Covariant Perturbation Theory

- **Covariant** = takes same form in all coordinate systems
- **Invariant** = takes the same value in all coordinate systems
- Fundamental equations are covariant: *Einstein equations*, covariant conservation of stress-energy tensor:

\[ G_{\mu\nu} = 8\pi G T_{\mu\nu} \]
\[ \nabla_\mu T^{\mu\nu} = 0 \]

- Components such as \( \rho \), velocity, curvature etc are *not invariant* under a coordinate change
- Between any two fully specified coordinates, Jacobian \( \partial x^\mu / \partial \tilde{x}^\nu \) is invertible - so perturbations in given gauge can be written in a covariant manner in terms of perturbations in an arbitrary gauge: called “gauge invariant” variables
Covariant Perturbation Theory

- In evolving perturbations we inevitably break explicit covariance by evolving conditions forward in a given time coordinate.

- Retain implicit covariance by allowing the freedom to choose an arbitrary time slicing and spatial coordinates threading constant time slices.

- Exploit covariance by choosing the specific slicing and threading (or “gauge”) according to what best matches problem.

- Preserve general covariance by keeping all free variables: 10 for each symmetric $4 \times 4$ tensor but blocked into $3 + 1$ “ADM” form.

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
5 & 6 & 7 \\
8 & 9 \\
10 & \\
\end{array}
\]
ADM 3+1 Split

- Since Einstein equations dynamically evolve the spacetime, to solve the initial value problem choose a slicing for the foliation and evolve the spatial metric forward: 3+1 ADM split

- Define most general line element: lapse $N$, shift $N^i$, 3-metric $h_{ij}$

\[ ds^2 = -N^2 d\phi^2 + h_{ij}(dx^i + N^i d\phi)(dx^j + N^j d\phi) \]

or equivalently the metric

\[ g_{00} = -N^2 + N^i N_i, \quad g_{0i} = h_{ij} N^j \equiv N_i, \quad g_{ij} = h_{ij} \]

and its inverse $g^{\mu\alpha} g_{\alpha\nu} = \delta^\mu_\nu$

\[ g^{00} = -1/N^2, \quad g^{0i} = N^i/N^2, \quad g^{ij} = h^{ij} - N^i N^j / N^2 \]

- Time coordinate $x^0 = \phi$ need not be cosmological time $t$ - could be any parameterization, e.g. conformal time, scalar field, \ldots
ADM 3+1 Split

- Useful to define the unit normal timelike vector $n_\mu n^\mu = -1$, orthogonal to constant time surfaces $n_\mu \propto \partial_\mu \phi$
  
  $n_\mu = (-N, 0, 0, 0), \quad n^\mu = (1/N, -N^i/N)$

  where we have used $n^\mu = g^{\mu\nu} n_\nu$

- Interpretation: lapse of proper time along normal, shift of spatial coordinates with respect to normal

- In GR (and most scalar-tensor EFT extensions), the lapse and shift are non-dynamical and just define the coordinates or gauge

- Dynamics in evolving the spatial metric forwards
ADM 3+1 Split

- Projecting 4D tensors onto the normal direction utilizes $n^\mu n_\nu$, e.g.
  $$-n^\mu n_\nu V^\nu$$

- Projecting 4D tensors onto the 3D tensors involves the complement through the induced metric
  $$h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu,$$
  $$h^{\mu\nu} V^\nu = (\delta_{\mu\nu} + n_\mu n_\nu) V^\nu = V^\mu + n^\mu n_\nu V^\nu$$

  e.g. in the preferred slicing
  $$\tilde{V}^\mu = h^{\mu\nu} V^\nu = (\delta_{\mu\nu} + n_\mu n_\nu) V^\nu = (0, V^i + N^i V^0)$$

  whose spatial indices are raised an lowered by $h_{ij}$:
  $$\tilde{V}_i = g_{i\nu} \tilde{V}^\nu = h_{ij} \tilde{V}^j$$
• 3-surface embedded in 4D, so there is both an intrinsic curvature associated with $h_{ij}$ and an extrinsic curvature which is the spatial projection of the gradient of $n^\mu$

$$K_{\mu\nu} = h_\mu^\alpha h_\nu^\beta n_{\alpha;\beta}$$

• $K_{\mu\nu}$ symmetric since the antisymmetric projection (or vorticity) vanishes by construction since $n_\mu = -N\phi_{;\mu}$
ADM 3+1 Split

- Likewise split the spacetime curvature $^{(4)}R$ into intrinsic and extrinsic pieces via Gauss-Codazzi relation

$$^{(4)}R = K_{\mu\nu}K^{\mu\nu} - (K_\mu^\mu)^2 + ^{(3)}R + 2(K_\nu^\nu n^\mu - n^\alpha n_\alpha^\mu ;\mu) ;\mu$$

Last piece is total derivative so Einstein Hilbert action is equivalent to keeping first three pieces

- No explicit dependence on slicing and threading $N, N^i$ - any preferred slicing is picked out by the matter distribution not by general relativity

- Beyond GR we can embed a preferred slicing by making the Lagrangian an explicit function of $N$ - will return to this in the effective field theory of inflation, dark energy
ADM 3+1 Split

- Trace $K_{\mu}^{\mu} = n^{\mu} ;_{\mu} \equiv \theta$ is expansion

- Avoid confusion with FRW notation for intrinsic curvature:
  \((3) R = 6K/a^2\)

- The anisotropic part is known as the shear

  $$\sigma_{\mu\nu} = K_{\mu\nu} - \frac{\theta}{3} h_{\mu\nu}$$

- For the FRW background the shear vanishes and the expansion
  \(\theta = 3H\)
ADM 3+1 Split

