$$

Relic Neutrinos

- Relic number density (zero chemical potential; now required by oscillations & BBN)

$$n_\nu = n_\gamma \frac{3}{4}\frac{4}{11} = 112\mathrm{cm}^{-3}$$

- Relic energy density assuming one species with finite m_ν:

$\rho_\nu = m_\nu n_\nu$

$$\rho_\nu = 112 \frac{m_\nu}{\mathrm{eV}}\,\mathrm{eV\,cm}^{-3} \qquad \rho_c = 1.05 \times 10^4 h^2\,\mathrm{eV cm}^{-3}$$

$$\Omega_\nu h^2 = \frac{m_\nu}{93.7\mathrm{eV}}$$

- Candidate for dark matter? an eV mass neutrino goes non relativistic around $z \sim 1000$ and retains a substantial velocity dispersion σ_ν.

Hot Dark Matter

- Momenta for a nonrelativistic species redshifts like temperature for a relativistic one, so average momentum is still given by

$$\langle q \rangle = 3T_\nu = m\sigma_\nu$$

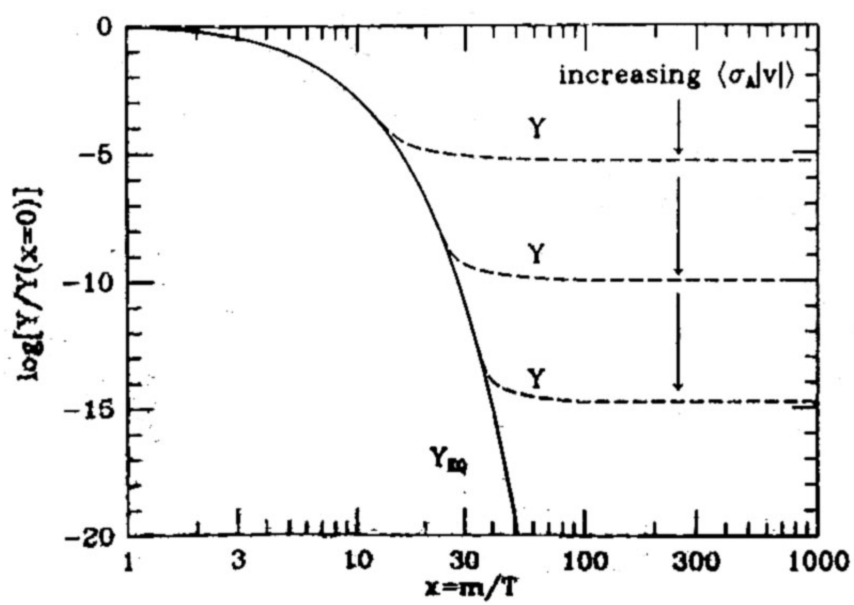
$$\sigma_\nu = 3\left(\frac{m_\nu}{1\mathrm{eV}}\right)^{-1}\left(\frac{T_\nu}{1\mathrm{eV}}\right) = 3\left(\frac{m_\nu}{1\mathrm{eV}}\right)^{-1}\left(\frac{T_\nu}{10^4\mathrm{K}}\right)$$

$$= 6\times 10^{-4}\left(\frac{m_\nu}{1\mathrm{eV}}\right)^{-1} = 200\mathrm{km/s}\left(\frac{m_\nu}{1\mathrm{eV}}\right)^{-1}$$

- Of order the rotation velocity of galactic halos and higher at higher redshift - small objects can't form: top down structure formation – not observed – must not constitute the bulk of the dark matter

Cold Dark Matter

- Problem with neutrinos is they decouple while relativistic and hence have a comparable number density to photons - for a reasonable energy density, the mass must be small



- The equilibrium distribution for a non-relativistic species declines exponentially beyond the mass threshold

$$n = g(\frac{mT}{2\pi})^{3/2} e^{-m/T}$$

- Exponential will eventually win soon after $T < m$, suppressing annihilation rates

