

Astro 4PT

Lecture Notes Set 2

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# References

- General References

**Basics:** Kolb & Turner, Early Universe

**Structure Formation:** Liddle & Lyth, The Primordial Density Perturbation

- Inflationary Perturbations

**Covariant Evolution:** Mukhanov, Feldman, Brandenberger (1992), Phys. Rep., 215, 203

**Particle Physics Models:** Lyth, Riotto (1999), Phys. Rep., 314, 1

**String Inspired Models:** Burgess arXiv:0708.2865

**Nongaussianity:** Bartolo, Komatsu, Matarrese, Riotto (2004), Phys. Rep., 402, 103

**Isocurvature & Nongaussianity:** Malik & Wands (2009), Phys. Rep., 475, 1

# Scalar Fields

- Stress-energy tensor of a scalar field

$$T^{\mu}_{\nu} = \nabla^{\mu}\varphi \nabla_{\nu}\varphi - \frac{1}{2}(\nabla^{\alpha}\varphi \nabla_{\alpha}\varphi + 2V)\delta^{\mu}_{\nu}.$$

- For the background  $\langle\phi\rangle \equiv \phi_0$  ( $a^{-2}$  from conformal time)

$$\rho_{\phi} = \frac{1}{2}a^{-2}\dot{\phi}_0^2 + V, \quad p_{\phi} = \frac{1}{2}a^{-2}\dot{\phi}_0^2 - V$$

- So for kinetic dominated  $w_{\phi} = p_{\phi}/\rho_{\phi} \rightarrow 1$
- And potential dominated  $w_{\phi} = p_{\phi}/\rho_{\phi} \rightarrow -1$
- A slowly rolling (potential dominated) scalar field can accelerate the expansion and so solve the horizon problem or act as a dark energy candidate

# Equation of Motion

- Can use general equations of motion dictated by stress energy conservation

$$\dot{\rho}_\phi = -3(\rho_\phi + p_\phi)\frac{\dot{a}}{a},$$

to obtain the equation of motion of the background field  $\phi$

$$\ddot{\phi}_0 + 2\frac{\dot{a}}{a}\dot{\phi}_0 + a^2V' = 0,$$

- In terms of time instead of conformal time

$$\frac{d^2\phi_0}{dt^2} + 3H\frac{d\phi_0}{dt} + V' = 0$$

- Field rolls down potential hill but experiences “Hubble friction” to create slow roll. In slow roll  $3Hd\phi_0/dt \approx -V'$  and so kinetic energy is determined by field position  $\rightarrow$  adiabatic – both kinetic and potential energy determined by single degree of freedom  $\phi_0$

# Equation of Motion

- Likewise for the perturbations  $\phi = \phi_0 + \phi_1$

$$\delta\rho_\phi = a^{-2}(\dot{\phi}_0\dot{\phi}_1 - \dot{\phi}_0^2 A) + V'\phi_1,$$

$$\delta p_\phi = a^{-2}(\dot{\phi}_0\dot{\phi}_1 - \dot{\phi}_0^2 A) - V'\phi_1,$$

$$(\rho_\phi + p_\phi)(v_\phi - B) = a^{-2}k\dot{\phi}_0\phi_1,$$

$$p_\phi\pi_\phi = 0,$$

- For comoving slicing where  $v_\phi = B$ ,  $\delta p_\phi = \delta\rho_\phi$  so the sound speed is  $\delta p_\phi/\delta\rho_\phi = 1$ . Very useful for solving system since in this gauge everything is specified by  $w(a)$
- More generally, the sound speed in this slicing depends on the form of the kinetic term not the potential – k-essence, DBI inflation

# Equation of Motion

- Scalar field fluctuations are stable inside the horizon and are a good candidate for the smooth dark energy
- More generally, continuity and Euler equations imply

$$\ddot{\phi}_1 = -2\frac{\dot{a}}{a}\dot{\phi}_1 - (k^2 + a^2V'')\phi_1 + (\dot{A} - 3\dot{H}_L - kB)\dot{\phi}_0 - 2Aa^2V'.$$