- Fully decompose the 4-tensor $n_{\mu;\nu}$ by adding normal components

\[
\begin{align*}
n_{\mu;\nu} &= K_{\mu\nu} - n_\mu n^\alpha h_\nu{}^\beta n_{\alpha;\beta} - h_\mu{}^\alpha n_\nu n^\beta n_{\alpha;\beta} + n_\mu n^\alpha n_\nu n^\beta n_{\alpha;\beta} \\
&= K_{\mu\nu} - h_\mu{}^\alpha n_\nu n^\beta n_{\alpha;\beta} = K_{\mu\nu} - n_\nu n^\beta n_{\mu;\beta} - n_\mu n^\alpha n_\nu n^\beta n_{\alpha;\beta} \\
&= K_{\mu\nu} - n_\nu n^\beta n_{\mu;\beta} = K_{\mu\nu} - a_\mu n_\nu
\end{align*}
\]

where we have used \([ (n_\mu n^\mu)_{;\nu} = 0 \rightarrow n^\mu n_{\mu;\nu} = 0 ]\)

- Here the directional derivative of the normal along the normal or “acceleration” is

\[
a_\mu = (n_{\mu;\beta}) n^\beta
\]
ADM 3+1 Split

- Since

\[ n_{i:j} = K_{ij} = -\Gamma_{ij}^\mu n_\mu = \Gamma_{ij}^0 N \]

in terms of the ADM variables

\[ K_{ij} = \frac{1}{2N} (\partial_t h_{ij} - N_{j|i} - N_{i|j}) \]

where | denotes the covariant derivative with respect to \( h_{ij} \)

- Extrinsic curvature acts like a "velocity" term for \( h_{ij} \) moving the metric from one slice to another with the coordinate freedom of the lapse and shift

- Initial value problem in GR: define \( h_{ij} \) and \( \dot{h}_{ij} \) on the spacelike surface and integrate forwards, with lapse and shift defining the temporal and spatial coordinates
ADM 3+1 Split

- Beyond GR we can extend this logic by constructing a general theory with some scalar whose constant surfaces define the normal and the time coordinate - build the most general action that retains spatial diffeomorphism invariance out of the ADM geometric objects.

  \[ h_{ij} = a^2 \gamma_{ij} \]

  \( \rightarrow \) EFT of inflation and dark energy: return to this in inflation discussion.

- For linear perturbation theory in GR, ADM looks simpler since we can linearize metric fluctuations and take out the global scale factor in the spatial tensors for convenience.

- ADM language useful in defining the geometric meaning of gauge choices in defining the time slicing and spatial threading.
Metric Perturbations

• First define the slicing (lapse function $A$, shift function $B^i$)

$$
g^{00} = -a^{-2}(1 - 2A),
$$
$$
g^{0i} = -a^{-2}B^i,
$$

• ADM correspondence: perturbed lapse $N = a(1 + A)$ between 3-surfaces whereas shift $N^i = -B^i$ defines the shift of 3-coordinates of the surfaces: note that for convenience spatial indices on $B^i$ are lowered by $\gamma_{ij}$ not $h_{ij}$

• This absorbs 1+3=4 free variables in the metric, remaining 6 is in the spatial surfaces which we parameterize as

$$
g^{ij} = a^{-2}(\gamma^{ij} - 2H_L \gamma^{ij} - 2H_T^{ij}).
$$

Here (1) $H_L$ a perturbation to the scale factor; (5) $H_T^{ij}$ a trace-free distortion to spatial metric
Curvature Perturbation

- Curvature perturbation on the 3D slice

\[
\delta^{(3) R} = -\frac{4}{a^2} (\nabla^2 + 3K) H_L + \frac{2}{a^2} \nabla_i \nabla_j H_T^{ij}
\]

- Note that we will often loosely refer to \( H_L \) as the “curvature perturbation”

- We will see that many representations have \( H_T = 0 \)

- It is easier to work with a dimensionless quantity

- Curvature perturbation is a 3-scalar in the ADM split and a Scalar in the SVT decomposition
Matter Tensor

- Likewise expand the matter stress energy tensor around a homogeneous density $\rho$ and pressure $p$:

$$
T_{0}^{0} = -\rho - \delta \rho ,
$$
$$
T_{i}^{0} = (\rho + p)(v_{i} - B_{i}) ,
$$
$$
T_{0}^{i} = -(\rho + p)v^{i} ,
$$
$$
T_{i}^{j} = (p + \delta p)\delta_{i}^{j} + p\Pi_{i}^{j} ,
$$

- (1) $\delta \rho$ a density perturbation; (3) $v_{i}$ a vector velocity, (1) $\delta p$ a pressure perturbation; (5) $\Pi_{i}^{j}$ an anisotropic stress perturbation

- So far this is fully general and applies to any type of matter or coordinate choice including non-linearities in the matter, e.g. scalar fields, cosmological defects, exotic dark energy.
Counting Variables

20 Variables (10 metric; 10 matter)
−10 Einstein equations
−4 Conservation equations
+4 Bianchi identities
−4 Gauge (coordinate choice 1 time, 3 space)

6 Free Variables

• Without loss of generality these can be taken to be the 6 components of the matter stress tensor

• For the background, specify \( p(a) \) or equivalently \( w(a) \equiv p(a)/\rho(a) \) the equation of state parameter.
Homogeneous Einstein Equations

- Einstein (Friedmann) equations:

\[
\left(\frac{1}{a} \frac{da}{dt}\right)^2 = -\frac{K}{a^2} + \frac{8\pi G}{3} \rho \quad [= \left(\frac{1}{a} \frac{\dot{a}}{a}\right)^2 = H^2]
\]

\[
\frac{1}{a} \frac{d^2a}{dt^2} = -\frac{4\pi G}{3} (\rho + 3p) \quad [= \frac{1}{a^2} \frac{d}{d\eta} \frac{\dot{a}}{a} = \frac{1}{a^2} \frac{d}{d\eta} (aH)]
\]

so that \( w \equiv p/\rho < -1/3 \) for acceleration

- Conservation equation \( \nabla^\mu T_{\mu\nu} = 0 \) implies

\[
\frac{\dot{\rho}}{\rho} = -3(1 + w) \frac{\dot{a}}{a}
\]

overdots are conformal time but equally true with coordinate time
Homogeneous Einstein Equations

