Astro 321

Set 2: Thermal History

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Macro vs Micro Description

- In the first set of notes, we used a macroscopic description.
- **Gravity** only cares about bulk properties: energy density, momentum density, pressure, anisotropic stress – stress tensor
- Matter and radiation is composed of particles whose properties can be described by their phase space distribution or occupation function
- Macroscopic properties are integrals or moments of the phase space distribution
- Particle interactions involve the evolution of the phase space distribution
- Rapid interactions drive distribution to thermal equilibrium but must compete with the expansion rate of universe
- **Freeze out**, the origin of species
Brief Thermal History
Astro-Particle Dictionary

Astro and physics literature use different words to describe same thing:

- Specific intensity $I_\nu \leftrightarrow$ phase space distribution $f$
- Surface brightness conservation $\leftrightarrow$ Liouville equation
- Absorption, emission, scattering $\leftrightarrow$ Collision term
- Einstein relations for absorption, stimulated and spontaneous emission $\leftrightarrow$ matrix element determines strength of all interactions of a given type
- Radiative transfer equation $\leftrightarrow$ Boltzmann equation
- Optically thin conditions $\leftrightarrow$ freezeout of interactions

We take physics notation but the content is the same as in other astro courses but placed in an expanding universe context.
Allowed Particle States

• Counting momentum states with momentum $q$ and de Broglie wavelength

$$\lambda = \frac{h}{q} = \frac{2\pi\hbar}{q}$$

• In a discrete volume $L^3$ there is a discrete set of states that satisfy periodic boundary conditions

• We will hereafter set $\hbar = c = 1$

• As in Fourier analysis

$$e^{2\pi i x / \lambda} = e^{iqx} = e^{iq(x+L)} \rightarrow e^{iql} = 1$$
Fitting in a Box

- Periodicity yields a discrete set of allowed states

\[ L q = 2\pi m_i, \quad m_i = 1, 2, 3... \]

\[ q_i = \frac{2\pi}{L} m_i \]

- In each of 3 directions

\[ \sum_{m_x, m_y, m_z} \rightarrow \int d^3 m \]

- The differential number of allowed momenta in the volume

\[ d^3 m = \left( \frac{L}{2\pi} \right)^3 d^3 q \]
Density of States

- The total number of states allows for a number of internal degrees of freedom, e.g. spin, quantified by the degeneracy factor $g$

- Total density of states:

$$\frac{dN_s}{V} = \frac{g}{V} d^3 m = \frac{g}{(2\pi)^3} d^3 q$$

- If all states were occupied by a single particle, then particle density

$$n_s = \frac{N_s}{V} = \frac{1}{V} \int dN_s = \int \frac{g}{(2\pi)^3} d^3 q$$
The distribution function $f$ quantifies the occupation of the allowed momentum states

$$n = \frac{N}{V} = \frac{1}{V} \int f \, dN_s = \int \frac{g}{(2\pi)^3} f \, d^3q$$

$f$, aka phase space occupation number, also quantifies the density of particles per unit phase space $dN/(\Delta x)^3(\Delta q)^3$

For photons, the spin degeneracy $g = 2$ accounting for the 2 polarization states

Energy $E(q) = (q^2 + m^2)^{1/2}$

Momentum → frequency $q = 2\pi/\lambda = 2\pi\nu = \omega = E$ (where $m = 0$ and $\lambda\nu = c = 1$)
Bulk Properties

- Integrals over the distribution function define the bulk properties of the collection of particles

- Number density

\[ n(x, t) \equiv \frac{N}{V} = g \int \frac{d^3q}{(2\pi)^3} f \]

- Energy density

\[ \rho(x, t) = g \int \frac{d^3q}{(2\pi)^3} E(q)f \]

where \( E^2 = q^2 + m^2 \)

- Momentum density

\[ (\rho + p)v(x, t) = g \int \frac{d^3q}{(2\pi)^3} q f \]
Vacuum Energy

- We have assumed here that the state of zero particles has zero energy.
- In QFT, like the simple harmonic oscillator in ordinary quantum mechanics, there is a zero point energy to the ground state.
- For bosons, $\hbar \omega/2 = E(q)/2$, so the most naive version of the cosmological constant problem is that $\rho \propto M^4$ where $M = M_{\text{Pl}} = 1/\sqrt{8\pi G}$ if the theory applies out to the Planck scale.
- The critical energy density $\rho_c = 3H_0^2/8\pi G \approx 8 \times 10^{-47}\hbar^2\text{GeV}^4$ is more than $10^{120}$ off $M_{\text{Pl}}^4 \approx 2 \times 10^{76}$ GeV$^4$.
- Note that $p_{\text{vac}} \approx \rho_{\text{vac}}/3$ so this fixed momentum cutoff calculation is a bit too naive since we know that $p_{\text{vac}} = -\rho_{\text{vac}}$. 
Vacuum Energy

- A Lorentz invariant renormalization scheme corrects this to

\[ \rho_{\text{vac}} = \frac{m^4}{64\pi^2} \ln\left(\frac{m^2}{\mu^2}\right) \]

where \( \mu \) is some renormalization scale

- But even if there are no mass states above the known standard model bosons, e.g. Higgs boson of \( m \approx 125 \text{ GeV} \), this is way off, even though it helps by some 68 orders of magnitude!