- Alternately this follows from perturbing the Klein Gordon equation  $\nabla_\mu \nabla^\mu \phi = V'$

# Inflationary Perturbations

- Classical equations of motion for scalar field inflaton determine the evolution of scalar field fluctuations generated by quantum fluctuations
- Since the curvature  $\mathcal{R}$  on comoving slicing is conserved in the absence of stress fluctuations (i.e. outside the apparent horizon, calculate this and we're done no matter what happens in between inflation and the late universe (reheating etc.)
- But in the comoving slicing  $\phi_1 = 0!$  no scalar-field perturbation
- Solution: solve the scalar field equation in the dual gauge where the curvature  $H_L + H_T/3 = 0$  (“spatially flat” slicing) and transform the result to comoving slicing

# Transformation to Comoving Slicing

- Scalar field transforms as scalar field

$$\tilde{\phi}_1 = \phi_1 - \dot{\phi}_0 T$$

- To get to comoving slicing  $\tilde{\phi}_1 = 0$ ,  $T = \phi_1 / \dot{\phi}_0$ , and  $\tilde{H}_T = H_T + kL$  so

$$\mathcal{R} = H_L + \frac{H_T}{3} - \frac{\dot{a}}{a} \frac{\phi_1}{\dot{\phi}_0}$$

- Transformation particularly simple from a spatially flat slicing where  $H_L + H_T/3 = 0$ , i.e. spatially unperturbed metric

$$\mathcal{R} = -\frac{\dot{a}}{a} \frac{\phi_1}{\dot{\phi}_0}$$

# Spatially Flat Gauge

- Spatially Flat (flat slicing, isotropic threading):

$$\begin{aligned}\tilde{H}_L + \tilde{H}_T/3 &= \tilde{H}_T = 0 \\ A_f &= \tilde{A}, B_f = \tilde{B} \\ T &= \left(\frac{\dot{a}}{a}\right)^{-1} \left(H_L + \frac{1}{3}H_T\right) \\ L &= -H_T/k\end{aligned}$$

- Einstein Poisson and Momentum

$$\begin{aligned}-3\left(\frac{\dot{a}}{a}\right)^2 A_f + \frac{\dot{a}}{a}k B_f &= 4\pi G a^2 \delta\rho_\phi, \\ \frac{\dot{a}}{a}A_f - \frac{K}{k^2}(k B_f) &= 4\pi G a^2 (\rho_\phi + p_\phi)(v_\phi - B_f)/k,\end{aligned}$$

- Conservation

$$\ddot{\phi}_1 = -2\frac{\dot{a}}{a}\dot{\phi}_1 - (k^2 + a^2 V'')\phi_1 + (\dot{A}_f - k B_f)\dot{\phi}_0 - 2A_f a^2 V'.$$

# Spatially Flat Gauge

- For modes where  $|k^2/K| \gg 1$  we obtain

$$\frac{\dot{a}}{a} A_f = 4\pi G a^2 \dot{\phi}_0 \phi_1 ,$$

$$\frac{\dot{a}}{a} k B_f = 4\pi G [\dot{\phi}_0 \dot{\phi}_1 - \dot{\phi}_0^2 A_f + a^2 V' \phi_1 + 3 \frac{\dot{a}}{a} \dot{\phi}_0 \phi_1]$$

so combining  $\dot{A}_f - k B_f$  eliminates the  $\dot{\phi}_1$  term

- The metric source to the scalar field equation can be reexpressed in terms of the field perturbation and background quantities

$$(\dot{A}_f - k B_f) \dot{\phi}_0 - 2 A_f a^2 V' - a^2 V'' \phi_1 = f(\eta) \phi_1$$