WIMP Miracle

- Freezeout when annihilation rate equal expansion rate $\Gamma \propto \sigma_A$, increasing annihilation cross section decreases abundance

$$\Gamma = n\langle\sigma_A v\rangle = H$$

$$H \propto T^2 \sim m^2$$

$$\rho_{\rm freeze} = mn \propto \frac{m^3}{\langle\sigma_A v\rangle}$$

$$\rho_c = \rho_{\rm freeze}(T/T_0)^{-3} \propto \frac{1}{\langle\sigma_A v\rangle}$$

independently of the mass of the CDM particle

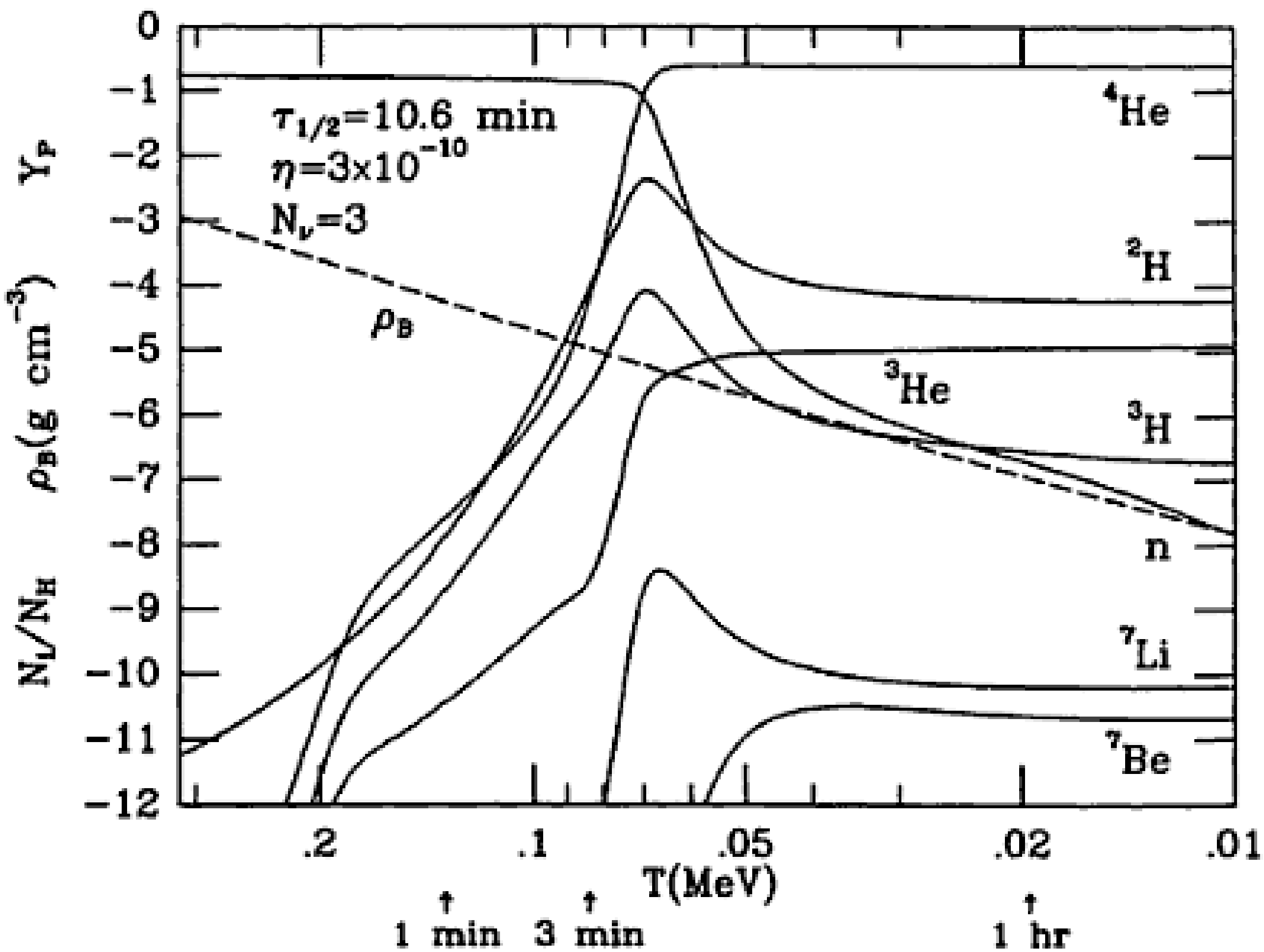
- Plug in some typical numbers for supersymmetric candidates or WIMPs (weakly interacting massive particles) of $\langle\sigma_A v\rangle \approx 10^{-36}$ cm^2 and restore the proportionality constant $\Omega_c h^2$ is of the right order of magnitude (~ 0.1)!