- Caveat: fermions contribute negatively to the vacuum energy so if supersymmetry is unbroken would cancel

- But supersymmetry is clearly broken at low energies and has yet to be seen at LHC - so taking this as a lower limit on the scale of supersymmetry breaking the vacuum energy \( m > 1 \text{ TeV} \), \( m^4 \) is still 60 orders of magnitude off.
Bulk Properties

- Pressure: particles bouncing off a surface of area $A$ in a volume spanned by $L_x$: per momentum state

$$p_q = \frac{F}{A} = \frac{N_{\text{part}} \Delta q}{A \Delta t}$$

($\Delta q = 2|q_x|$, $\Delta t = 2L_x/v_x$)

$$= \frac{N_{\text{part}} |q_x||v_x|}{V} = \frac{N_{\text{part}} |q||v|}{V}$$

($v = \gamma mv/\gamma m = q/E$)

$$= \frac{N_{\text{part}} q^2}{V \frac{3E}{3}}$$
Bulk Properties

- So that summed over occupied momenta states

\[ p(x, t) = g \int \frac{d^3q}{(2\pi)^3} \frac{|q|^2}{3E(q)} f \]

- Pressure is just one of the quadratic in \( q \) moments, in particular the isotropic one

- The remaining 5 components are the anisotropic stress (vanishes in the background)

\[ \pi^{ij}(x, t) = g \int \frac{d^3q}{(2\pi)^3} \frac{3q^iq_j - q^2\delta^i_j}{3E(q)} f \]

- We shall see that these are related to the 5 quadrupole moments of the angular distribution
Bulk Properties

- These are more generally the components of the stress-energy tensor

\[
T_{\mu \nu} = g \int \frac{d^3 q}{(2\pi)^3} \frac{q^\mu q^\nu}{E(q)} f
\]

- 0-0: energy density
- 0-\(i\): momentum density
- \(i - i\): pressure
- \(i \neq j\): anisotropic stress
- In the FRW background cosmology, isotropy requires that there be only a net energy density and pressure
Observable Properties

- Only get to measure luminous properties of the universe. For photons mass \( m = 0, g = 2 \) (units: \( J \ m^{-3} \))

\[
\rho(x, t) = 2 \int \frac{d^3q}{(2\pi)^3} q f = 2 \int dq d\Omega \left( \frac{q}{2\pi} \right)^3 f
\]

- Spectral energy density (per unit frequency \( q = h\nu = \hbar 2\pi\nu = 2\pi\nu \), solid angle)

\[
u_\nu = \frac{d\rho}{d\nu d\Omega} = 2(2\pi)\nu^3 f
\]

- Photons travelling at speed of light so that \( u_\nu = I_\nu = 4\pi\nu^3 f \) the specific intensity or brightness, energy flux across a surface, units of \( W \ m^{-2} \ Hz^{-1} \ sr^{-1} \) (SI); ergs \( s^{-1} \ cm^{-2} \ Hz^{-1} \ sr^{-1} \) (cgs)
Diffuse Extragalactic Light

- $\nu I_\nu$ peaks in the microwave mm-cm region: CMB black body $T = 2.725 \pm 0.002 K$ or $n_\gamma = 410 \text{ cm}^{-3}$, $\Omega_\gamma = 2.47 \times 10^{-5} h^{-2}$.
 Observable Properties

- Integrate over frequencies for total intensity

\[ I = \int d\nu I_\nu = \int d\ln \nu \nu I_\nu \]

\( \nu I_\nu \) often plotted since it shows peak under a log plot; \( I \) and \( \nu I_\nu \) have units of \( \text{W m}^{-2} \text{ sr}^{-1} \) and is independent of choice of frequency unit

- Flux density (specific flux): integrate over the solid angle of a radiation source, units of \( \text{W m}^{-2} \text{ Hz}^{-1} \) or Jansky = \( 10^{-26} \text{ W m}^{-2} \text{ Hz}^{-1} \)

\[ F_\nu = \int_{\text{source}} I_\nu d\Omega \]

a.k.a. spectral energy distribution
Observable Properties

- Flux integrate over frequency, units of W m$^{-2}$

\[ F = \int d\ln \nu \nu F_\nu \]

- Flux in a frequency band $S_b$ measured in terms of magnitudes (optical), set to some standard zero point per band

\[ m_b - m_{\text{norm}} = 2.5 \log_{10}(F_{\text{norm}}/F_b) \approx \ln(F_{\text{norm}}/F_b) \]

- Luminosity: integrate over area assuming isotropic emission or beaming factor, units of W

\[ L = 4\pi d_L^2 F \]
The Liouville Equation

- In absence of interactions and changes to the momentum, particle conservation implies that the phase space distribution is invariant along the trajectory of the particles.