- Single closed form 2nd order ODE for  $\phi_1$

# Mukhanov Equation

- Equation resembles a damped oscillator equation with a particular dispersion relation

$$\ddot{\phi}_1 + 2\frac{\dot{a}}{a}\dot{\phi}_1 + [k^2 + f(\eta)]\phi_1$$

- $f(\eta)$  involves terms with  $\dot{\phi}_0$ ,  $V'$ ,  $V''$  implying that for a sufficiently flat potential  $f(\eta)$  represents a small correction
- Transform out the background expansion  $u \equiv a\phi_1$

$$\dot{u} = \dot{a}\phi + a\dot{\phi}$$

$$\ddot{u} = \ddot{a}\phi_1 + 2\dot{a}\dot{\phi}_1 + a\ddot{\phi}_1$$

$$\ddot{u} + [k^2 - \frac{\ddot{a}}{a} + f(\eta)]u = 0$$

- Note Friedmann equations say if  $p = -\rho$ ,  $\ddot{a}/a = 2(\dot{a}/a)^2$

# Mukhanov Equation

- Using the background Einstein and scalar field equations, this source term can be expressed in a surprisingly compact form

$$\ddot{u} + \left[ k^2 - \frac{\ddot{z}}{z} \right] u = 0$$

- and

$$z \equiv \frac{a\dot{\phi}_0}{\dot{a}/a}$$

- This equation is sometimes called the “Mukhanov Equation” and is both exact (in linear theory) and compact
- For large  $k$  (subhorizon), this looks like a free oscillator equation which can be quantized
- Let’s examine the relationship between  $z$  and the slow roll parameters

# Slow Roll Parameters

- Rewrite equations of motion in terms of slow roll parameters but do not require them to be small or constant.
- Deviation from de Sitter expansion

$$\begin{aligned}\epsilon &\equiv \frac{3}{2}(1 + w_\phi) \\ &= \frac{\frac{3}{2}(d\phi_0/dt)^2/V}{1 + \frac{1}{2}(d\phi_0/dt)^2/V}\end{aligned}$$

- Deviation from overdamped limit of  $d^2\phi_0/dt^2 = 0$

$$\begin{aligned}\delta &\equiv \frac{d^2\phi_0/dt^2}{H d\phi_0/dt} \\ &= \frac{\ddot{\phi}_0}{\dot{\phi}_0} \left(\frac{\dot{a}}{a}\right)^{-1} - 1\end{aligned}$$

# Slow Roll Parameters

- Friedmann equations:

$$\left(\frac{\dot{a}}{a}\right)^2 = 4\pi G \dot{\phi}_0^2 \epsilon^{-1}$$

$$\frac{d}{d\eta} \left(\frac{\dot{a}}{a}\right) = \frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 = \left(\frac{\dot{a}}{a}\right)^2 (1 - \epsilon)$$

Take derivative of first equation, divide through by  $(\dot{a}/a)^2$

$$2\frac{\dot{a}}{a}(1 - \epsilon) = 2\frac{\ddot{\phi}_0}{\dot{\phi}_0} - \frac{\dot{\epsilon}}{\epsilon}$$

- Replace  $\ddot{\phi}_0$  with  $\delta$

$$\dot{\epsilon} = 2\epsilon(\delta + \epsilon)\frac{\dot{a}}{a}$$

- Evolution of  $\epsilon$  is second order in parameters

# Slow Roll parameters

- Returning to the Mukhanov equation

$$\ddot{u} + [k^2 + g(\eta)]u = 0$$

where

$$\begin{aligned} g(\eta) &\equiv f(\eta) + \epsilon - 2 \\ &= - \left( \frac{\dot{a}}{a} \right)^2 [2 + 3\delta + 2\epsilon + (\delta + \epsilon)(\delta + 2\epsilon)] - \frac{\dot{a}}{a} \dot{\delta} \\ &= - \frac{\ddot{z}}{z} \end{aligned}$$