- Counting exercise:

\[
\begin{align*}
20 & \quad \text{Variables (10 metric; 10 matter)} \\
-17 & \quad \text{Homogeneity and Isotropy} \\
-2 & \quad \text{Einstein equations} \\
-1 & \quad \text{Conservation equations} \\
+1 & \quad \text{Bianchi identities} \\
\hline
1 & \quad \text{Free Variables}
\end{align*}
\]

without loss of generality choose ratio of homogeneous & isotropic component of the stress tensor to the density \( w(a) = p(a)/\rho(a) \).
Acceleration Implies Negative Pressure

- Role of stresses in the background cosmology

- Homogeneous Einstein equations $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ imply the two Friedmann equations (flat universe, or associating curvature $\rho_K = -3K/8\pi Ga^2$)

\[
\left( \frac{1}{a} \frac{da}{dt} \right)^2 = \frac{8\pi G}{3} \rho
\]

\[
\frac{1}{a} \frac{d^2a}{dt^2} = -\frac{4\pi G}{3} (\rho + 3p)
\]

so that the total equation of state $w \equiv p/\rho < -1/3$ for acceleration

- Conservation equation $\nabla^\mu T_{\mu\nu} = 0$ implies

\[
\frac{\dot{\rho}}{\rho} = -3(1 + w) \frac{\dot{a}}{a}
\]

so that $\rho$ must scale more slowly than $a^{-2}$
Scalar, Vector, Tensor

- In linear perturbation theory, perturbations may be separated by their transformation properties under 3D rotation and translation.

- The eigenfunctions of the Laplacian operator form a complete set

  \[ \nabla^2 Q^{(0)} = -k^2 Q^{(0)} \quad \text{S}, \]
  \[ \nabla^2 Q^{(\pm 1)}_i = -k^2 Q^{(\pm 1)}_i \quad \text{V}, \]
  \[ \nabla^2 Q^{(\pm 2)}_{ij} = -k^2 Q^{(\pm 2)}_{ij} \quad \text{T}, \]

- Vector and tensor modes satisfy divergence-free and transverse-traceless conditions

  \[ \nabla^i Q^{(\pm 1)}_i = 0 \]
  \[ \nabla^i Q^{(\pm 2)}_{ij} = 0 \]
  \[ \gamma^{ij} Q^{(\pm 2)}_{ij} = 0 \]
Vector and Tensor Quantities

- A scalar mode carries with it associated vector (curl-free) and tensor (longitudinal) quantities
- A vector mode carries and associated tensor (trace and divergence free) quantities
- A tensor mode has only a tensor (trace and divergence free)
- These are built from the mode basis out of covariant derivatives and the metric

\[
Q_{i}^{(0)} = -k^{-1} \nabla_{i} Q^{(0)},
\]

\[
Q_{ij}^{(0)} = (k^{-2} \nabla_{i} \nabla_{j} + \frac{1}{3} \gamma_{ij}) Q^{(0)},
\]

\[
Q_{ij}^{(\pm1)} = - \frac{1}{2k} [\nabla_{i} Q_{j}^{(\pm1)} + \nabla_{j} Q_{i}^{(\pm1)}],
\]
Perturbation $k$-Modes

- For the $k$th eigenmode, the scalar components become
  \[ A(x) = A(k) Q^{(0)}, \quad H_L(x) = H_L(k) Q^{(0)}, \]
  \[ \delta \rho(x) = \delta \rho(k) Q^{(0)}, \quad \delta p(x) = \delta p(k) Q^{(0)}, \]
  the vectors components become
  \[ B_i(x) = \sum_{m=-1}^1 B^{(m)}(k) Q_i^{(m)}, \quad v_i(x) = \sum_{m=-1}^1 v^{(m)}(k) Q_i^{(m)}, \]
  and the tensors components
  \[ H_{Tij}(x) = \sum_{m=-2}^2 H_{T}^{(m)}(k) Q_{ij}^{(m)}, \quad \Pi_{ij}(x) = \sum_{m=-2}^2 \Pi^{(m)}(k) Q_{ij}^{(m)}, \]

- Note that the curvature perturbation only involves scalars
  \[ \delta \left[ {^{(3)} R} \right] = \frac{4}{a^2} \left( k^2 - 3K \right) \left( H_L^{(0)} + \frac{1}{3} H_T^{(0)} \right) Q^{(0)} \]
Spatially Flat Case

- For a spatially flat background metric, harmonics are related to plane waves:

\[ Q^{(0)} = \exp(i k \cdot x) \]

\[ Q_{i}^{(\pm 1)} = \frac{-i}{\sqrt{2}}(\hat{e}_1 \pm i \hat{e}_2)_i \exp(i k \cdot x) \]

\[ Q_{ij}^{(\pm 2)} = -\sqrt{\frac{3}{8}}(\hat{e}_1 \pm i \hat{e}_2)_i (\hat{e}_1 \pm i \hat{e}_2)_j \exp(i k \cdot x) \]

where \( \hat{e}_3 \parallel k \). Chosen as spin states, c.f. polarization.

- For vectors, the harmonic points in a direction orthogonal to \( k \) suitable for the **vortical component** of a vector
Spatially Flat Case

- Tensor harmonics are the transverse traceless gauge representation
- Tensor amplitude related to the more traditional

\[ h_+[(e_1)_i(e_1)_j - (e_2)_i(e_2)_j], \quad h_x[(e_1)_i(e_2)_j + (e_2)_i(e_1)_j] \]

as

\[ h_+ \pm ih_x = -\sqrt{6}H_T^{(\mp 2)} \]

- \( H_T^{(\pm 2)} \) proportional to the right and left circularly polarized amplitudes of gravitational waves with a normalization that is convenient to match the scalar and vector definitions
Covariant Scalar Equations

- DOF counting exercise

8 Variables (4 metric; 4 matter)
-4 Einstein equations
-2 Conservation equations
+2 Bianchi identities
-2 Gauge (coordinate choice 1 time, 1 space)

