Astro 321

Set 3: Relativistic Perturbation Theory

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Covariant Perturbation Theory

- Covariant = takes same form in all coordinate systems
- Invariant = takes the same value in all coordinate systems
- Fundamental equations are covariant: Einstein equations, covariant conservation of stress-energy tensor:

$$\begin{aligned} G_{\mu\nu} &= 8\pi G T_{\mu\nu} \\ \nabla_\mu T^{\mu\nu} &= 0 \end{aligned}$$

- Components such as ρ, velocity, curvature etc are *not invariant* under a coordinate change
- Between any two fully specified coordinates, Jacobian $\partial x^\mu / \partial \tilde{x}^\nu$ is invertible - so perturbations in given gauge can be written in a covariant manner in terms of perturbations in an arbitrary gauge: called "gauge invariant" variables

Covariant Perturbation Theory

- In evolving perturbations we inevitably break explicit covariance by evolving conditions forward in a given time coordinate
- Retain implicit covariance by allowing the freedom to choose an arbitrary time slicing and spatial coordinates threading constant time slices
- Exploit covariance by choosing the specific slicing and threading (or "gauge") according to what best matches problem
- Preserve general covariance by keeping all free variables: 10 for each symmetric 4×4 tensor but blocked into $3+1$ "ADM" form

1	2	3	4
	5	6	7
		8	9
			10

ADM 3+1 Split

- Since Einstein equations dynamically evolve the spacetime, to solve the initial value problem choose a slicing for the foliation and evolve the spatial metric forward: 3+1 ADM split
- Define most general line element: lapse N, shift N^i, 3-metric h_{ij}

$$ds^2 = -N^2 d\phi^2 + h_{ij}(dx^i + N^i d\phi)(dx^j + N^j d\phi)$$

or equivalently the metric

$$g_{00} = -N^2 + N^i N_i, \quad g_{0i} = h_{ij} N^j \equiv N_i, \quad g_{ij} = h_{ij}$$

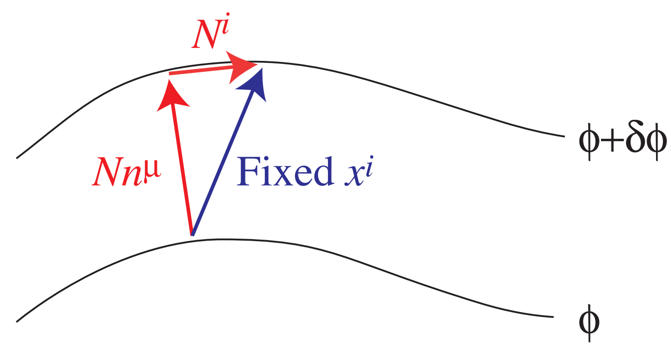
and its inverse $g^{\mu\alpha} g_{\alpha\nu} = \delta^\mu{}_\nu$

$$g^{00} = -1/N^2, \quad g^{0i} = N^i/N^2, \quad g^{ij} = h^{ij} - N^i N^j/N^2$$

- Time coordinate $x^0 = \phi$ need not be cosmological time t - could be any parameterization, e.g. conformal time, scalar field, ...

ADM 3+1 Split

- Useful to define the unit normal timelike vector $n_\mu n^\mu = -1$, orthogonal to constant time surfaces $n_\mu \propto \partial_\mu \phi$



$$n_\mu = (-N, 0, 0, 0), \quad n^\mu = (1/N, -N^i/N)$$

where we have used $n^\mu = g^{\mu\nu} n_\nu$

- Interpretation: lapse of proper time along normal, shift of spatial coordinates with respect to normal
- In GR (and most scalar-tensor EFT extensions), the lapse and shift are non-dynamical and just define the coordinates or gauge
- Dynamics in evolving the spatial metric forwards

ADM 3+1 Split

- Projecting 4D tensors onto the normal direction utilizes $n^\mu n_\nu$, e.g.

$$-n^\mu n_\nu V^\nu$$

- Projecting 4D tensors onto the 3D tensors involves the complement through the induced metric

$$h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu,$$
$$h^\mu{}_\nu V^\nu = (\delta^\mu{}_\nu + n^\mu n_\nu)V^\nu = V^\mu + n^\mu n_\nu V^\nu$$

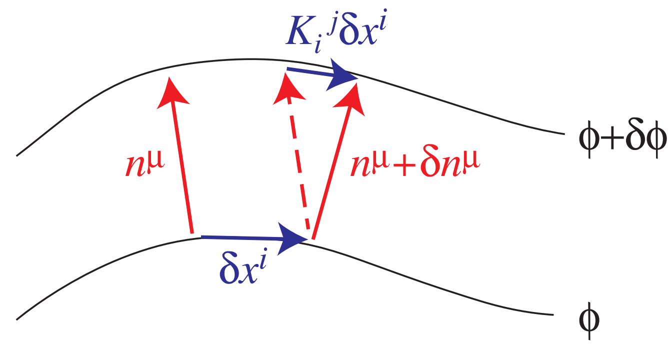
e.g. in the preferred slicing

$$\tilde{V}^\mu = h^\mu{}_\nu V^\nu = (\delta^\mu{}_\nu + n^\mu n_\nu)V^\nu = (0, V^i + N^i V^0)$$

whose spatial indices are raised an lowered by h_{ij}:
$\tilde{V}_i = g_{i\nu}\tilde{V}^\nu = h_{ij}\tilde{V}^j$

ADM 3+1 Split

- 3-surface embedded in 4D, so there is both an intrinsic curvature associated with h_{ij} and an extrinsic curvature which is the spatial projection of the gradient of n^μ



$$K_{\mu\nu} = h_\mu^{\ \alpha} h_\nu^{\ \beta} n_{\alpha;\beta}$$

- $K_{\mu\nu}$ symmetric since the antisymmetric projection (or vorticity) vanishes by construction since $n_\mu = -N\phi_{;\mu}$

ADM 3+1 Split

- Likewise split the spacetime curvature ${}^{(4)}R$ into intrinsic and extrinsic pieces via Gauss-Codazzi relation

$$ {}^{(4)}R = K_{\mu\nu}K^{\mu\nu} - (K_{\mu}{}^{\mu})^2 + {}^{(3)}R + 2(K_{\nu}{}^{\nu}n^{\mu} - n^{\alpha}n^{\mu}{}_{;\alpha})_{;\mu} $$