- Follow an element in $\Delta x$ with spread $\Delta q$. For example, for non-relativistic particles a spread in velocity of $\Delta v = \Delta q/m$.

- After a time $\delta t$ the low velocity tail will lag the high velocity tail by $\delta x = \Delta v \delta t = \Delta q \delta t/m$.

- For ultrarelativistic particles $v = c = 1$ and $\Delta v = 0$, so obviously true.
Liouville Equation

- The phase space element can shear but preserves area $\Delta x \Delta q$
- This remains true under Lorentz and even a general coordinate transform
- Therefore $df/dt = 0$ or $f$ is conserved when evaluated along the path of the particles
- Liouville Equation: $f \propto I_\nu / \nu^3$ and $ds = cdt$

$$\frac{df}{dt} = 0 \rightarrow \frac{dI_\nu}{ds} = 0$$

if frequency is also conserved on the path
Liouville Equation

- In general, expand out the total derivative

\[
\frac{df}{dt} = \frac{\partial f}{\partial t} + \sum_i \left( \frac{dx_i}{dt} \frac{\partial f}{\partial x_i} + \frac{dq_i}{dt} \frac{\partial f}{\partial q_i} \right) = 0
\]

- The spatial gradient terms are responsible for flow of particles in and out of a fixed volume

- The momentum amplitude derivative terms are responsible for redshift effects

- The momentum direction derivative terms are responsible for gravitational lensing
Liouville Equation

- Liouville theorem states that the phase space distribution function is conserved along a trajectory in the absence of particle interactions

\[
\frac{df}{dt} = \left[ \frac{\partial}{\partial t} + \frac{dq}{dt} \frac{\partial}{\partial q} + \frac{dx}{dt} \frac{\partial}{\partial x} \right] f = 0
\]

- Subtlety in expanding universe is that the de Broglie wavelength of particles changes with the expansion so that

\[ q \propto a^{-1} \]

- Homogeneous and isotropic limit

\[
\frac{\partial f}{\partial t} + \frac{dq}{dt} \frac{\partial f}{\partial q} = \frac{\partial f}{\partial t} - H(a) \frac{\partial f}{\partial \ln q} = 0
\]
Energy Density Evolution

- Integrate Liouville equation over $g \int d^3q/(2\pi)^3 E$ to form

\[
\frac{\partial \rho}{\partial t} = H(a)g \int \frac{d^3q}{(2\pi)^3} E q \frac{\partial}{\partial q} f
\]

\[
= H(a)g \int \frac{d\Omega}{(2\pi)^3} \int dq d^3q E \frac{\partial}{\partial q} f
\]

\[
= -H(a)g \int \frac{d\Omega}{(2\pi)^3} \int dq \frac{d(q^3E)}{dq} f
\]

\[
= -H(a)g \int \frac{d\Omega}{(2\pi)^3} \int dq (3q^2 E + q^3 \frac{dE}{dq}) f
\]

\[
d(E^2 = q^2 + m^2) \rightarrow EdE = q dq
\]

\[
= -3H(a)g \int \frac{d^3q}{(2\pi)^3} (E + \frac{q^2}{3E}) f = -3H(a)(\rho + p)
\]

as derived previously from energy conservation
Boltzmann Equation

• Boltzmann equation says that Liouville theorem must be modified to account for collisions

\[
\frac{Df}{Dt} = C[f]
\]

• Heuristically

\[
C[f] = \text{particle sources} - \text{sinks}
\]

• Collision term: integrate over phase space of incoming particles, connect to outgoing state with some interaction strength
Boltzmann Equation

- Form:

\[ C[f] = \frac{1}{E} \int d(\text{phase space}) [\text{energy-momentum conservation}] \times |M|^2 [\text{emission} - \text{absorption}] \]

- Matrix element \( M \), assumed T [or CP] invariant

- (Lorentz invariant) phase space element

\[ \int d(\text{phase space}) = \prod_{i} \frac{g_i}{(2\pi)^3} \int \frac{d^3 q_i}{2E_i} \]

- Energy conservation: \((2\pi)^4 \delta^{(4)}(q_1 + q_2 + \ldots)\)
Boltzmann Equation

- Emission - absorption term involves the particle occupation of the various states

- For concreteness: take $f$ to be the photon distribution function

- Interaction ($\gamma + \sum i \leftrightarrow \sum \mu$); sums are over all incoming and outgoing other particles