Axions

- Alternate solution: keep light particle but not created in thermal equilibrium
- Example: axion dark matter - particle that solves the strong CP problem
- Inflation sets initial conditions, fluctuation from potential minimum
- Once Hubble scale smaller than the mass scale, field unfreezes
- Coherent oscillations of the axion field - condensate state. Can be very light $m \ll 1\text{eV}$ and yet remain cold.
- Same reason a quintessence dark energy candidate must be lighter than the Hubble scale today

Big Bang Nucleosynthesis

- Integrating the Boltzmann equation for nuclear processes during first few minutes leads to synthesis and freezeout of light elements



Big Bang Nucleosynthesis

- Most of light element synthesis can be understood through nuclear statistical equilibrium and reaction rates
- Equilibrium abundance of species with mass number A and charge Z (Z protons and $A-Z$ neutrons)

$$n_A = g_A(\frac{m_A T}{2\pi})^{3/2} e^{(\mu_A - m_A)/T}$$

- In chemical equilibrium with protons and neutrons

$$\mu_A = Z\mu_p + (A-Z)\mu_n$$

$$n_A = g_A(\frac{m_A T}{2\pi})^{3/2} e^{-m_A/T} e^{(Z\mu_p + (A-Z)\mu_n)/T}$$

Big Bang Nucleosynthesis

- Eliminate chemical potentials with n_p, n_n

$$e^{\mu_p/T} = \frac{n_p}{g_p}\left(\frac{2\pi}{m_p T}\right)^{3/2} e^{m_p/T}$$

$$e^{\mu_n/T} = \frac{n_n}{g_n}\left(\frac{2\pi}{m_n T}\right)^{3/2} e^{m_n/T}$$

$$n_A = g_A g_p^{-Z} g_n^{Z-A} (\frac{m_A T}{2\pi})^{3/2} \left(\frac{2\pi}{m_p T}\right)^{3Z/2} \left(\frac{2\pi}{m_n T}\right)^{3(A-Z)/2}$$
$$\times\, e^{-m_A/T} e^{(Z\mu_p+(A-Z)\mu_n)/T} n_p^Z n_n^{A-Z}$$

$$(g_p = g_n = 2; m_p \approx m_n = m_b = m_A/A)$$

$$(B_A = Z m_p + (A-Z) m_n - m_A)$$

$$= g_A 2^{-A} \left(\frac{2\pi}{m_b T}\right)^{3(A-1)/2} A^{3/2} n_p^Z n_n^{A-Z} e^{B_A/T}$$

Big Bang Nucleosynthesis

- Convenient to define abundance fraction

$$X_A \equiv A\frac{n_A}{n_b} = Ag_A 2^{-A}\left(\frac{2\pi}{m_b T}\right)^{3(A-1)/2} A^{3/2} n_p^Z n_n^{A-Z} n_b^{-1} e^{B_A/T}$$

$$= Ag_A 2^{-A}\left(\frac{2\pi n_b^{2/3}}{m_b T}\right)^{3(A-1)/2} A^{3/2} e^{B_A/T} X_p^Z X_n^{A-Z}$$

$$(n_\gamma = \frac{2}{\pi^2} T^3 \zeta(3) \qquad \eta_{b\gamma} \equiv n_b/n_\gamma)$$