  Last piece is total derivative so Einstein Hilbert action is equivalent to keeping first three pieces

- No explicit dependence on slicing and threading N, N^i - any preferred slicing is picked out by the matter distribution not by general relativity

- Beyond GR we can embed a preferred slicing by making the Lagrangian an explicit function of N - will return to this in the effective field theory of inflation, dark energy

ADM 3+1 Split

- Trace ${K_\mu}^\mu = {n^\mu}_{;\mu} \equiv \theta$ is expansion
- Avoid confusion with FRW notation for intrinsic curvature: ${}^{(3)}R = 6K/a^2$
- The anisotropic part is known as the shear

$$\sigma_{\mu\nu} = K_{\mu\nu} - \frac{\theta}{3} h_{\mu\nu}$$

- For the FRW background the shear vanishes and the expansion $\theta = 3H$

ADM 3+1 Split

- Fully decompose the 4-tensor $n_{\mu;\nu}$ by adding normal components

$$\begin{aligned} n_{\mu;\nu} &= K_{\mu\nu} - n_\mu n^\alpha h_\nu{}^\beta n_{\alpha;\beta} - h_\mu{}^\alpha n_\nu n^\beta n_{\alpha;\beta} + n_\mu n^\alpha n_\nu n^\beta n_{\alpha;\beta} \\ &= K_{\mu\nu} - h_\mu{}^\alpha n_\nu n^\beta n_{\alpha;\beta} = K_{\mu\nu} - n_\nu n^\beta n_{\mu;\beta} - n_\mu n^\alpha n_\nu n^\beta n_{\alpha;\beta} \\ &= K_{\mu\nu} - n_\nu n^\beta n_{\mu;\beta} = K_{\mu\nu} - a_\mu n_\nu \end{aligned}$$

where we have used $[(n_\mu n^\mu)_{;\nu} = 0 \rightarrow n^\mu n_{\mu;\nu} = 0]$

- Here the directional derivative of the normal along the normal or "acceleration" is

$$a_\mu = (n_{\mu;\beta}) n^\beta$$

ADM 3+1 Split

- Since

$$n_{i;j} = K_{ij} = -\Gamma^{\mu}_{ij} n_{\mu} = \Gamma^{0}_{ij} N$$

in terms of the ADM variables

$$K_{ij} = \frac{1}{2N}(\partial_t h_{ij} - N_{j|i} - N_{i|j})$$

where $|$ denotes the covariant derivative with respect to h_{ij}

- Extrinsic curvature acts like a "velocity" term for h_{ij} moving the metric from one slice to another with the coordinate freedom of the lapse and shift
- Initial value problem in GR: define h_{ij} and $\dot{h}_{ij}$ on the spacelike surface and integrate forwards, with lapse and shift defining the temporal and spatial coordinates

ADM 3+1 Split

- Beyond GR we can extend this logic by constructing a general theory with some scalar whose constant surfaces define the normal and the time coordinate - build the most general action that retains spatial diffeomorphism invariance out of the ADM geometric objects

  $\rightarrow$ EFT of inflation and dark energy: return to this in inflation discussion

- For linear perturbation theory in GR, ADM looks simpler since we can linearize metric fluctuations and take out the global scale factor in the spatial tensors for convenience $h_{ij} = a^2\gamma_{ij}$
- ADM language useful in defining the geometric meaning of gauge choices in defining the time slicing and spatial threading

Metric Perturbations

- First define the slicing (lapse function A, shift function B^i)

$$
\begin{aligned}
g^{00} &= -a^{-2}(1-2A)\,, \\
g^{0i} &= -a^{-2}B^i\,,
\end{aligned}
$$

- ADM correspondence: perturbed lapse $N = a(1+A)$ between 3-surfaces whereas shift $N^i = -B^i$ defines the shift of 3-coordinates of the surfaces: note that for convenience spatial indices on B^i are lowered by γ_{ij} not h_{ij}
- This absorbs 1+3=4 free variables in the metric, remaining 6 is in the spatial surfaces which we parameterize as

$$
g^{ij} = a^{-2}(\gamma^{ij} - 2H_L\gamma^{ij} - 2H_T^{ij})\,.
$$

here (1) H_L a perturbation to the scale factor; (5) H_T^{ij} a trace-free distortion to spatial metric

Curvature Perturbation

- Curvature perturbation on the 3D slice

$$\delta[^{(3)}R] = -\frac{4}{a^2}\left(\nabla^2 + 3K\right) H_L + \frac{2}{a^2}\nabla_i\nabla_j H_T^{ij}$$

- Note that we will often loosely refer to H_L as the "curvature perturbation"
- We will see that many representations have $H_T = 0$
- It is easier to work with a dimensionless quantity
- Curvature perturbation is a 3-scalar in the ADM split and a Scalar in the SVT decomposition

Matter Tensor

- Likewise expand the matter stress energy tensor around a homogeneous density ρ and pressure p:

$$
\begin{aligned}
T^0{}_0 &= -\rho - \delta\rho\,, \\
T^0{}_i &= (\rho + p)(v_i - B_i)\,, \\
T_0{}^i &= -(\rho + p)v^i\,, \\
T^i{}_j &= (p + \delta p)\delta^i{}_j + p\Pi^i{}_j\,,
\end{aligned}
$$

- (1) $\delta\rho$ a density perturbation; (3) v_i a vector velocity, (1) δp a pressure perturbation; (5) Π_{ij} an anisotropic stress perturbation
- So far this is fully general and applies to any type of matter or coordinate choice including non-linearities in the matter, e.g. scalar fields, cosmological defects, exotic dark energy.

Counting Variables

20	Variables (10 metric; 10 matter)
-10	Einstein equations
-4	Conservation equations
$+4$	Bianchi identities
-4	Gauge (coordinate choice 1 time, 3 space)
6	Free Variables

- Without loss of generality these can be taken to be the 6 components of the matter stress tensor
- For the background, specify $p(a)$ or equivalently $w(a) \equiv p(a)/\rho(a)$ the equation of state parameter.