2 Free Variables

without loss of generality choose scalar components of the stress tensor $\delta p, \Pi$. 
Covariant Scalar Equations

- **Einstein equations** (suppressing 0) superscripts

\[
(k^2 - 3K)[H_L + \frac{1}{3}H_T] - 3(\frac{\dot{a}}{a})^2 A + 3\frac{\dot{a}}{a} \dot{H}_L + \frac{\dot{a}}{a} k B = 4\pi G a^2 \delta \rho, \quad 00 \text{ Poisson Equation}
\]

\[
\frac{\dot{a}}{a} A - \dot{H}_L - \frac{1}{3} \dot{H}_T - \frac{K}{k^2} (kB - \dot{H}_T)
= 4\pi G a^2 (\rho + p)(v - B)/k, \quad 0i \text{ Momentum Equation}
\]

\[
\left[2\frac{\ddot{a}}{a} - 2 \left(\frac{\dot{a}}{a}\right)^2 + \frac{\dot{a}}{a} \frac{d}{d\eta} - \frac{k^2}{3}\right] A - \left[\frac{d}{d\eta} + \frac{\dot{a}}{a}\right] (\dot{H}_L + \frac{1}{3}kB) = 4\pi G a^2 (\delta p + \frac{1}{3} \delta \rho), \quad ii \text{ Acceleration Equation}
\]

\[
k^2(A + H_L + \frac{1}{3}H_T) + \left(\frac{d}{d\eta} + 2\frac{\dot{a}}{a}\right) (kB - \dot{H}_T)
= -8\pi G a^2 p\Pi, \quad ij \text{ Anisotropy Equation}
\]
Covariant Scalar Equations

- Poisson and acceleration equations are the perturbed generalization of the Friedmann equations
- Momentum and anisotropy equations are new to the perturbed metric
- Poisson and momentum equations in the ADM language take the form of constraints on the shift and lapse respectively - leaving the spatial metric components as dynamical
- Like the Friedmann equations, the 4 equation are redundant given the 2 energy-momentum conservation equations
- Choose a gauge and set of equations to simplify the given problem
Covariant Scalar Equations

- Conservation equations: continuity and Navier Stokes

\[
\left[ \frac{d}{d\eta} + \frac{3}{a} \dot{a} \right] \delta \rho + \frac{3}{a} \dot{a} \delta p = -(\rho + p)(kv + 3\dot{H}_L),
\]

\[
\left[ \frac{d}{d\eta} + \frac{4}{a} \dot{a} \right] \left[ (\rho + p) \frac{(v - B)}{k} \right] = \delta p - \frac{2}{3} (1 - 3 \frac{K}{k^2}) p \Pi + (\rho + p)A,
\]

- Equations are not independent since \( \nabla_\mu G^{\mu\nu} = 0 \) via the Bianchi identities.

- Related to the ability to choose a coordinate system or “gauge” to represent the perturbations.
Covariant Vector Equations

- Einstein equations

\[(1 - 2K/k^2)(kB^{(\pm 1)} - \dot{H}_T^{(\pm 1)})\]

\[= 16\pi Ga^2(\rho + p)(v^{(\pm 1)} - B^{(\pm 1)})/k,\]

\[
\left[\frac{d}{d\eta} + \frac{2\dot{a}}{a}\right] (kB^{(\pm 1)} - \dot{H}_T^{(\pm 1)})
\]

\[= -8\pi Ga^2p\Pi^{(\pm 1)}.
\]

- Conservation Equations

\[
\left[\frac{d}{d\eta} + 4\frac{\dot{a}}{a}\right] [(\rho + p)(v^{(\pm 1)} - B^{(\pm 1)})/k]
\]

\[= -\frac{1}{2}(1 - 2K/k^2)p\Pi^{(\pm 1)},\]

- Gravity provides no source to vorticity → decay
Covariant Vector Equations

- DOF counting exercise

8 Variables (4 metric; 4 matter)
-4 Einstein equations
-2 Conservation equations
+2 Bianchi identities
-2 Gauge (coordinate choice 1 time, 1 space)

2 Free Variables

without loss of generality choose vector components of the stress tensor $\Pi^{(\pm 1)}$. 
Covariant Tensor Equation

• Einstein equation

\[
\left[ \frac{d^2}{d\eta^2} + 2\frac{\dot{a}}{a} \frac{d}{d\eta} + (k^2 + 2K) \right] H^{(\pm 2)}_T = 8\pi Ga^2 p\Pi^{(\pm 2)}.
\]

• DOF counting exercise

4 Variables (2 metric; 2 matter)

-2 Einstein equations

-0 Conservation equations

+0 Bianchi identities

-0 Gauge (coordinate choice 1 time, 1 space)

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2 Free Variables

wlog choose tensor components of the stress tensor \( \Pi^{(\pm 2)} \).
Arbitrary Dark Components

- Total stress energy tensor can be broken up into individual pieces.
- Dark components interact only through gravity and so satisfy separate conservation equations.
- Einstein equation source remains the sum of components.
- To specify an arbitrary dark component, give the behavior of the stress tensor: 6 components: $\delta p$, $\Pi^{(i)}$, where $i = -2, ..., 2$.
- Many types of dark components (dark matter, scalar fields, massive neutrinos,..) have simple forms for their stress tensor in terms of the energy density, i.e. described by equations of state.
- An equation of state for the background $w = p/\rho$ is not sufficient to determine the behavior of the perturbations.
Geometry of Gauge Choice

- Geometry of the gauge or time slicing and spatial threading
- For perturbations larger than the horizon, a local observer should just see a different (separate) FRW universe
- Scalar equations should be equivalent to an appropriately remapped Friedmann equation
- ADM recap: unit normal vector $n^\mu$ to constant time hypersurfaces
  
  $n_\mu dx^\mu = n_0 d\eta$, $n^\mu n_\mu = -1$, to linear order in metric

  $n_0 = -a(1 + AQ)$, $n_i = 0$

  $n^0 = a^{-1}(1 - AQ)$, $n^i = -BQ^i$

- Intrinsic 3-geometry of $\delta g_{ij}$, changes in the normal vector $n_{\mu;\nu}$ that define the extrinsic curvature
Geometric Quantities