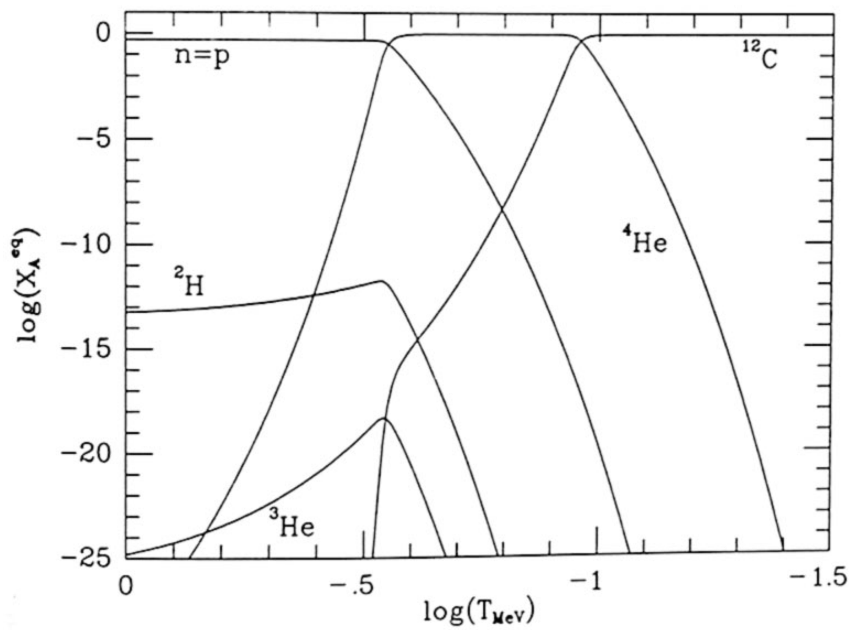
$$= A^{5/2} g_A 2^{-A}\left[\left(\frac{2\pi T}{m_b}\right)^{3/2} \frac{2\zeta(3)\eta_{b\gamma}}{\pi^2}\right]^{A-1} e^{B_A/T} X_p^Z X_n^{A-Z}$$

Deuterium

- Deuterium $A = 2$, $Z = 1$, $g_2 = 3$, $B_2 = 2.225$ MeV

$$X_2 = \frac{3}{\pi^2}\left(\frac{4\pi T}{m_b}\right)^{3/2}\eta_{b\gamma}\zeta(3)e^{B_2/T}X_pX_n$$

- Deuterium
  "bottleneck" is mainly
  due to the low baryon-photon
  number of the universe
  $\eta_{b\gamma} \sim 10^{-9}$, secondarily due
  to the low binding energy B_2