Homogeneous Einstein Equations

- Einstein (Friedmann) equations:

$$\left(\frac{1}{a}\frac{da}{dt}\right)^2 = -\frac{K}{a^2} + \frac{8\pi G}{3}\rho \quad [= \left(\frac{1}{a}\frac{\dot{a}}{a}\right)^2 = H^2]$$

$$\frac{1}{a}\frac{d^2a}{dt^2} = -\frac{4\pi G}{3}(\rho + 3p) \quad [= \frac{1}{a^2}\frac{d}{d\eta}\frac{\dot{a}}{a} = \frac{1}{a^2}\frac{d}{d\eta}(aH)]$$

so that $w \equiv p/\rho < -1/3$ for acceleration

- Conservation equation $\nabla^\mu T_{\mu\nu} = 0$ implies

$$\frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a}$$

overdots are conformal time but equally true with coordinate time

Homogeneous Einstein Equations

- Counting exercise:

20	Variables (10 metric; 10 matter)
-17	Homogeneity and Isotropy
-2	Einstein equations
-1	Conservation equations
$+1$	Bianchi identities
1	Free Variables

without loss of generality choose ratio of homogeneous & isotropic component of the stress tensor to the density $w(a) = p(a)/\rho(a)$.

Acceleration Implies Negative Pressure

- Role of stresses in the background cosmology
- Homogeneous Einstein equations $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ imply the two Friedmann equations (flat universe, or associating curvature $\rho_K = -3K/8\pi G a^2$)

$$\left(\frac{1}{a}\frac{da}{dt}\right)^2 = \frac{8\pi G}{3}\rho$$
$$\frac{1}{a}\frac{d^2a}{dt^2} = -\frac{4\pi G}{3}(\rho + 3p)$$

  so that the total equation of state $w \equiv p/\rho < -1/3$ for acceleration
- Conservation equation $\nabla^\mu T_{\mu\nu} = 0$ implies

$$\frac{\dot\rho}{\rho} = -3(1+w)\frac{\dot a}{a}$$

  so that ρ must scale more slowly than a^{-2}

Scalar, Vector, Tensor

- In linear perturbation theory, perturbations may be separated by their transformation properties under 3D rotation and translation.
- The eigenfunctions of the Laplacian operator form a complete set

$$
\begin{aligned}
\nabla^2 Q^{(0)} &= -k^2 Q^{(0)} && \mathrm{S}\,, \\
\nabla^2 Q_i^{(\pm 1)} &= -k^2 Q_i^{(\pm 1)} && \mathrm{V}\,, \\
\nabla^2 Q_{ij}^{(\pm 2)} &= -k^2 Q_{ij}^{(\pm 2)} && \mathrm{T}\,,
\end{aligned}
$$

- Vector and tensor modes satisfy divergence-free and transverse-traceless conditions

$$
\begin{aligned}
\nabla^i Q_i^{(\pm 1)} &= 0 \\
\nabla^i Q_{ij}^{(\pm 2)} &= 0 \\
\gamma^{ij} Q_{ij}^{(\pm 2)} &= 0
\end{aligned}
$$

Vector and Tensor Quantities

- A scalar mode carries with it associated vector (curl-free) and tensor (longitudinal) quantities
- A vector mode carries and associated tensor (trace and divergence free) quantities
- A tensor mode has only a tensor (trace and divergence free)
- These are built from the mode basis out of covariant derivatives and the metric

$$
\begin{aligned}
Q_i^{(0)} &= -k^{-1}\nabla_i Q^{(0)}\,, \\
Q_{ij}^{(0)} &= (k^{-2}\nabla_i\nabla_j + \frac{1}{3}\gamma_{ij})Q^{(0)}\,, \\
Q_{ij}^{(\pm 1)} &= -\frac{1}{2k}[\nabla_i Q_j^{(\pm 1)} + \nabla_j Q_i^{(\pm 1)}]\,,
\end{aligned}
$$

Perturbation k-Modes

- For the kth eigenmode, the scalar components become

$$
\begin{aligned}
A(\mathbf{x}) &= A(k)\, Q^{(0)} \,, & H_L(\mathbf{x}) &= H_L(k)\, Q^{(0)} \,, \\
\delta\rho(\mathbf{x}) &= \delta\rho(k)\, Q^{(0)} \,, & \delta p(\mathbf{x}) &= \delta p(k)\, Q^{(0)} \,,
\end{aligned}
$$

the vectors components become

$$
B_i(\mathbf{x}) = \sum_{m=-1}^{1} B^{(m)}(k)\, Q_i^{(m)} \,, \qquad v_i(\mathbf{x}) = \sum_{m=-1}^{1} v^{(m)}(k)\, Q_i^{(m)} \,,
$$

and the tensors components

$$
H_{Tij}(\mathbf{x}) = \sum_{m=-2}^{2} H_T^{(m)}(k)\, Q_{ij}^{(m)} \,, \quad \Pi_{ij}(\mathbf{x}) = \sum_{m=-2}^{2} \Pi^{(m)}(k)\, Q_{ij}^{(m)} \,,
$$

- Note that the curvature perturbation only involves scalars

$$
\delta[{}^{(3)}R] = \frac{4}{a^2}(k^2 - 3K)(H_L^{(0)} + \frac{1}{3}H_T^{(0)})Q^{(0)}
$$

Spatially Flat Case

- For a spatially flat background metric, harmonics are related to plane waves:

$$
\begin{aligned}
Q^{(0)} &= \exp(i\mathbf{k}\cdot\mathbf{x}) \\
Q_i^{(\pm 1)} &= \frac{-i}{\sqrt{2}}(\hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2)_i \exp(i\mathbf{k}\cdot\mathbf{x}) \\
Q_{ij}^{(\pm 2)} &= -\sqrt{\frac{3}{8}}(\hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2)_i(\hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2)_j \exp(i\mathbf{k}\cdot\mathbf{x})
\end{aligned}
$$

  where $\hat{\mathbf{e}}_3 \parallel \mathbf{k}$. Chosen as spin states, c.f. polarization.