- [emission-absorption] $+$ $=$ boson; $-$ $=$ fermion

$$\Pi_i \Pi_\mu f_\mu (1 \pm f_i)(1 \pm f) - \Pi_i \Pi_\mu (1 \pm f_\mu) f_i f$$
Boltzmann Equation

- **Photon Emission:** \( f_\mu (1 \pm f_i)(1 + f) \)
  - \( f_\mu \): proportional to number of emitters
  - \( (1 \pm f_i) \): if final state is occupied and a fermion, process blocked; if boson the process enhanced
  - \( (1 + f) \): final state factor for photons: “1”: spontaneous emission (remains if \( f = 0 \)); “+f”: stimulated and proportional to the occupation of final photon

- **Photon Absorption:** \( -(1 \pm f_\mu)f_if \)
  - \( (1 \pm f_\mu) \): if final state is occupied and fermion, process blocked; if boson the process enhanced
  - \( f_i \): proportional to number of absorbers
  - \( f \): proportional to incoming photons
Boltzmann Equation

• If interactions are rapid they will establish an equilibrium distribution where the distribution functions no longer change
\[ C[f_{eq}] = 0 \]

• Solve by inspection
\[ \Pi_i \Pi_{\mu} f_\mu (1 \pm f_i)(1 \pm f) - \Pi_i \Pi_{\mu}(1 \pm f_\mu)f_i f = 0 \]

• Try \( f_a = (e^{E_a/T} \mp 1)^{-1} \) so that \( 1 \pm f_a = e^{-E_a/T}(e^{E_a/T} \mp 1)^{-1} \)
\[ e^{-\sum(E_i+E)/T} - e^{-\sum E_\mu/T} = 0 \]

and energy conservation says \( E + \sum E_i = \sum E_\mu \), so identity is satisfied if the constant \( T \) is the same for all species, i.e. are in thermal equilibrium
Boltzmann Equation

- If the interaction does not create or destroy particles then the distribution

\[ f_{eq} = \left( e^{(E-\mu)/T} \mp 1 \right)^{-1} \]

also solves the equilibrium equation: e.g. a scattering type reaction

\[ \gamma_E + i \rightarrow \gamma_{E'} + j \]

where \( i \) and \( j \) represent the same collection of particles but with different energies after the scattering

\[ \sum (E_i - \mu_i) + (E - \mu) = \sum (E_j - \mu_j) + (E' - \mu) \]

since \( \mu_i = \mu_j \) for each particle

- Not surprisingly, this is the Fermi-Dirac distribution for fermions and the Bose-Einstein distribution for bosons
Boltzmann Equation

- More generally, equilibrium is satisfied if the sum of the chemical potentials on both sides of the interaction are equal, $\gamma + i \leftrightarrow \nu$

$$\sum \mu_i + \mu = \sum \mu_{\nu}$$

i.e. the law of mass action is satisfied

- If interactions that create or destroy particles are in equilibrium then this law says that the chemical potential will vanish: e.g. $\gamma + e^- \rightarrow 2\gamma + e^-$

$$\mu_e + \mu = \mu_e + 2\mu \rightarrow \mu = 0$$

so that the chemical potential is driven to zero if particle number is not conserved in interaction
Maxwell Boltzmann Distribution

- For the nonrelativistic limit $E = m + \frac{1}{2}q^2/m$, nondegenerate limit $(E - \mu)/T \gg 1$ so both distributions go to the Maxwell-Boltzmann distribution

$$f_{eq} = \exp[-(m - \mu)/T] \exp(-q^2/2mT)$$

- Here it is even clearer that the chemical potential $\mu$ is the normalization parameter for the number density of particles whose number is conserved.

- $\mu$ and $n$ can be used interchangably
Poor Man’s Boltzmann Equation

- Non expanding medium

\[ \frac{\partial f}{\partial t} = \Gamma (f - f_{eq}) \]

where \( \Gamma \) is some rate for collisions

- Add in expansion in a homogeneous medium

\[ \frac{\partial f}{\partial t} + \frac{dq}{dt} \frac{\partial f}{\partial q} = \Gamma (f - f_{eq}) \]

\[ (q \propto a^{-1} \rightarrow \frac{1}{q} \frac{dq}{dt} = -\frac{1}{a} \frac{da}{dt} = H) \]

\[ \frac{\partial f}{\partial t} - H \frac{\partial f}{\partial \ln q} = \Gamma (f - f_{eq}) \]

- So equilibrium will be maintained if collision rate exceeds expansion rate \( \Gamma > H \)
Non-Relativistic Bulk Properties