and recall

$$z \equiv a \left( \frac{\dot{a}}{a} \right)^{-1} \dot{\phi}_0$$

# Slow Roll Limit

- Slow roll  $\epsilon \ll 1$ ,  $\delta \ll 1$ ,  $\dot{\delta} \ll \frac{\dot{a}}{a}$

$$\ddot{u} + \left[ k^2 - 2 \left( \frac{\dot{a}}{a} \right)^2 \right] u = 0$$

- or for conformal time measured from the end of inflation

$$\tilde{\eta} = \eta - \eta_{\text{end}}$$

$$\tilde{\eta} = \int_{a_{\text{end}}}^a \frac{da}{Ha^2} \approx -\frac{1}{aH}$$

- Compact, slow-roll equation:

$$\ddot{u} + \left[ k^2 - \frac{2}{\tilde{\eta}^2} \right] u = 0$$

# Slow Roll Limit

- Slow roll equation has the exact solution:

$$u = A\left(k \mp \frac{i}{\tilde{\eta}}\right)e^{\mp ik\tilde{\eta}}$$

- For  $|k\tilde{\eta}| \gg 1$  (early times, inside Hubble length) behaves as free oscillator

$$\lim_{|k\tilde{\eta}| \rightarrow \infty} u = Ake^{\mp ik\tilde{\eta}}$$

- Normalization  $A$  will be set by origin in quantum fluctuations of free field

# Slow Roll Limit

- For  $|k\tilde{\eta}| \ll 1$  (late times,  $\gg$  Hubble length) fluctuation freezes in

$$\lim_{|k\tilde{\eta}| \rightarrow 0} u = \mp \frac{\dot{A}}{\tilde{\eta}} = \mp i H a A$$

$$\phi_1 = \mp i H A$$

$$\mathcal{R} = \pm i H A \left( \frac{\dot{a}}{a} \right) \frac{1}{\dot{\phi}_0}$$

- Slow roll replacement

$$\left( \frac{\dot{a}}{a} \right)^2 \frac{1}{\dot{\phi}_0^2} = \frac{8\pi G a^2 V}{3} \frac{3}{2a^2 V \epsilon} = \frac{4\pi G}{\epsilon}$$

- Curvature power spectrum

$$\Delta_{\mathcal{R}}^2 \equiv \frac{k^3 |\mathcal{R}|^2}{2\pi^2} = 8\pi G \frac{H^2}{(2\pi)^2 \epsilon} k^3 A^2$$

# Quantum Fluctuations

- Simple harmonic oscillator  $\ll$  Hubble length

$$\ddot{u} + k^2 u = 0$$

- Quantize the simple harmonic oscillator

$$\hat{u} = u(k, \eta) \hat{a} + u^*(k, \eta) \hat{a}^\dagger$$

where  $u(k, \eta)$  satisfies classical equation of motion and the creation and annihilation operators satisfy

$$[a, a^\dagger] = 1, \quad a|0\rangle = 0$$

- Normalize wavefunction  $[\hat{u}, d\hat{u}/d\eta] = i$

$$u(k, \eta) = \frac{1}{\sqrt{2k}} e^{-ik\tilde{\eta}}$$

# Quantum Fluctuations

- Zero point fluctuations of ground state

$$\begin{aligned}\langle u^2 \rangle &= \langle 0 | u^\dagger u | 0 \rangle \\ &= \langle 0 | (u^* \hat{a}^\dagger + u \hat{a}) (u \hat{a} + u^* \hat{a}^\dagger) | 0 \rangle \\ &= \langle 0 | \hat{a} \hat{a}^\dagger | 0 \rangle |u(k, \tilde{\eta})|^2 \\ &= \langle 0 | [\hat{a}, \hat{a}^\dagger] + \hat{a}^\dagger \hat{a} | 0 \rangle |u(k, \tilde{\eta})|^2 \\ &= |u(k, \tilde{\eta})|^2 = \frac{1}{2k}\end{aligned}$$