- Expansion of spatial volume per proper time is given by 4-divergence

\[ n_\mu^\mu \equiv \theta = 3H(1 - AQ) + \frac{k}{a}BQ + \frac{3}{a}\dot{H}_LQ \]

- Other pieces of \( n_{\mu;\nu} \) give the vorticity, shear and acceleration

\[ n_{\mu;\nu} \equiv \omega_{\mu\nu} + \sigma_{\mu\nu} + \frac{\theta}{3}h_{\mu\nu} - a_\mu n_\nu \]

\[ h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu \]
\[ \omega_{\mu\nu} = h_\mu^\alpha h_\nu^\beta (n_\alpha;\beta - n_\beta;\alpha) = 0 \]
\[ \sigma_{\mu\nu} = \frac{1}{2} h_\mu^\alpha h_\nu^\beta (n_\alpha;\beta + n_\beta;\alpha) - \frac{1}{3} \theta h_{\mu\nu} \]
\[ a_\mu = n_{\mu;\alpha} n^\alpha \]

- Recall \( n_\mu \) is a special timelike vector normal to the constant time surfaces, the vorticity vanishes by construction
Geometric Quantities

- Remaining perturbed quantities are the spatial shear and acceleration (0 components vanish)

\[ \sigma_{ij} = a(\dot{H}_T - kB)Q_{ij} \]
\[ a_i = -kA Q_i \]

- Recall that the extrinsic curvature \( K_{ij} = \sigma_{ij} + \theta h_{\mu\nu}/3 \)

- Intrinsic curvature of the 3-surface determined by 3-metric \( h_{ij} \)

\[ \delta^{(3)} R = \frac{4}{a^2} (k^2 - 3K)(H_L + \frac{H_T}{3}) \]

- E-foldings of the local expansion \( \ln a_L \) are given

\[ \ln a_L = \int d\tau \frac{1}{3} \theta = \int d\eta \left( \frac{\dot{a}}{a} + \dot{H}_L Q + \frac{1}{3} kB Q \right) \]

where we have used \( d\tau = (1 + AQ) ad\eta \)
Separate Universe

- Notice that

\[
\frac{d}{d\eta} \delta \ln a_L = \dot{H}_L + \frac{\dot{H}_T}{3} - \frac{1}{3}(\dot{H}_T - kB)
\]

so that if the shear is negligible the change in efolds tracks the change in curvature

- Shear vanishes in the FRW background; uniform efolding gives constant curvature

- Underlying principle: local observer should find long wavelength perturbations are indistinguishable from a change in the background FRW quantities

- Perturbation equations take the form of Friedmann equations once rescaled
Time Slicing

- Constant time surfaces can be defined according to what geometry is helpful for the problem at hand

- Common choices:
  - Uniform eefolding: $\dot{H}_L + kB/3 = 0$
  - Shear free: $\dot{H}_T - kB = 0$
  - Zero lapse pert or acceleration, $A = 0$
  - Uniform expansion: $-3HA + (3\dot{H}_L + kB) = 0$
  - Comoving: $v = B$

- For the background all of these conditions hold.

- For perturbations each define a choice of slicing

- Can define the validity of the separate universe principle as the coexistence of comoving and zero lapse slicing
Time Slicing

- Comoving slicing is more formally called velocity orthogonal slicing since constant time surfaces are orthogonal to the matter 4-velocity $V^\mu$:

\[ h^\mu_\nu V^\nu = (\delta^\mu_\nu + n^\mu n_\nu) V^\nu = (0, V^i + N^i V^0) = 0 \]

\[ \rightarrow V^i = vQ^i = B^i = BQ^i \]

- Should not be confused with comoving (threading) where the 3-velocity $v = 0$ unless the shift $B$ also vanishes
Gauge

- Metric and matter fluctuations take on different values in different coordinate system.
- No such thing as a “gauge invariant” density perturbation!
- General coordinate transformation:

\[
\tilde{\eta} = \eta + T \\
\tilde{x}^i = x^i + L^i
\]

free to choose \((T, L^i)\) to simplify equations or physics – corresponds to a choice of slicing and threading in ADM.

- Decompose these into scalar \(T\), \(L^{(0)}\) and vector harmonics \(L^{(\pm 1)}\).
Gauge

- $g_{\mu\nu}$ and $T_{\mu\nu}$ transform as tensors, so components in different frames can be related

$$\tilde{g}_{\mu\nu}(\tilde{\eta}, \tilde{x}^i) = \frac{\partial x^\alpha}{\partial \tilde{x}^\mu} \frac{\partial x^\beta}{\partial \tilde{x}^\nu} g_{\alpha\beta}(\eta, x^i)$$

$$= \frac{\partial x^\alpha}{\partial \tilde{x}^\mu} \frac{\partial x^\beta}{\partial \tilde{x}^\nu} g_{\alpha\beta}(\tilde{\eta} - TQ, \tilde{x}^i - LQ^i)$$

- Fluctuations are compared at the same coordinate positions (not same space time positions) between the two gauges
- For example with a $TQ$ perturbation, an event labeled with $\tilde{\eta} =$const. and $\tilde{x} =$const. represents a different time with respect to the underlying homogeneous and isotropic background
Gauge Transformation