- For vectors, the harmonic points in a direction orthogonal to $\mathbf{k}$ suitable for the vortical component of a vector

Spatially Flat Case

- Tensor harmonics are the transverse traceless gauge representation
- Tensor amplitude related to the more traditional

$$h_+[(\mathbf{e}_1)_i(\mathbf{e}_1)_j - (\mathbf{e}_2)_i(\mathbf{e}_2)_j]\,, \qquad h_\times[(\mathbf{e}_1)_i(\mathbf{e}_2)_j + (\mathbf{e}_2)_i(\mathbf{e}_1)_j]$$

  as

$$h_+ \pm i h_\times = -\sqrt{6} H_T^{(\mp 2)}$$

- $H_T^{(\pm 2)}$ proportional to the right and left circularly polarized amplitudes of gravitational waves with a normalization that is convenient to match the scalar and vector definitions

Covariant Scalar Equations

- DOF counting exercise

8	Variables (4 metric; 4 matter)
-4	Einstein equations
-2	Conservation equations
$+2$	Bianchi identities
-2	Gauge (coordinate choice 1 time, 1 space)
——	
2	Free Variables

without loss of generality choose scalar components of the stress tensor δp, Π .

Covariant Scalar Equations

- Einstein equations (suppressing 0) superscripts

$$
\begin{aligned}
&(k^2 - 3K)[H_L + \frac{1}{3}H_T] - 3(\frac{\dot a}{a})^2 A + 3\frac{\dot a}{a}\dot H_L + \frac{\dot a}{a}kB = \\
&\qquad = 4\pi G a^2 \delta\rho\,, \quad \text{00 Poisson Equation} \\
&\frac{\dot a}{a}A - \dot H_L - \frac{1}{3}\dot H_T - \frac{K}{k^2}(kB - \dot H_T) \\
&\qquad = 4\pi G a^2 (\rho + p)(v - B)/k\,, \quad 0i \text{ Momentum Equation} \\
&\left[2\frac{\ddot a}{a} - 2\left(\frac{\dot a}{a}\right)^2 + \frac{\dot a}{a}\frac{d}{d\eta} - \frac{k^2}{3}\right] A - \left[\frac{d}{d\eta} + \frac{\dot a}{a}\right](\dot H_L + \frac{1}{3}kB) \\
&\qquad = 4\pi G a^2 (\delta p + \frac{1}{3}\delta\rho)\,, \quad ii \text{ Acceleration Equation} \\
&k^2(A + H_L + \frac{1}{3}H_T) + \left(\frac{d}{d\eta} + 2\frac{\dot a}{a}\right)(kB - \dot H_T) \\
&\qquad = -8\pi G a^2 p\Pi\,, \quad ij \text{ Anisotropy Equation}
\end{aligned}
$$

Covariant Scalar Equations

- Poisson and acceleration equations are the perturbed generalization of the Friedmann equations
- Momentum and anisotropy equations are new to the perturbed metric
- Poisson and momentum equations in the ADM language take the form of constraints on the shift and lapse respectively - leaving the spatial metric components as dynamical
- Like the Friedmann equations, the 4 equation are redundant given the 2 energy-momentum conservation equations
- Choose a gauge and set of equations to simplify the given problem

Covariant Scalar Equations

- Conservation equations: continuity and Navier Stokes

$$
\begin{aligned}
\left[\frac{d}{d\eta} + 3\frac{\dot a}{a}\right]\delta\rho + 3\frac{\dot a}{a}\delta p &= -(\rho+p)(kv + 3\dot H_L)\,, \\
\left[\frac{d}{d\eta} + 4\frac{\dot a}{a}\right]\left[(\rho+p)\frac{(v-B)}{k}\right] &= \delta p - \frac{2}{3}(1-3\frac{K}{k^2})p\Pi + (\rho+p)A\,,
\end{aligned}
$$

- Equations are not independent since $\nabla_\mu G^{\mu\nu} = 0$ via the Bianchi identities.
- Related to the ability to choose a coordinate system or "gauge" to represent the perturbations.

Covariant Vector Equations

- Einstein equations

$$
\begin{aligned}
(1 - 2K/k^2)(kB^{(\pm 1)} - \dot{H}_T^{(\pm 1)}) \\
&= 16\pi G a^2 (\rho + p)(v^{(\pm 1)} - B^{(\pm 1)})/k \,, \\
\left[\frac{d}{d\eta} + 2\frac{\dot{a}}{a} \right] (kB^{(\pm 1)} - \dot{H}_T^{(\pm 1)}) \\
&= -8\pi G a^2 p \Pi^{(\pm 1)} \,.
\end{aligned}
$$

- Conservation Equations

$$
\begin{aligned}
\left[\frac{d}{d\eta} + 4\frac{\dot{a}}{a} \right] [(\rho + p)(v^{(\pm 1)} - B^{(\pm 1)})/k] \\
&= -\frac{1}{2}(1 - 2K/k^2) p \Pi^{(\pm 1)} \,,
\end{aligned}
$$

- Gravity provides no source to vorticity $\rightarrow$ decay

Covariant Vector Equations

- DOF counting exercise

8	Variables (4 metric; 4 matter)
-4	Einstein equations
-2	Conservation equations
$+2$	Bianchi identities
-2	Gauge (coordinate choice 1 time, 1 space)
2	Free Variables

without loss of generality choose vector components of the stress tensor $\Pi^{(\pm 1)}$.

Covariant Tensor Equation

- Einstein equation

$$\left[\frac{d^2}{d\eta^2} + 2\frac{\dot{a}}{a}\frac{d}{d\eta} + (k^2 + 2K)\right] H_T^{(\pm 2)} = 8\pi G a^2 p \Pi^{(\pm 2)} .$$

- DOF counting exercise

4	Variables (2 metric; 2 matter)
-2	Einstein equations
-0	Conservation equations
$+0$	Bianchi identities
-0	Gauge (coordinate choice 1 time, 1 space)
——	
2	Free Variables

wlog choose tensor components of the stress tensor $\Pi^{(\pm 2)}$.