- **Number density**

\[ n = g e^{-\frac{(m-\mu)}{T}} \frac{4\pi}{(2\pi)^3} \int_0^\infty q^2 dq \exp\left(-\frac{q^2}{2mT}\right) \]

\[ = g e^{-\frac{(m-\mu)}{T}} \frac{2^{3/2}}{2\pi^2} (mT)^{3/2} \int_0^\infty x^2 dx \exp(-x^2) \]

\[ = g \left(\frac{mT}{2\pi}\right)^{3/2} e^{-\frac{(m-\mu)}{T}} \]

- **Energy density** \( E = m \rightarrow \rho = mn \)

- **Pressure** \( \frac{q^2}{3E} = \frac{q^2}{3m} \rightarrow p = nT \), ideal gas law
Ultra-Relativistic Bulk Properties

- Chemical potential $\mu = 0, \zeta(3) \approx 1.202$

- Number density

$$n_{\text{boson}} = gT^3 \frac{\zeta(3)}{\pi^2} \quad \zeta(n + 1) \equiv \frac{1}{n!} \int_0^{\infty} \frac{x^n}{e^x - 1} dx$$

$$n_{\text{fermion}} = \frac{3}{4} gT^3 \frac{\zeta(3)}{\pi^2}$$

- Energy density

$$\rho_{\text{boson}} = gT^4 \frac{3}{\pi^2} \zeta(4) = gT^4 \frac{\pi^2}{30}$$

$$\rho_{\text{fermion}} = \frac{7}{8} gT^4 \frac{3}{\pi^2} \zeta(4) = \frac{7}{8} gT^4 \frac{\pi^2}{30}$$

- Pressure $q^2/3E = E/3 \rightarrow p = \rho/3, \ w_r = 1/3$
Entropy Density

- First law of thermodynamics

\[ dS = \frac{1}{T}(d\rho(T)V + p(T)dV) \]

so that

\[ \left. \frac{\partial S}{\partial V} \right|_T = \frac{1}{T}[\rho(T) + p(T)] \]
\[ \left. \frac{\partial S}{\partial T} \right|_V = \frac{V}{T} \frac{d\rho}{dT} \]

- Since \( S(V, T) \propto V \) is extensive

\[ S = \frac{V}{T}[\rho(T) + p(T)] \]
\[ \sigma = S/V = \frac{1}{T}[\rho(T) + p(T)] \]
### Entropy Density

- Integrability condition \( dS/dVdT = dS/dTdV \) relates the evolution of entropy density.

\[
\frac{d\sigma}{dT} = \frac{1}{T} \frac{d\rho}{dT}
\]

\[
\frac{d\sigma}{dt} = \frac{1}{T} \frac{d\rho}{dt} = \frac{1}{T} \left[ -3(\rho + p) \right] \frac{d\ln a}{dt}
\]

\[
\frac{d\ln \sigma}{dt} = -3 \frac{d\ln a}{dt}
\]

- Comoving entropy density is conserved in thermal equilibrium.

- For ultra relativisitc bosons \( s_{\text{boson}} = 3.602n_{\text{boson}} \); for fermions factor of \( 7/8 \) from energy density.

\[
g_* = \sum_{\text{bosons}} g_b + \frac{7}{8} \sum g_f
\]
Neutrino Freezeout

- Neutrino equilibrium maintained by weak interactions, e.g.
  \[ e^+ + e^- \leftrightarrow \nu + \bar{\nu} \]

- Weak interaction cross section
  \[ T_{10} = \frac{T}{10^{10} K} \sim \frac{T}{1 \text{MeV}} \]

- Weak interaction cross section
  \[ \sigma_w \sim G_F^2 E^2_\nu \approx 4 \times 10^{-44} T_{10}^2 \text{cm}^2 \]

- Rate
  \[ \Gamma = n_\nu \sigma_w = H \quad \text{at} \quad T_{10} \sim 3 \quad \text{or} \quad t \sim 0.2 \text{s} \]

- After neutrino freezeout, electrons and positrons annihilate dumping their entropy into the photons

- Before \( g_* \): \( \gamma, e^+, e^- = 2 + \frac{7}{8} (2 + 2) = \frac{11}{2} \)

- After \( g_* \): \( \gamma = 2 \); so conservation of entropy gives

\[ g_* T^3 \bigg|_{\text{initial}} = g_* T^3 \bigg|_{\text{final}} \quad T_\nu = \left( \frac{4}{11} \right)^{1/3} T_\gamma \]
Relic Neutrinos

- Relic number density (zero chemical potential; now required by oscillations & BBN)

\[ n_\nu = n_\gamma \frac{3}{4} \frac{4}{11} = 112 \text{cm}^{-3} \]

- Relic energy density assuming one species with finite \( m_\nu \):

\[ \rho_\nu = m_\nu n_\nu \]

\[ \rho_\nu = 112 \frac{m_\nu}{\text{eV}} \text{ eV cm}^{-3} \quad \rho_c = 1.05 \times 10^4 h^2 \text{ eV cm}^{-3} \]

\[ \Omega_\nu h^2 = \frac{m_\nu}{93.7 \text{eV}} \]