- Scalar Metric:

\[
\begin{align*}
\tilde{A} &= A - \dot{T} - \frac{\dot{a}}{a} T, \\
\tilde{B} &= B + \dot{L} + kT, \\
\tilde{H}_L &= H_L - \frac{k}{3} L - \frac{\dot{a}}{a} T, \\
\tilde{H}_T &= H_T + kL, \\
\tilde{H}_L + \frac{1}{3} \tilde{H}_T &= H_L + \frac{1}{3} H_T - \frac{\dot{a}}{a} T
\end{align*}
\]

curvature perturbation depends on slicing not threading

- Scalar Matter (\(J\)th component):

\[
\begin{align*}
\delta \tilde{\rho}_J &= \delta \rho_J - \rho_J \dot{T}, \\
\delta \tilde{p}_J &= \delta p_J - \rho_J \dot{T}, \\
\tilde{v}_J &= v_J + \dot{L},
\end{align*}
\]

density and pressure likewise depend on slicing only
Gauge Transformation

- Vector:

\[
\tilde{B}^{(\pm 1)} = B^{(\pm 1)} + \dot{L}^{(\pm 1)}, \\
\tilde{H}_T^{(\pm 1)} = H_T^{(\pm 1)} + kL^{(\pm 1)}, \\
\tilde{v}_J^{(\pm 1)} = v_J^{(\pm 1)} + \dot{L}^{(\pm 1)},
\]

- Spatial vector has no background component hence no dependence on slicing at first order

Tensor: no dependence on slicing or threading at first order

- Gauge transformations and covariant representation can be extended to higher orders

- A coordinate system is fully specified if there is an explicit prescription for \((T, L^i)\) or for scalars \((T, L)\)
Slicing

Common choices for slicing $T$: set something geometric to zero

- Proper time slicing $A = 0$: proper time between slices corresponds to coordinate time $- T$ allows $c/a$ freedom
- Comoving (velocity orthogonal) slicing: $v - B = 0$, slicing is orthogonal to matter 4 velocity $- T$ fixed
- Newtonian (shear free) slicing: $\dot{H}_T - kB = 0$, expansion rate is isotropic, shear free, $T$ fixed
- Uniform expansion slicing: $-(\dot{a}/a)A + \dot{H}_L + kB/3 = 0$, perturbation to the volume expansion rate $\theta$ vanishes, $T$ fixed
- Flat (constant curvature) slicing, $\delta^{(3)}R = 0$, $(H_L + H_T/3 = 0)$, $T$ fixed
- Constant density slicing, $\delta\rho_I = 0$, $T$ fixed
Threading

- Threading specifies the relationship between constant spatial coordinates between slices and is determined by $L$.

  Typically involves a condition on $v$, $B$, $H_T$

  - Orthogonal threading $B = 0$, constant spatial coordinates orthogonal to slicing (zero shift), allows $\delta L = c$ translational freedom.

  - Comoving threading $v = 0$, allows $\delta L = c$ translational freedom.

  - Isotropic threading $H_T = 0$, fully fixes $L$.
Newtonian (Longitudinal) Gauge

- Newtonian (shear free slicing, isotropic threading):

\[
\begin{align*}
\tilde{B} &= \tilde{H}_T = 0 \\
\Psi &\equiv \tilde{A} \quad \text{(Newtonian potential)} \\
\Phi &\equiv \tilde{H}_L \quad \text{(Newtonian curvature)} \\
L &= -\frac{H_T}{k} \\
T &= -\frac{B}{k} + \frac{\dot{H}_T}{k^2}
\end{align*}
\]

**Good:** intuitive Newtonian like gravity; matter and metric algebraically related; commonly chosen for analytic CMB and lensing work

**Bad:** numerically unstable
Newtonian (Longitudinal) Gauge

- Newtonian (shear free) slicing, isotropic threading $B = H_T = 0$:

\[
(k^2 - 3K) \Phi = 4\pi G a^2 \left[ \delta \rho + 3 \frac{\dot{a}}{a} (\rho + p)v/k \right] \quad \text{Poisson + Momentum}
\]

\[
k^2(\Psi + \Phi) = -8\pi G a^2 p \Pi \quad \text{Anisotropy}
\]

so $\Psi = -\Phi$ if anisotropic stress $\Pi = 0$ and

\[
\left[ \frac{d}{d\eta} + 3\frac{\dot{a}}{a} \right] \delta \rho + 3\frac{\dot{a}}{a} \delta p = - (\rho + p)(kv + 3\dot{\Phi}),
\]

\[
\left[ \frac{d}{d\eta} + 4\frac{\dot{a}}{a} \right] (\rho + p)v = k\delta p - \frac{2}{3} (1 - 3\frac{K}{k^2}) p k\Pi + (\rho + p) k\Psi,
\]

- Newtonian competition between stress (pressure and viscosity) and potential gradients

- Note: Poisson source is the density perturbation on comoving slicing
Comoving Gauge

- Comoving gauge (comoving slicing, isotropic threading)

\[ \tilde{B} = \tilde{v} \quad (T^0_i = 0) \]
\[ H_T = 0 \]
\[ \xi = \tilde{A} \]
\[ \mathcal{R} = \tilde{H}_L \quad \text{(comoving curvature)} \]
\[ \Delta = \tilde{\delta} \quad \text{(total density pert)} \]
\[ T = (v - B)/k \]
\[ L = -H_T/k \]

**Good:** Algebraic relations between matter and metric; comoving curvature perturbation obeys conservation law

**Bad:** Non-intuitive threading involving \( v \)
Comoving Gauge

- Euler equation becomes an algebraic relation between stress and potential

\[(\rho + p)\dot{\xi} = -\delta p + \frac{2}{3} \left(1 - \frac{3K}{k^2}\right) p\Pi\]

- Einstein equation lacks momentum density source

\[
\frac{\dot{a}}{a} \dot{\xi} - \dot{\mathcal{R}} - \frac{K}{k^2} k\nu = 0
\]

Combine: \(\mathcal{R}\) is conserved if stress fluctuations negligible, e.g. above the horizon if \(|K| \ll H^2\)

\[
\dot{\mathcal{R}} + K\nu/k = \frac{\dot{a}}{a} \left[ -\frac{\delta p}{\rho + p} + \frac{2}{3} \left(1 - \frac{3K}{k^2}\right) \frac{p}{\rho + p} \Pi \right] \rightarrow 0
\]
“Gauge Invariant” Approach

- Gauge transformation rules allow variables which take on a geometric meaning in one choice of slicing and threading to be accessed from variables on another choice.