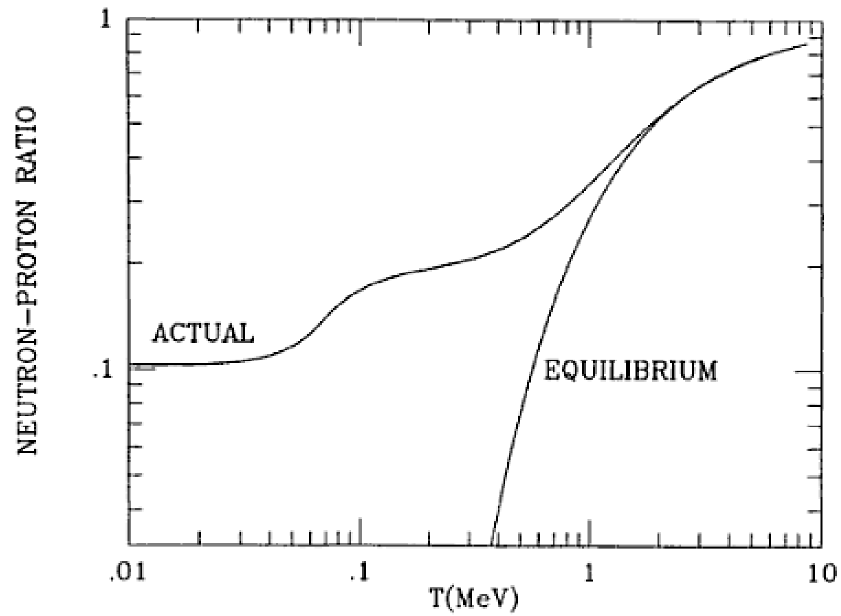


Deuterium

- $X_2/X_pX_n \approx \mathcal{O}(1)$ at $T \approx 100$keV or 10^9 K, much lower than the binding energy B_2
- Most of the deuterium formed then goes through to helium via $\mathrm{D} + \mathrm{D} \to {}^3\mathrm{He} + p$, ${}^3\mathrm{He} + \mathrm{D} \to {}^4\mathrm{He} + n$
- Deuterium freezes out as number abundance becomes too small to maintain reactions $n_D =$ const. independent of n_b
- The deuterium freezeout fraction $n_D/n_b \propto \eta_{b\gamma}^{-1} \propto (\Omega_b h^2)^{-1}$ and so is fairly sensitive to the baryon density.
- Observations of the ratio in quasar absorption systems give $\Omega_b h^2 \approx 0.02$

Helium

- Essentially all neutrons around during nucleosynthesis end up in Helium
- In equilibrium, the neutron-to-proton ratio is determined by the mass difference $Q = m_n - m_p = 1.293$ MeV



$$\frac{n_n}{n_p} = \exp[-Q/T]$$

Helium

- Equilibrium is maintained through weak interactions, e.g. $n \leftrightarrow p + e^- + \bar{\nu}$, $\nu + n \leftrightarrow p + e^-$, $e^+ + n \leftrightarrow p + \bar{\nu}$ with rate

$$\frac{\Gamma}{H} \approx \frac{T}{0.8\mathrm{MeV}}$$

- Freezeout fraction

$$\frac{n_n}{n_p} = \exp[-1.293/0.8] \approx 0.2$$

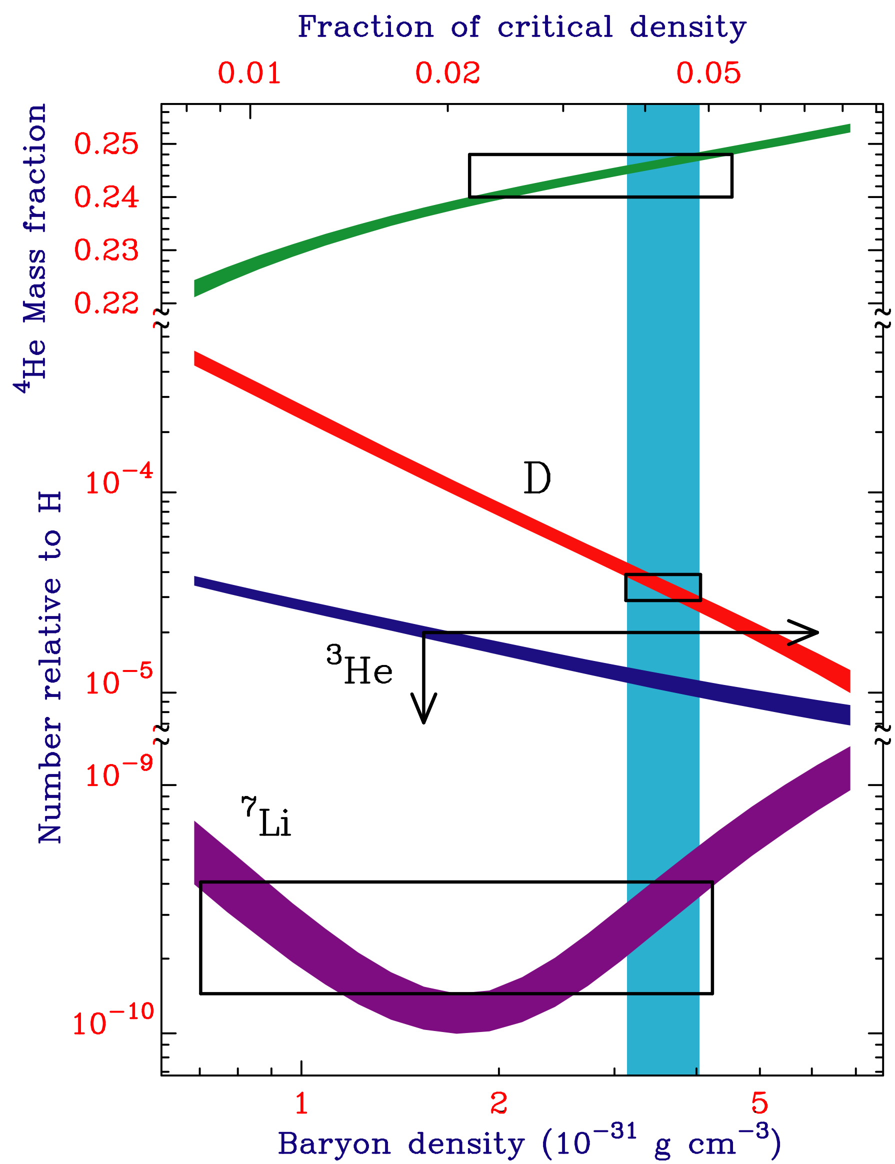
- Finite lifetime of neutrons brings this to $\sim 1/7$ by 10^9K
- Helium mass fraction