- Candidate for dark matter? an eV mass neutrino goes non relativistic around \( z \sim 1000 \) and retains a substantial velocity dispersion \( \sigma_\nu \).
Hot Dark Matter

- Momenta for a nonrelativistic species redshifts like temperature for a relativistic one, so average momentum is still given by

\[ \langle q \rangle = 3T_\nu = m\sigma_\nu \]

\[ \sigma_\nu = 3 \left( \frac{m_\nu}{1\text{eV}} \right)^{-1} \left( \frac{T_\nu}{1\text{eV}} \right) = 3 \left( \frac{m_\nu}{1\text{eV}} \right)^{-1} \left( \frac{T_\nu}{10^4\text{K}} \right) \]

\[ = 6 \times 10^{-4} \left( \frac{m_\nu}{1\text{eV}} \right)^{-1} = 200\text{km/s} \left( \frac{m_\nu}{1\text{eV}} \right)^{-1} \]

- Of order the rotation velocity of galactic halos and higher at higher redshift - small objects can’t form: top down structure formation – not observed – must not constitute the bulk of the dark matter
• Problem with neutrinos is they decouple while relativistic and hence have a comparable number density to photons - for a reasonable energy density, the mass must be small

• The equilibrium distribution for a non-relativistic species declines exponentially beyond the mass threshold

\[ n = g \left( \frac{mT}{2\pi} \right)^{3/2} e^{-m/T} \]

• Exponential will eventually win soon after \( T < m \), suppressing annihilation rates
WIMP Miracle

• Freezeout when annihilation rate equal expansion rate $\Gamma \propto \sigma_A$, increasing annihilation cross section decreases abundance

$$\Gamma = n\langle \sigma_A v \rangle = H$$

$$H \propto T^2 \sim m^2$$

$$\rho_{\text{freeze}} = mn \propto \frac{m^3}{\langle \sigma_A v \rangle}$$

$$\rho_c = \rho_{\text{freeze}}(T/T_0)^{-3} \propto \frac{1}{\langle \sigma_A v \rangle}$$

independently of the mass of the CDM particle

• Plug in some typical numbers for supersymmetric candidates or WIMPs (weakly interacting massive particles) of $\langle \sigma_A v \rangle \approx 10^{-36}$ cm$^2$ and restore the proportionality constant $\Omega_c h^2$ is of the right order of magnitude ($\sim 0.1$)!
Axions

- Alternate solution: keep light particle but not created in thermal equilibrium
- Example: axion dark matter - particle that solves the strong CP problem
- Inflation sets initial conditions, fluctuation from potential minimum
- Once Hubble scale smaller than the mass scale, field unfreezes
- Coherent oscillations of the axion field - condensate state. Can be very light $m \ll 1\text{eV}$ and yet remain cold.
- Same reason a quintessence dark energy candidate must be lighter than the Hubble scale today
Big Bang Nucleosynthesis

- Integrating the Boltzmann equation for nuclear processes during first few minutes leads to synthesis and freezeout of light elements
Big Bang Nucleosynthesis

- Most of light element synthesis can be understood through nuclear statistical equilibrium and reaction rates.

- Equilibrium abundance of species with mass number $A$ and charge $Z$ ($Z$ protons and $A - Z$ neutrons)

$$n_A = g_A \left( \frac{m_A T}{2\pi} \right)^{3/2} e^{\frac{(\mu_A - m_A)}{T}}$$

- In chemical equilibrium with protons and neutrons

$$\mu_A = Z\mu_p + (A - Z)\mu_n$$

$$n_A = g_A \left( \frac{m_A T}{2\pi} \right)^{3/2} e^{-m_A/T} e^{(Z\mu_p + (A - Z)\mu_n)/T}$$
Big Bang Nucleosynthesis

- Eliminate chemical potentials with $n_p, n_n$

$$e^{\mu_p/T} = \frac{n_p}{g_p} \left( \frac{2\pi}{m_p T} \right)^{3/2} e^{m_p/T}$$

$$e^{\mu_n/T} = \frac{n_n}{g_n} \left( \frac{2\pi}{m_n T} \right)^{3/2} e^{m_n/T}$$

$$n_A = g_A g_p^{-Z} g_n^{Z-A} \left( \frac{m_A T}{2\pi} \right)^{3/2} \left( \frac{2\pi}{m_p T} \right)^{3Z/2} \left( \frac{2\pi}{m_n T} \right)^{3(A-Z)/2}$$

$$\times e^{-m_A/T} e^{(Z\mu_p + (A-Z)\mu_n)/T} n_p^Z n_n^{A-Z}$$

$$(g_p = g_n = 2; m_p \approx m_n = m_b = m_A/A)$$

$$(B_A = Zm_p + (A - Z)m_n - m_A)$$

$$= g_A 2^{-A} \left( \frac{2\pi}{m_b T} \right)^{3(A-1)/2} A^{3/2} n_p^Z n_n^{A-Z} e^{B_A/T}$$
Big Bang Nucleosynthesis