Arbitrary Dark Components

- Total stress energy tensor can be broken up into individual pieces
- Dark components interact only through gravity and so satisfy separate conservation equations
- Einstein equation source remains the sum of components.
- To specify an arbitrary dark component, give the behavior of the stress tensor: 6 components: δp, $\Pi^{(i)}$, where $i = -2, ..., 2$.
- Many types of dark components (dark matter, scalar fields, massive neutrinos,..) have simple forms for their stress tensor in terms of the energy density, i.e. described by equations of state.
- An equation of state for the background $w = p/\rho$ is *not* sufficient to determine the behavior of the perturbations.

Geometry of Gauge Choice

- Geometry of the gauge or time slicing and spatial threading
- For perturbations larger than the horizon, a local observer should just see a different (separate) FRW universe
- Scalar equations should be equivalent to an appropriately remapped Friedmann equation
- ADM recap: unit normal vector n^μ to constant time hypersurfaces $n_\mu dx^\mu = n_0 d\eta$, $n^\mu n_\mu = -1$, to linear order in metric

$$\begin{aligned} n_0 &= -a(1+AQ), \qquad n_i = 0 \\ n^0 &= a^{-1}(1-AQ), \qquad n^i = -BQ^i \end{aligned}$$

- Intrinsic 3-geometry of δg_{ij}, changes in the normal vector $n_{\mu;\nu}$ that define the extrinsic curvature

Geometric Quantities

- Expansion of spatial volume per proper time is given by 4-divergence

$$n^{\mu}{}_{\mu} \equiv \theta = 3H(1 - AQ) + \frac{k}{a}BQ + \frac{3}{a}\dot{H}_L Q$$

- Other pieces of $n_{\mu;\nu}$ give the vorticity, shear and acceleration

$$n_{\mu;\nu} \equiv \omega_{\mu\nu} + \sigma_{\mu\nu} + \frac{\theta}{3}h_{\mu\nu} - a_\mu n_\nu$$

$$\begin{aligned}
h_{\mu\nu} &= g_{\mu\nu} + n_\mu n_\nu \\
\omega_{\mu\nu} &= h_\mu{}^\alpha h_\nu{}^\beta (n_{\alpha;\beta} - n_{\beta;\alpha}) = 0 \\
\sigma_{\mu\nu} &= \frac{1}{2}h_\mu{}^\alpha h_\nu{}^\beta (n_{\alpha;\beta} + n_{\beta;\alpha}) - \frac{1}{3}\theta h_{\mu\nu} \\
a_\mu &= n_{\mu;\alpha} n^\alpha
\end{aligned}$$

- Recall n_μ is a special timelike vector normal to the constant time surfaces, the vorticity vanishes by construction

Geometric Quantities

- Remaining perturbed quantities are the spatial shear and acceleration (0 components vanish)

$$
\begin{aligned}
\sigma_{ij} &= a(\dot{H}_T - kB)Q_{ij} \\
a_i &= -kAQ_i
\end{aligned}
$$

- Recall that the extrinsic curvature $K_{ij} = \sigma_{ij} + \theta h_{\mu\nu}/3$
- Intrinsic curvature of the 3-surface determined by 3-metric h_{ij}

$$
\delta[^{(3)}R] = \frac{4}{a^2}(k^2 - 3K)(H_L + \frac{H_T}{3})
$$

- E-foldings of the local expansion $\ln a_L$ are given

$$
\ln a_L = \int d\tau \frac{1}{3}\theta = \int d\eta \left(\frac{\dot{a}}{a} + \dot{H}_L Q + \frac{1}{3}kBQ \right)
$$

where we have used $d\tau = (1 + AQ)a d\eta$

Separate Universe

- Notice that
$$\frac{d}{d\eta}\delta \ln a_L = \dot H_L + \frac{\dot H_T}{3} - \frac{1}{3}(\dot H_T - kB)$$
so that if the shear is negligible the change in efolds tracks the change in curvature
- Shear vanishes in the FRW background; uniform efolding gives constant curvature
- Underlying principle: local observer should find long wavelength perturbations are indistingishable from a change in the background FRW quantities
- Perturbation equations take the form of Friedmann equations once rescaled

Time Slicing

- Constant time surfaces can be defined according to what geometry is helpful for the problem at hand
- Common choices:

  Uniform efolding: $\dot{H}_L + kB/3 = 0$

  Shear free: $\dot{H}_T - kB = 0$

  Zero lapse pert or acceleration, $A = 0$

  Uniform expansion: $-3HA + (3\dot{H}_L + kB) = 0$

  Comoving: $v = B$

- For the background all of these conditions hold.
- For perturbations each define a choice of slicing
- Can define the validity of the separate universe principle as the coexistence of comoving and zero lapse slicing

Time Slicing

- Comoving slicing is more formally called velocity orthogonal slicing since constant time surfaces are orthogonal to the matter 4-velocity V^μ:

$$h^\mu{}_\nu V^\nu = (\delta^\mu{}_\nu + n^\mu n_\nu)V^\nu = (0, V^i + N^i V^0) = 0$$

$$\rightarrow V^i = vQ^i = B^i = BQ^i$$

- Should not be confused with comoving (threading) where the 3-velocity $v = 0$ unless the shift B also vanishes

Gauge

- Metric and matter fluctuations take on different values in different coordinate system
- No such thing as a "gauge invariant" density perturbation!
- General coordinate transformation:

$$
\begin{aligned}
\tilde{\eta} &= \eta + T \\
\tilde{x}^i &= x^i + L^i
\end{aligned}
$$

free to choose (T, L^i) to simplify equations or physics – corresponds to a choice of slicing and threading in ADM.

- Decompose these into scalar T, $L^{(0)}$ and vector harmonics $L^{(\pm 1)}$.