$$\begin{aligned} Y_{\mathrm{He}} &= \frac{4n_{He}}{n_b} = \frac{4(n_n/2)}{n_n + n_p} \\ &= \frac{2n_n/n_p}{1 + n_n/n_p} \approx \frac{2/7}{8/7} \approx \frac{1}{4} \end{aligned}$$

Helium

- Depends mainly on the expansion rate during BBN - measure number of relativistic species
- Traces of ^{7}Li as well. Measured abundances in reasonable agreement with deuterium measure $\Omega_b h^2 = 0.02$ but the detailed interpretation is still up for debate

Light Elements



Burles, Nollett, Turner (1999)

Baryogenesis

- What explains the small, but non-zero, baryon-to-photon ratio?

$$\eta_{b\gamma} = n_b/n_\gamma \approx 3 \times 10^{-8}\Omega_b h^2 \approx 6 \times 10^{-10}$$

- Must be a slight excess of baryons b to anti-baryons $\bar{b}$ that remains after annihilation
- Sakharov conditions
  - Baryon number violation: some process must change the net baryon number
  - CP violation: process which produces b and $\bar{b}$ must differ in rate
  - Out of equilibrium: else equilibrium distribution with vanishing chemical potential (processes exist which change baryon number) gives equal numbers for b and $\bar{b}$
- Expanding universe provides 3; physics must provide 1,2

Baryogenesis

- Example: out of equilibrium decay of some heavy boson X, $\bar{X}$
- Suppose X decays through 2 channels with baryon number b_1 and b_2 with branching ratio r and $1-r$ leading to a change in the baryon number per decay of