- Convenient to define abundance fraction

\[ X_A \equiv A \frac{n_A}{n_b} = A g_A 2^{-A} \left( \frac{2\pi}{m_b T} \right)^{3(A-1)/2} A^{3/2} n_p n_n^{A-Z} n_b^{-1} e^{B_A/T} \]

\[ = A g_A 2^{-A} \left( \frac{2\pi n_b^{2/3}}{m_b T} \right)^{3(A-1)/2} A^{3/2} e^{B_A/T} X_p X_n^{A-Z} \]

\[ (n_\gamma = \frac{2}{\pi^2} T^3 \zeta(3)) \quad \eta_{b\gamma} \equiv n_b/n_\gamma \]

\[ = A^{5/2} g_A 2^{-A} \left[ \left( \frac{2\pi T}{m_b} \right)^{3/2} \frac{2\zeta(3)\eta_{b\gamma}}{\pi^2} \right]^{A-1} X_p X_n^{A-Z} \]
Deuterium

- Deuterium $A = 2, Z = 1, g_2 = 3, B_2 = 2.225 \text{ MeV}$

$$X_2 = \frac{3}{\pi^2} \left( \frac{4 \pi T}{m_b} \right)^{3/2} \eta_{b \gamma} \zeta(3) e^{B_2/T} X_p X_n$$

- Deuterium
  "bottleneck" is mainly due to the low baryon-photon number of the universe
  $\eta_{b \gamma} \sim 10^{-9}$, secondarily due to the low binding energy $B_2$
Deuterium

- $X_2/X_pX_n \approx \mathcal{O}(1)$ at $T \approx 100$keV or $10^9$ K, much lower than the binding energy $B_2$

- Most of the deuterium formed then goes through to helium via $D + D \rightarrow ^3\text{He} + p$, $^3\text{He} + D \rightarrow ^4\text{He} + n$

- Deuterium freezes out as number abundance becomes too small to maintain reactions $n_D = \text{const.}$ independent of $n_b$

- The deuterium freezeout fraction $n_D/n_b \propto \eta_{b\gamma}^{-1} \propto (\Omega_b h^2)^{-1}$ and so is fairly sensitive to the baryon density.

- Observations of the ratio in quasar absorption systems give $\Omega_b h^2 \approx 0.02$
Essentially all neutrons around during nucleosynthesis end up in Helium.

In equilibrium, the neutron-to-proton ratio is determined by the mass difference:

\[ Q = m_n - m_p = 1.293 \text{ MeV} \]

\[ \frac{n_n}{n_p} = \exp\left[-\frac{Q}{T}\right] \]
Helium

- Equilibrium is maintained through weak interactions, e.g.
  \[ n \leftrightarrow p + e^- + \bar{\nu}, \nu + n \leftrightarrow p + e^-, e^+ + n \leftrightarrow p + \bar{\nu} \]  with rate
  \[ \frac{\Gamma}{H} \approx \frac{T}{0.8\text{MeV}} \]

- Freezeout fraction
  \[ \frac{n_n}{n_p} = \exp[-1.293/0.8] \approx 0.2 \]

- Finite lifetime of neutrons brings this to \( \sim 1/7 \) by \( 10^9\text{K} \)

- Helium mass fraction
  \[ Y_{\text{He}} = \frac{4n_{He}}{n_b} = \frac{4(n_n/2)}{n_n + n_p} = \frac{2n_n/n_p}{1 + n_n/n_p} \approx \frac{2/7}{8/7} \approx \frac{1}{4} \]
Helium

- Depends mainly on the expansion rate during BBN - measure number of relativistic species
- Traces of $^7\text{Li}$ as well. Measured abundances in reasonable agreement with deuterium measure $\Omega_b h^2 = 0.02$ but the detailed interpretation is still up for debate
Baryogenesis

- What explains the small, but non-zero, baryon-to-photon ratio?

\[ \eta_{b\gamma} = \frac{n_b}{n_\gamma} \approx 3 \times 10^{-8} \Omega_b h^2 \approx 6 \times 10^{-10} \]

- Must be a slight excess of baryons \( b \) to anti-baryons \( \bar{b} \) that remains after annihilation

- Sakharov conditions
  - Baryon number violation: some process must change the net baryon number
  - CP violation: process which produces \( b \) and \( \bar{b} \) must differ in rate
  - Out of equilibrium: else equilibrium distribution with vanishing chemical potential (processes exist which change baryon number) gives equal numbers for \( b \) and \( \bar{b} \)