Gauge

- $g_{\mu\nu}$ and $T_{\mu\nu}$ transform as tensors, so components in different frames can be related

$$
\begin{aligned}
\tilde{g}_{\mu\nu}(\tilde{\eta}, \tilde{x}^i) &= \frac{\partial x^\alpha}{\partial \tilde{x}^\mu} \frac{\partial x^\beta}{\partial \tilde{x}^\nu} g_{\alpha\beta}(\eta, x^i) \\
&= \frac{\partial x^\alpha}{\partial \tilde{x}^\mu} \frac{\partial x^\beta}{\partial \tilde{x}^\nu} g_{\alpha\beta}(\tilde{\eta} - TQ, \tilde{x}^i - LQ^i)
\end{aligned}
$$

- Fluctuations are compared at the same coordinate positions (not same space time positions) between the two gauges
- For example with a TQ perturbation, an event labeled with $\tilde{\eta}$ =const. and $\tilde{x}$ =const. represents a different time with respect to the underlying homogeneous and isotropic background

Gauge Transformation

- Scalar Metric:

$$
\begin{aligned}
\tilde{A} &= A - \dot{T} - \frac{\dot{a}}{a}T\,,\\
\tilde{B} &= B + \dot{L} + kT\,,\\
\tilde{H}_L &= H_L - \frac{k}{3}L - \frac{\dot{a}}{a}T\,,\\
\tilde{H}_T &= H_T + kL\,, \qquad \tilde{H}_L + \frac{1}{3}\tilde{H}_T = H_L + \frac{1}{3}H_T - \frac{\dot{a}}{a}T
\end{aligned}
$$

  curvature perturbation depends on slicing not threading

- Scalar Matter (Jth component):

$$
\begin{aligned}
\delta\tilde{\rho}_J &= \delta\rho_J - \dot{\rho}_J T,\\
\delta\tilde{p}_J &= \delta p_J - \dot{p}_J T,\\
\tilde{v}_J &= v_J + \dot{L},
\end{aligned}
$$

  density and pressure likewise depend on slicing only

Gauge Transformation

- Vector:

$$\begin{aligned}
\tilde{B}^{(\pm 1)} &= B^{(\pm 1)} + \dot{L}^{(\pm 1)}, \\
\tilde{H}_T^{(\pm 1)} &= H_T^{(\pm 1)} + kL^{(\pm 1)}, \\
\tilde{v}_J^{(\pm 1)} &= v_J^{(\pm 1)} + \dot{L}^{(\pm 1)},
\end{aligned}$$

- Spatial vector has no background component hence no dependence on slicing at first order

  Tensor: no dependence on slicing or threading at first order

- Gauge transformations and covariant representation can be extended to higher orders

- A coordinate system is fully specified if there is an explicit prescription for (T, L^i) or for scalars (T, L)

Slicing

Common choices for slicing T: set something geometric to zero

- Proper time slicing $A = 0$: proper time between slices corresponds to coordinate time – T allows c/a freedom
- Comoving (velocity orthogonal) slicing: $v - B = 0$, slicing is orthogonal to matter 4 velocity - T fixed
- Newtonian (shear free) slicing: $\dot{H}_T - kB = 0$, expansion rate is isotropic, shear free, T fixed
- Uniform expansion slicing: $-(\dot{a}/a)A + \dot{H}_L + kB/3 = 0$, perturbation to the volume expansion rate θ vanishes, T fixed
- Flat (constant curvature) slicing, $\delta^{(3)}R = 0$, $(H_L + H_T/3 = 0)$, T fixed
- Constant density slicing, $\delta\rho_I = 0$, T fixed

Threading

- Threading specifies the relationship between constant spatial coordinates between slices and is determined by L

  Typically involves a condition on v, B, H_T

  - Orthogonal threading $B = 0$, constant spatial coordinates orthogonal to slicing (zero shift), allows $\delta L = c$ translational freedom
  - Comoving threading $v = 0$, allows $\delta L = c$ translational freedom.
  - Isotropic threading $H_T = 0$, fully fixes L

Newtonian (Longitudinal) Gauge

- Newtonian (shear free slicing, isotropic threading):

$$\begin{aligned}
\tilde{B} &= \tilde{H}_T = 0 \\
\Psi &\equiv \tilde{A} \quad \text{(Newtonian potential)} \\
\Phi &\equiv \tilde{H}_L \quad \text{(Newtonian curvature)} \\
L &= -H_T/k \\
T &= -B/k + \dot{H}_T/k^2
\end{aligned}$$

Good: intuitive Newtonian like gravity; matter and metric algebraically related; commonly chosen for analytic CMB and lensing work

Bad: numerically unstable

Newtonian (Longitudinal) Gauge

- Newtonian (shear free) slicing, isotropic threading $B = H_T = 0$:

$$
\begin{aligned}
(k^2 - 3K)\Phi &= 4\pi G a^2 \left[\delta\rho + 3\frac{\dot a}{a}(\rho + p)v/k\right] \quad \text{Poisson + Momentum} \\
k^2(\Psi + \Phi) &= -8\pi G a^2 p \Pi \quad \text{Anisotropy}
\end{aligned}
$$

so $\Psi = -\Phi$ if anisotropic stress $\Pi = 0$ and

$$
\begin{aligned}
\left[\frac{d}{d\eta} + 3\frac{\dot a}{a}\right]\delta\rho + 3\frac{\dot a}{a}\delta p &= -(\rho + p)(kv + 3\dot\Phi)\,, \\
\left[\frac{d}{d\eta} + 4\frac{\dot a}{a}\right](\rho + p)v &= k\delta p - \frac{2}{3}(1 - 3\frac{K}{k^2})p\, k\Pi + (\rho + p)\, k\Psi\,,
\end{aligned}
$$

- Newtonian competition between stress (pressure and viscosity) and potential gradients
- Note: Poisson source is the density perturbation on comoving slicing

Comoving Gauge

- Comoving gauge (comoving slicing, isotropic threading)

$$
\begin{aligned}
\tilde{B} &= \tilde{v} \quad (T_i^0 = 0) \\
H_T &= 0 \\
\xi &= \tilde{A} \\
\mathcal{R} &= \tilde{H}_L \quad \text{(comoving curvature)} \\
\Delta &= \tilde{\delta} \quad \text{(total density pert)} \\
T &= (v - B)/k \\
L &= -H_T/k
\end{aligned}
$$

Good: Algebraic relations between matter and metric; comoving curvature perturbation obeys conservation law