$$rb_1 + (1-r)b_2$$

- And $\bar{X}$ to $-b_1$ and $-b_2$ with ratio $\bar{r}$ and $1-\bar{r}$

$$-\bar{r}b_1 - (1-\bar{r})b_2$$

- Net production

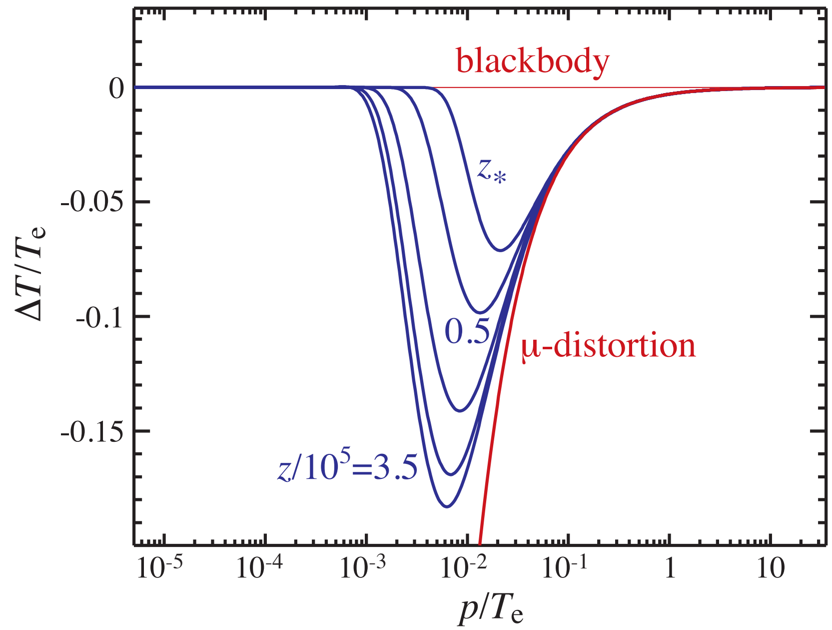
$$\Delta b = (r-\bar{r})(b_1 - b_2)$$

Baryogenesis

- Condition 1: $b_1 \neq 0$, $b_2 \neq 0$
- Condition 2: $\bar{r} \neq r$
- Condition 3: out of equilibrium decay
- GUT and electroweak (instanton) baryogenesis mechanisms exist
- Active subject of research

Black Body Formation

- After $z \sim 10^6$, photon creating processes $\gamma + e^- \leftrightarrow 2\gamma + e^-$ and bremmstrahlung $e^- + p \leftrightarrow e^- + p + \gamma$ drop out of equilibrium for photon energies $E \sim T$.



- Compton scattering remains effective in redistributing energy via exchange with electrons
- Out of equilibrium processes like decays leave residual photon chemical potential imprint
- Observed black body spectrum places tight constraints on any that might dump energy into the CMB

Recombination

- Maxwell-Boltzmann distribution determines the equilibrium distribution for reactions, e.g. big-bang nucleosynthesis, recombination:

$$p + e^- \leftrightarrow H + \gamma$$

$$\frac{n_p n_e}{n_H} \approx e^{-B/T} \left(\frac{m_e T}{2\pi} \right)^{3/2} e^{(\mu_p + \mu_e - \mu_H)/T}$$

where $B = m_p + m_e - m_H = 13.6$eV is the binding energy, $g_p = g_e = \frac{1}{2} g_H = 2$, and $\mu_p + \mu_e = \mu_H$ in equilibrium

- Define ionization fraction

$$n_p = n_e = x_e n_b$$

$$n_H = n_b - n_p = (1 - x_e) n_b$$

Recombination

- Saha Equation

$$\frac{n_e n_p}{n_H n_b} = \frac{x_e^2}{1 - x_e} = \frac{1}{n_b}\left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-B/T}$$

- Naive guess of $T_* = B$ wrong due to the low baryon-photon ratio – $T_* \approx 0.3$eV so recombination at $z_* \approx 1000$
- But the photon-baryon ratio is very low