- Expanding universe provides 3; physics must provide 1,2
Baryogenesis

- Example: out of equilibrium decay of some heavy boson $X, \bar{X}$

- Suppose $X$ decays through 2 channels with baryon number $b_1$ and $b_2$ with branching ratio $r$ and $1 - r$ leading to a change in the baryon number per decay of

$$rb_1 + (1 - r)b_2$$

- And $\bar{X}$ to $-b_1$ and $-b_2$ with ratio $\bar{r}$ and $1 - \bar{r}$

$$-\bar{r}b_1 - (1 - \bar{r})b_2$$

- Net production

$$\Delta b = (r - \bar{r})(b_1 - b_2)$$
Baryogenesis

- Condition 1: $b_1 \neq 0, b_2 \neq 0$
- Condition 2: $\bar{r} \neq r$
- Condition 3: out of equilibrium decay
- GUT and electroweak (instanton) baryogenesis mechanisms exist
- Active subject of research
Black Body Formation

- After $z \sim 10^6$, photon creating processes $\gamma + e^- \leftrightarrow 2\gamma + e^-$ and bremmstrahlung $e^- + p \leftrightarrow e^- + p + \gamma$ drop out of equilibrium for photon energies $E \sim T$.
- Compton scattering remains effective in redistributing energy via exchange with electrons.
- Out of equilibrium processes like decays leave residual photon chemical potential imprint.
- Observed black body spectrum places tight constraints on any that might dump energy into the CMB.
Recombination

- Maxwell-Boltzmann distribution determines the equilibrium distribution for reactions, e.g. big-bang nucleosynthesis, recombination:

\[ p + e^- \leftrightarrow H + \gamma \]

\[
\frac{n_p n_e}{n_H} \approx e^{-B/T} \left( \frac{m_e T}{2\pi} \right)^{3/2} e^{(\mu_p + \mu_e - \mu_H)/T}
\]

where \( B = m_p + m_e - m_H = 13.6 \text{eV} \) is the binding energy, \( g_p = g_e = \frac{1}{2} g_H = 2 \), and \( \mu_p + \mu_e = \mu_H \) in equilibrium

- Define ionization fraction

\[
\begin{align*}
    n_p & = n_e = x_e n_b \\
    n_H & = n_b - n_p = (1 - x_e) n_b
\end{align*}
\]
Recombination

- Saha Equation

\[ \frac{n_e n_p}{n_H n_b} = \frac{x_e^2}{1 - x_e} = \frac{1}{n_b} \left( \frac{m_e T}{2\pi} \right)^{3/2} e^{-B/T} \]

- Naive guess of \( T_\ast = B \) wrong due to the low baryon-photon ratio

\(- T_\ast \approx 0.3 \text{eV} \) so recombination at \( z_\ast \approx 1000 \)

- But the photon-baryon ratio is very low

\[ \eta_{b\gamma} \equiv \frac{n_b}{n_\gamma} \approx 3 \times 10^{-8} \Omega_b h^2 \]
Recombination

- **Eliminate** in favor of $\eta_{b\gamma}$ and $B/T$ through

$$n_\gamma = 0.244T^3, \quad \frac{m_e}{B} = 3.76 \times 10^4$$

- Big coefficient

$$\frac{x_e^2}{1 - x_e} = 3.16 \times 10^{15} \left(\frac{B}{T}\right)^{3/2} e^{-B/T}$$

$$T = 1/3\text{eV} \rightarrow x_e = 0.7, \quad T = 0.3\text{eV} \rightarrow x_e = 0.2$$

- **Further delayed** by inability to maintain equilibrium since net is through $2\gamma$ process and redshifting out of line

- Free electrons freezeout as recombination rate drops below Hubble - low ionization tail has observable impact on CMB power spectrum and is used to constrain energy injection from dark matter annihilation, primordial black holes, etc.
Recombination

Redshift $z$

Scale factor $a$

Ionization fraction

Saha

2-level
Inhomogeneous Universe

- Boltzmann equation defines how particles propagate and interact in the spacetime metric: “geometry tells matter how to move”

- Beyond the background FRW metric, fluctuations in the metric are inhomogeneous and anisotropic (but statistically homogeneous and isotropic in ensemble average)

- Particle numbers are still conserved in the absence of collisions if the spacetime metric is fluctuating adiabatically: generalize \( df/dt = 0 \) to changes in the momentum due to gravitational forces, redshift and lensing

- Matter tells spacetime how to curve: perturbed stress energy or low order moments of \( f \) determine the geometry and its evolution – joint solution

- Solve for the low order moments with the Boltzmann equation or an “equation of state” closure condition for the moment hierarchy
Inhomogeneous Universe

- To be continued with relativistic perturbation theory defining the joint evolution...