Bad: Non-intuitive threading involving v

Comoving Gauge

- Euler equation becomes an algebraic relation between stress and potential

$$(\rho + p)\xi = -\delta p + \frac{2}{3}\left(1 - \frac{3K}{k^2}\right) p\Pi$$

- Einstein equation lacks momentum density source

$$\frac{\dot{a}}{a}\xi - \dot{\mathcal{R}} - \frac{K}{k^2}kv = 0$$

Combine: $\mathcal{R}$ is conserved if stress fluctuations negligible, e.g. above the horizon if $|K| \ll H^2$

$$\dot{\mathcal{R}} + Kv/k = \frac{\dot{a}}{a}\left[-\frac{\delta p}{\rho + p} + \frac{2}{3}\left(1 - \frac{3K}{k^2}\right)\frac{p}{\rho + p}\Pi\right] \to 0$$

"Gauge Invariant" Approach

- Gauge transformation rules allow variables which take on a geometric meaning in one choice of slicing and threading to be accessed from variables on another choice
- Functional form of the relationship between the variables is gauge invariant (*not* the variable values themselves! – i.e. equation is *covariant*)
- E.g. comoving curvature and density perturbations

$$
\begin{aligned}
\mathcal{R} &= H_L + \frac{1}{3} H_T - \frac{\dot{a}}{a}(v - B)/k \\
\Delta\rho &= \delta\rho + 3(\rho + p)\frac{\dot{a}}{a}(v - B)/k
\end{aligned}
$$

Newtonian-Comoving Hybrid

- With the gauge in(*or co*)variant approach, express variables of one gauge in terms of those in another – allows a mixture in the equations of motion
- Example: Newtonian curvature and comoving density

$$(k^2 - 3K)\Phi = 4\pi G a^2 \rho \Delta$$

  ordinary Poisson equation then implies Φ approximately constant if stresses negligible.

- Example: Exact Newtonian curvature above the horizon derived through comoving curvature conservation

  Gauge transformation

$$\Phi = \mathcal{R} + \frac{\dot{a}}{a}\frac{v}{k}$$

Hybrid "Gauge Invariant" Approach

Einstein equation to eliminate velocity

$$\frac{\dot{a}}{a}\Psi - \dot{\Phi} = 4\pi G a^2(\rho + p)v/k$$

Friedmann equation with no spatial curvature

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}a^2\rho$$

With $\dot{\Phi} = 0$ and $\Psi \approx -\Phi$

$$\frac{\dot{a}}{a}\frac{v}{k} = -\frac{2}{3(1+w)}\Phi$$

Newtonian-Comoving Hybrid

Combining gauge transformation with velocity relation

$$\Phi = \frac{3+3w}{5+3w}\mathcal{R}$$

Usage: calculate $\mathcal{R}$ from inflation determines Φ for any choice of matter content or causal evolution.

- Example: Scalar field ("quintessence" dark energy) equations in comoving gauge imply a sound speed $\delta p/\delta\rho = 1$ independent of potential $V(\phi)$. Solve in synchronous gauge.

Synchronous Gauge

- Synchronous: (proper time slicing, orthogonal threading)

$$
\begin{aligned}
\tilde{A} &= \tilde{B} = 0 \\
\eta_T &\equiv -\tilde{H}_L - \frac{1}{3}\tilde{H}_T \\
h_L &\equiv 6H_L \\
T &= a^{-1}\int d\eta a A + c_1 a^{-1} \\
L &= -\int d\eta (B + kT) + c_2
\end{aligned}
$$

Good: stable, the choice of numerical codes and separate universe constructs

Bad: residual gauge freedom in constants c_1, c_2 must be specified as an initial condition, intrinsically relativistic, threading conditions breaks down beyond linear regime if c_1 is fixed to CDM comoving.

Synchronous Gauge

- The Einstein equations give

$$
\begin{aligned}
\dot\eta_T - \frac{K}{2k^2}(\dot h_L + 6\dot\eta_T) &= 4\pi G a^2(\rho + p)\frac{v}{k}\,, \\
\ddot h_L + \frac{\dot a}{a}\dot h_L &= -8\pi G a^2(\delta\rho + 3\delta p)\,, \\
-(k^2 - 3K)\eta_T + \frac{1}{2}\frac{\dot a}{a}\dot h_L &= 4\pi G a^2 \delta\rho
\end{aligned}
$$

[choose (1 & 2) or (1 & 3)] while the conservation equations give

$$
\begin{aligned}
\left[\frac{d}{d\eta} + 3\frac{\dot a}{a}\right]\delta\rho_J + 3\frac{\dot a}{a}\delta p_J &= -(\rho_J + p_J)(kv_J + \frac{1}{2}\dot h_L)\,, \\
\left[\frac{d}{d\eta} + 4\frac{\dot a}{a}\right](\rho_J + p_J)\frac{v_J}{k} &= \delta p_J - \frac{2}{3}(1 - 3\frac{K}{k^2})p_J\Pi_J\,.
\end{aligned}
$$

Synchronous Gauge

- Lack of a lapse A implies no gravitational forces in Navier-Stokes equation. Hence for stress free matter like cold dark matter, zero velocity initially implies zero velocity always.

- Choosing the momentum and acceleration Einstein equations is good since for CDM domination, curvature η_T is conserved and $\dot{h}_L$ is simple to solve for.

- Choosing the momentum and Poisson equations is good when the equation of state of the matter is complicated since δp is not involved. This is the choice of CAMB.

  Caution: since the curvature η_T appears and it has zero CDM source, subtle effects like dark energy perturbations are important everywhere

Spatially Flat Gauge

- Spatially Flat (flat slicing, isotropic threading):

$$
\begin{aligned}
\tilde{H}_L &= \tilde{H}_T = 0 \\
L &= -H_T/k \\
\tilde{A}\,, \tilde{B} &= \quad \text{metric perturbations} \\
T &= \left(\frac{\dot{a}}{a}\right)^{-1} \left(H_L + \frac{1}{3} H_T\right)
\end{aligned}
$$

  Good: eliminates spatial metric evolution in ADM and perturbation equations ; useful in inflationary calculations (Mukhanov et al)

  Bad: non-intuitive slicing (no curvature!) and threading

- Caution: perturbation evolution is governed by the behavior of stress fluctuations and an isotropic stress fluctuation δp is gauge dependent.