$$\eta_{b\gamma} \equiv n_b/n_\gamma \approx 3 \times 10^{-8}\Omega_b h^2$$

Recombination

- Eliminate in favor of $\eta_{b\gamma}$ and B/T through

$$n_\gamma = 0.244 T^3 \,, \qquad \frac{m_e}{B} = 3.76 \times 10^4$$

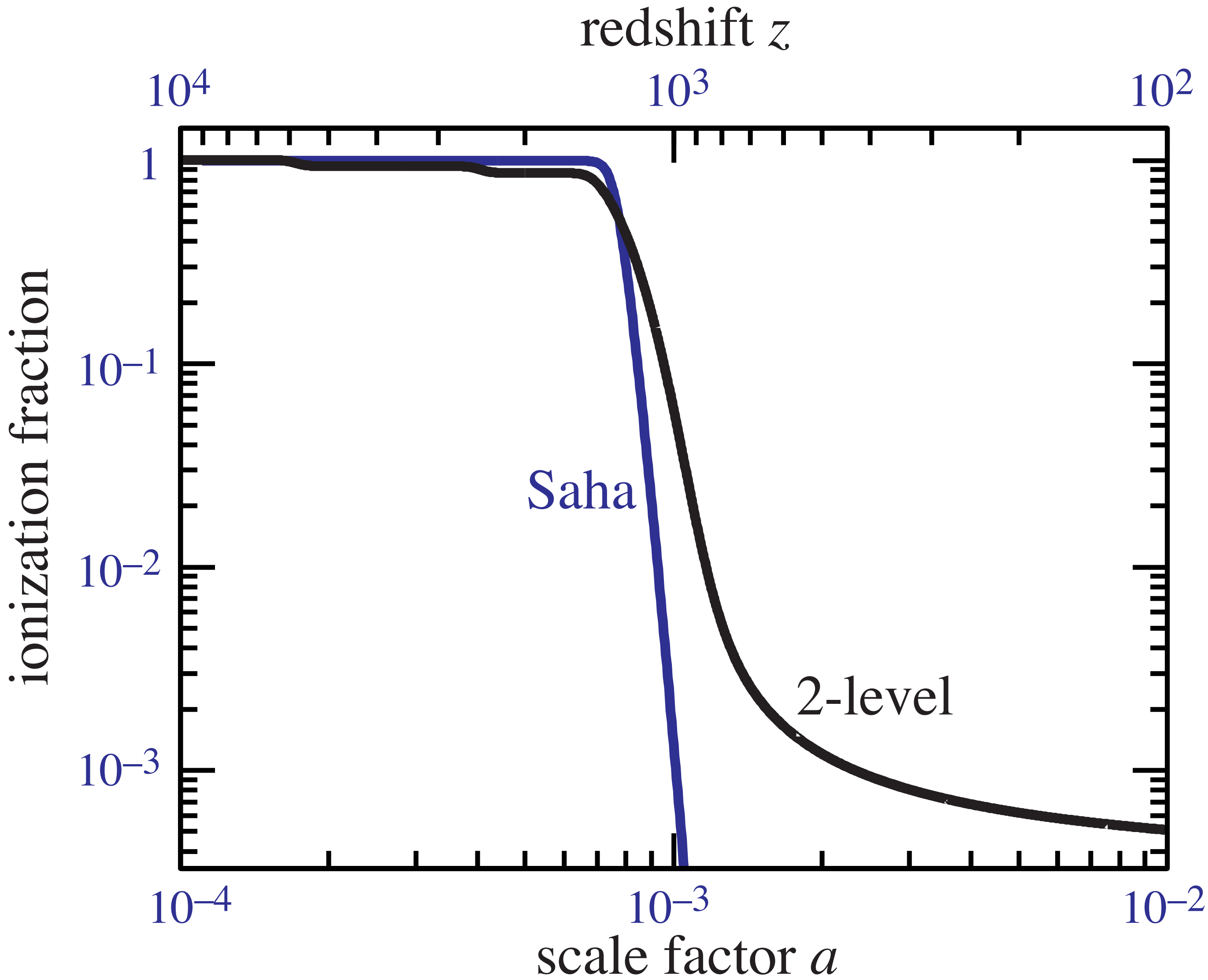
- Big coefficient

$$\frac{{x_e}^2}{1 - x_e} = 3.16 \times 10^{15} \left(\frac{B}{T} \right)^{3/2} e^{-B/T}$$

$T = 1/3\text{eV} \to x_e = 0.7$, $T = 0.3\text{eV} \to x_e = 0.2$

- Further delayed by inability to maintain equilibrium since net is through 2γ process and redshifting out of line
- Free electrons freezeout as recombination rate drops below Hubble - low ionization tail has observable impact on CMB power spectrum and is used to constrain energy injection from dark matter annihilation, primordial black holes, etc.

Recombination



Inhomogeneous Universe

- Boltzmann equation defines how particles propagate and interact in the spacetime metric: "geometry tells matter how to move"
- Beyond the background FRW metric, fluctuations in the metric are inhomogeneous and anisotropic (but *statistically* homogeneous and isotropic in ensemble average)
- Particle numbers are still conserved in the absence of collisions if the spacetime metric is fluctuating adiabatically: generalize $df/dt = 0$ to changes in the momentum due to gravitational forces, redshift and lensing
- Matter tells spacetime how to curve: perturbed stress energy or low order moments of f determine the geometry and its evolution – joint solution
- Solve for the low order moments with the Boltzmann equation or an "equation of state" closure condition for the moment hierarchy

Inhomogeneous Universe

- To be continued with relativistic perturbation theory defining the joint evolution...