- Functional form of the relationship between the variables is gauge invariant (not the variable values themselves! – i.e. equation is covariant)

- E.g. comoving curvature and density perturbations

\[
\mathcal{R} = H_L + \frac{1}{3} H_T - \frac{\dot{a}}{a} (v - B)/k
\]

\[
\Delta \rho = \delta \rho + 3(\rho + p) \frac{\dot{a}}{a} (v - B)/k
\]
Newtonian-Comoving Hybrid

- With the gauge in *(or co)*variant approach, express variables of one gauge in terms of those in another – allows a mixture in the equations of motion

- **Example:** Newtonian curvature and comoving density

\[(k^2 - 3K)\Phi = 4\pi Ga^2 \rho \Delta\]

ordinary Poisson equation then implies \(\Phi\) approximately constant if stresses negligible.

- **Example:** Exact Newtonian curvature above the horizon derived through comoving curvature conservation

Gauge transformation

\[\Phi = \mathcal{R} + \frac{\dot{a} \, v}{a \, k}\]
Hybrid “Gauge Invariant” Approach

Einstein equation to eliminate velocity

\[ \frac{\dot{a}}{a} \Psi - \dot{\Phi} = 4\pi G a^2 (\rho + p)v/k \]

Friedmann equation with no spatial curvature

\[ \left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} a^2 \rho \]

With \( \dot{\Phi} = 0 \) and \( \Psi \approx -\Phi \)

\[ \frac{\ddot{a}}{a} \frac{v}{k} = -\frac{2}{3(1 + w)} \Phi \]
Newtonian-Comoving Hybrid

Combining gauge transformation with velocity relation

\[ \Phi = \frac{3 + 3w}{5 + 3w} R \]

Usage: calculate \( R \) from inflation determines \( \Phi \) for any choice of matter content or causal evolution.

- **Example:** Scalar field (“quintessence” dark energy) equations in comoving gauge imply a sound speed \( \delta p/\delta \rho = 1 \) independent of potential \( V(\phi) \). Solve in synchronous gauge.
Synchronous Gauge

- Synchronous: (proper time slicing, orthogonal threading)

\[
\begin{align*}
\tilde{A} &= \tilde{B} = 0 \\
\eta_T &\equiv -\tilde{H}_L - \frac{1}{3}\tilde{H}_T \\
h_L &\equiv 6H_L \\
T &= a^{-1}\int d\eta A + c_1 a^{-1} \\
L &= -\int d\eta (B + kT) + c_2
\end{align*}
\]

**Good:** stable, the choice of numerical codes and separate universe constructs

**Bad:** residual gauge freedom in constants \(c_1, c_2\) must be specified as an initial condition, intrinsically relativistic, threading conditions breaks down beyond linear regime if \(c_1\) is fixed to CDM comoving.
Synchronous Gauge

- The Einstein equations give

\[ \dot{\eta}_T - \frac{K}{2k^2}(\dot{h}_L + 6\dot{\eta}_T) = 4\pi G a^2 (\rho + p) \frac{v}{k}, \]

\[ \ddot{h}_L + \frac{\dot{a}}{a} \dot{h}_L = -8\pi G a^2 (\delta \rho + 3\delta p), \]

\[- (k^2 - 3K) \dot{h}_T + \frac{1}{2} \frac{\dot{a}}{a} \dot{h}_L = 4\pi G a^2 \delta \rho \]

[choose (1 & 2) or (1 & 3)] while the conservation equations give

\[ \left[ \frac{d}{d\eta} + 3 \frac{\dot{a}}{a} \right] \delta \rho_J + 3 \frac{\dot{a}}{a} \delta p_J = -(\rho_J + p_J)(kv_J + \frac{1}{2} \dot{h}_L), \]

\[ \left[ \frac{d}{d\eta} + 4 \frac{\dot{a}}{a} \right] (\rho_J + p_J) \frac{v_J}{k} = \delta p_J - \frac{2}{3} (1 - 3 \frac{K}{k^2}) p_J \Pi_J. \]
Synchronous Gauge

- Lack of a lapse $A$ implies no gravitational forces in Navier-Stokes equation. Hence for stress free matter like cold dark matter, zero velocity initially implies zero velocity always.

- Choosing the momentum and acceleration Einstein equations is good since for CDM domination, curvature $\eta_T$ is conserved and $\dot{h}_L$ is simple to solve for.

- Choosing the momentum and Poisson equations is good when the equation of state of the matter is complicated since $\delta p$ is not involved. This is the choice of CAMB.

Caution: since the curvature $\eta_T$ appears and it has zero CDM source, subtle effects like dark energy perturbations are important everywhere
Spatially Flat Gauge

- Spatially Flat (flat slicing, isotropic threading):

  \[ \tilde{H}_L = \tilde{H}_T = 0 \]
  \[ L = \frac{-H_T}{k} \]
  \[ \tilde{A}, \tilde{B} = \text{metric perturbations} \]
  \[ T = \left( \frac{\dot{a}}{a} \right)^{-1} \left( H_L + \frac{1}{3} H_T \right) \]

  **Good:** eliminates spatial metric evolution in ADM and perturbation equations; useful in inflationary calculations (Mukhanov et al)

  **Bad:** non-intuitive slicing (no curvature!) and threading

- **Caution:** perturbation evolution is governed by the behavior of stress fluctuations and an isotropic stress fluctuation \( \delta p \) is gauge dependent.
Uniform Density Gauge

- Uniform density: (constant density slicing, isotropic threading)

\[ H_T = 0, \]
\[ \zeta_I \equiv H_L \]
\[ B_I \equiv B \]
\[ A_I \equiv A \]
\[ T = \frac{\delta \rho_I}{\dot{\rho}_I} \]
\[ L = -H_T/k \]