Uniform Density Gauge

- Uniform density: (constant density slicing, isotropic threading)

$$
\begin{aligned}
H_T &= 0\,, \\
\zeta_I &\equiv H_L \\
B_I &\equiv B \\
A_I &\equiv A \\
T &= \frac{\delta\rho_I}{\dot{\rho}_I} \\
L &= -H_T/k
\end{aligned}
$$

Good: Curvature conserved involves only stress energy conservation; simplifies isocurvature treatment

Bad: non intuitive slicing (no density pert! problems beyond linear regime) and threading

Uniform Density Gauge

- Einstein equations with I as the total or dominant species

$$(k^2 - 3K)\zeta_I - 3\left(\frac{\dot a}{a}\right)^2 A_I + 3\frac{\dot a}{a}\dot\zeta_I + \frac{\dot a}{a} k B_I = 0\,,$$

$$\frac{\dot a}{a} A_I - \dot\zeta_I - \frac{K}{k} B_I = 4\pi G a^2 (\rho + p)\frac{v - B_I}{k}\,,$$

- The conservation equations (if $J = I$ then $\delta\rho_J = 0$)

$$\left[\frac{d}{d\eta} + 3\frac{\dot a}{a}\right]\delta\rho_J + 3\frac{\dot a}{a}\delta p_J = -(\rho_J + p_J)(k v_J + 3\dot\zeta_I)\,,$$

$$\left[\frac{d}{d\eta} + 4\frac{\dot a}{a}\right](\rho_J + p_J)\frac{v_J - B_I}{k} = \delta p_J - \frac{2}{3}(1 - 3\frac{K}{k^2}) p_J \Pi_J + (\rho_J + p_J) A_I\,.$$

Uniform Density Gauge

- Conservation of curvature - single component I

$$\dot{\zeta}_I = -\frac{\dot{a}}{a} \frac{\delta p_I}{\rho_I + p_I} - \frac{1}{3} k v_I \, .$$

- Since $\delta\rho_I = 0$, δp_I is the non-adiabatic stress and curvature is constant as $k \to 0$ for adiabatic fluctuations $p_I(\rho_I)$.

- Note that this conservation law does not involve the Einstein equations at all: just local energy momentum conservation so it is valid for alternate theories of gravity

- Curvature on comoving slices $\mathcal{R}$ and ζ_I related by

$$\zeta_I = \mathcal{R} + \frac{1}{3} \frac{\rho \Delta_I}{(\rho_I + p_I)} \Big|_{\rm comoving} \, .$$

and coincide above the horizon for adiabatic fluctuations

Uniform Density Gauge

- Simple relationship to density fluctuations in the spatially flat gauge

$$\zeta_I = \frac{1}{3}\frac{\delta\tilde{\rho}_I}{(\rho_I + p_I)}\Big|_{\rm flat} .$$

- For each particle species $\delta\rho/(\rho + p) = \delta n/n$, the number density fluctuation
- Multiple ζ_J carry information about number density fluctuations between species
- ζ_J constant component by component outside horizon if each component is adiabatic $p_J(\rho_J)$.

Unitary Gauge

- Given a scalar field $\phi(x^i, \eta)$, choose a slicing so that the field is spatially uniform $\phi(x^i, \eta) = \phi(\eta)$ via the transformation

$$\delta\tilde{\phi} = \delta\phi - \dot{\phi}T \quad \rightarrow \quad T = \frac{\delta\phi}{\dot{\phi}_0}$$

- Specify threading, e.g. isotropic threading $L = -H_T/k$

  Good: Scalar field carried completely by the metric; EFT of inflation and scalar-tensor theories of gravity. Extensible to nonlinear perturbations as long as $\partial_\mu\phi$ remains timelike

  Bad: Preferred slicing retains only the spatial diffeomorphism invariance; can make full covariance and DOF counting obscure

- For a canonical scalar field, unitary and comoving gauge coincide

EFT of Dark Energy and Inflation

- Beyond linear theory, unitary gauge and ADM is useful to define most general Lagrangian and interaction terms for a scalar-tensor theory of gravity: so-called Effective Field Theory (EFT)
- Rule: broken temporal diffeomorphisms (preferred slicing) but spatial diffeomorphism invariance means explicit functions of unitary time and ADM spatial scalars allowed
- Typically also want second order in time derivatives to avoid Ostrogradsky ghost, lapse and shift non-dynamical

$$\mathcal{L}(N, K^i{}_j, R^i{}_j, \nabla^i; t)$$

  where the function is constructed out of spatial scalars (3D Riemann tensor can be expressed through 3 Ricci tensor and metric
- Recall that the extrinsic curvature carries a first time derivative of the spatial metric and spatial gradients of the shift

EFT of Dark Energy and Inflation

- This class includes quintesence, k-essence, $f(R)$, Horndeski, “beyond Horndeski”, Horava-Liftshiz gravity, ghost condensate
- Does not include theories where derivatives of the lapse N appear but the shift is still nondynamical due to hidden constraints - can be generalized
- GR: time diffeomorphism not broken so the Einstein-Hilbert Lagrangian in EFT language is given by the Gauss-Codazzi relation

$$^{(4)}R \to K_{\mu\nu}K^{\mu\nu} - K^2 + {}^{(3)}R$$

EFT of Dark Energy and Inflation

- Now consider the scalar field to pick out a particular foliation
- Simplest example k-essence where $\mathcal{L}(X, \phi)$ where $X = g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ (sometimes $-1/2\times$ to resemble kinetic energy)
- In unitary gauge ϕ is a function of the temporal coordinate only so $X = -\dot{\phi}^2/N^2$ so that

$$\mathcal{L}(X, \phi) \to \mathcal{L}(N, t)$$

- We will return to this case when considering inflationary non-Gaussianity: note that $g^{00} = -1/N^2$ so the EFT literature sometimes writes $\mathcal{L}(g^{00}, \ldots, t)$ (Cheung et al 2008)
- Unifying description for "building blocks" of dark energy (Gleyzes, Langois, Vernizzi 2015)

Vector Gauges

- Vector gauge depends only on threading L
- Poisson gauge: orthogonal threading $B^{(\pm 1)} = 0$, leaves constant L translational freedom
- Isotropic gauge: isotropic threading $H_T^{(\pm 1)} = 0$, fixes L
- To first order scalar and vector gauge conditions can be chosen separately
- More care required for second and higher order where scalars and vectors mix