**Good:** Curvature conserved involves only stress energy conservation; simplifies isocurvature treatment

**Bad:** non intuitive slicing (no density pert! problems beyond linear regime) and threading
Uniform Density Gauge

- Einstein equations with $I$ as the total or dominant species

\[
(k^2 - 3K)\zeta_I - 3 \left(\frac{\dot{a}}{a}\right)^2 A_I + 3\frac{\dot{a}}{a} \dot{\zeta}_I + \frac{\dot{a}}{a}k B_I = 0 ,
\]
\[
\frac{\dot{a}}{a} A_I - \dot{\zeta}_I - \frac{K}{k} B_I = 4\pi Ga^2(\rho + p)\frac{v - B_I}{k} ,
\]

- The conservation equations (if $J = I$ then $\delta \rho_J = 0$)

\[
\frac{d}{d\eta} + 3\frac{\dot{a}}{a} \left[ \delta \rho_J + 3\frac{\dot{a}}{a} \delta p_J \right] = -(\rho_J + p_J)(k v_J + 3\dot{\zeta}_I) ,
\]
\[
\frac{d}{d\eta} + 4\frac{\dot{a}}{a} (\rho_J + p_J)\frac{v_J - B_I}{k} = \delta p_J - \frac{2}{3}(1 - 3\frac{K}{k^2})p_J \Pi_J + (\rho_J + p_J)A_I .
\]
Uniform Density Gauge

- Conservation of curvature - single component $I$

$$\dot{\zeta}_I = -\frac{\dot{a}}{a} \frac{\delta p_I}{\rho_I + p_I} - \frac{1}{3} k v_I.$$

- Since $\delta \rho_I = 0$, $\delta p_I$ is the non-adiabatic stress and curvature is constant as $k \to 0$ for adiabatic fluctuations $p_I(\rho_I)$.

- Note that this conservation law does not involve the Einstein equations at all: just local energy momentum conservation so it is valid for alternate theories of gravity.

- Curvature on comoving slices $\mathcal{R}$ and $\zeta_I$ related by

$$\zeta_I = \mathcal{R} + \frac{1}{3} \frac{\rho \Delta I}{(\rho_I + p_I)} \bigg|_{\text{comoving}}.$$

and coincide above the horizon for adiabatic fluctuations.
Uniform Density Gauge

- Simple relationship to density fluctuations in the spatially flat gauge

\[ \zeta_I = \frac{1}{3} \frac{\delta \tilde{\rho}_I}{(\rho_I + p_I)} \bigg|_{\text{flat}}. \]

- For each particle species \( \delta \rho / (\rho + p) = \delta n / n \), the number density fluctuation

- Multiple \( \zeta_J \) carry information about number density fluctuations between species

- \( \zeta_J \) constant component by component outside horizon if each component is adiabatic \( p_J(\rho_J) \).
Unitary Gauge

- Given a scalar field $\phi(x^i, \eta)$, choose a slicing so that the field is spatially uniform $\phi(x^i, \eta) = \phi(\eta)$ via the transformation

$$\delta \tilde{\phi} = \delta \phi - \dot{\phi} T \quad \rightarrow \quad T = \frac{\delta \phi}{\dot{\phi}_0}$$

- Specify threading, e.g. isotropic threading $L = - H T / k$
  
  **Good:** Scalar field carried completely by the metric; EFT of inflation and scalar-tensor theories of gravity. Extensible to nonlinear perturbations as long as $\partial_\mu \phi$ remains timelike
  
  **Bad:** Preferred slicing retains only the spatial diffeomorphism invariance; can make full covariance and DOF counting obscure

- For a canonical scalar field, unitary and comoving gauge coincide
EFT of Dark Energy and Inflation

• Beyond linear theory, unitary gauge and ADM is useful to define most general Lagrangian and interaction terms for a scalar-tensor theory of gravity: so-called Effective Field Theory (EFT)

• Rule: broken temporal diffeomorphisms (preferred slicing) but spatial diffeomorphism invariance means explicit functions of unitary time and ADM spatial scalars allowed

• Typically also want second order in time derivatives to avoid Ostrogradsky ghost, lapse and shift non-dynamical

\[ \mathcal{L}(N, K^i_j, R^i_j, \nabla^i; t) \]

where the function is constructed out of spatial scalars (3D Riemann tensor can be expressed through 3 Ricci tensor and metric

• Recall that the extrinsic curvature carries a first time derivative of the spatial metric and spatial gradients of the shift
EFT of Dark Energy and Inflation

- This class includes quintessence, $k$-essence, $f(R)$, Horndeski, “beyond Horndeski”, Horava-Lifshitz gravity, ghost condensate

- Does not include theories where derivatives of the lapse $N$ appear but the shift is still nondynamical due to hidden constraints - can be generalized

- GR: time diffeomorphism not broken so the Einstein-Hilbert Lagrangian in EFT language is given by the Gauss-Codazzi relation

\[
(4) R \rightarrow K_{\mu\nu}K^{\mu\nu} - K^2 + (3) R
\]
EFT of Dark Energy and Inflation

• Now consider the scalar field to pick out a particular foliation

• Simplest example $k$-essence where $\mathcal{L}(X, \phi)$ where

  $X = g^{\mu \nu} \partial_\mu \phi \partial_\nu \phi$ (sometimes $-1/2 \times$ to resemble kinetic energy)

• In unitary gauge $\phi$ is a function of the temporal coordinate only so

  $X = -\dot{\phi}^2 / N^2$ so that

  $$\mathcal{L}(X, \phi) \rightarrow \mathcal{L}(N, t)$$

• We will return to this case when considering inflationary non-Gaussianity: note that $g^{00} = -1/N^2$ so the EFT literature sometimes writes $\mathcal{L}(g^{00}, \ldots, t)$ (Cheung et al 2008)

• Unifying description for “building blocks” of dark energy (Gleyzes, Langois, Vernizzi 2015)
Vector Gauges

- Vector gauge depends only on threading $L$
- Poisson gauge: orthogonal threading $B^{(\pm 1)} = 0$, leaves constant $L$ translational freedom
- Isotropic gauge: isotropic threading $H_{T}^{(\pm 1)} = 0$, fixes $L$
- To first order scalar and vector gauge conditions can be chosen separately
- More care required for second and higher order where scalars and vectors mix