- Classical equation of motion take this quantum fluctuation outside horizon where it freezes in. Slow roll equation
- So  $A = (2k^3)^{-1/2}$  and curvature power spectrum

$$\Delta_{\mathcal{R}}^2 = \frac{8\pi G}{2} \frac{H^2}{(2\pi)^2 \epsilon}, \quad \Delta_{\phi_1}^2 = \frac{H^2}{(2\pi)^2}$$

# Tilt

- Curvature power spectrum is scale invariant to the extent that  $H$  is constant
- Scalar spectral index

$$\begin{aligned}\frac{d \ln \Delta_{\mathcal{R}}^2}{d \ln k} &\equiv n_S - 1 \\ &= 2 \frac{d \ln H}{d \ln k} - \frac{d \ln \epsilon}{d \ln k}\end{aligned}$$

- Evaluate at horizon crossing where fluctuation freezes

$$\begin{aligned}\frac{d \ln H}{d \ln k} \Big|_{-k\tilde{\eta}=1} &= \frac{k}{H} \frac{dH}{d\tilde{\eta}} \Big|_{-k\tilde{\eta}=1} \frac{d\tilde{\eta}}{dk} \Big|_{-k\tilde{\eta}=1} \\ &= \frac{k}{H} (-aH^2\epsilon) \Big|_{-k\tilde{\eta}=1} \frac{1}{k^2} = -\epsilon\end{aligned}$$

where  $aH = -1/\tilde{\eta} = k$

# Tilt

- Evolution of  $\epsilon$

$$\frac{d \ln \epsilon}{d \ln k} = -\frac{d \ln \epsilon}{d \ln \tilde{\eta}} = -2(\delta + \epsilon) \frac{\dot{a}}{a} \tilde{\eta} = 2(\delta + \epsilon)$$

- Tilt in the slow-roll approximation

$$n_S = 1 - 4\epsilon - 2\delta$$

# Relationship to Potential

- To leading order in slow roll parameters

$$\begin{aligned}\epsilon &= \frac{\frac{3}{2}\dot{\phi}_0^2/a^2V}{1 + \frac{1}{2}\dot{\phi}_0^2/a^2V} \\ &\approx \frac{3}{2}\dot{\phi}_0^2/a^2V \\ &\approx \frac{3}{2a^2V} \frac{a^4V'^2}{9(\dot{a}/a)^2}, \quad \left(3\dot{\phi}_0 \frac{\dot{a}}{a} = -a^2V'\right) \\ &\approx \frac{1}{6} \frac{3}{8\pi G} \left(\frac{V'}{V}\right)^2, \quad \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}a^2V \\ &\approx \frac{1}{16\pi G} \left(\frac{V'}{V}\right)^2\end{aligned}$$

so  $\epsilon \ll 1$  is related to the first derivative of potential being small

# Relationship to Potential

- And

$$\delta = \frac{\ddot{\phi}_0}{\dot{\phi}_0} \left( \frac{\dot{a}}{a} \right)^{-1} - 1$$

$$\left( \dot{\phi}_0 \approx -a^2 \left( \frac{\dot{a}}{a} \right)^{-1} \frac{V'}{3} \right)$$

$$\left( \ddot{\phi}_0 \approx -\frac{a^2 V'}{3} (1 + \epsilon) + a^4 \left( \frac{\dot{a}}{a} \right)^{-2} \frac{V' V''}{9} \right)$$

$$\approx -\frac{1}{a^2 V' / 3} \left( -\frac{a^2 V'}{3} (1 + \epsilon) + \frac{a^2}{9} \frac{3}{8\pi G} \frac{V' V''}{V} \right) - 1 \approx \epsilon - \frac{1}{8\pi G} \frac{V''}{V}$$

so  $\delta$  is related to second derivative of potential being small. Very flat potential.

# Relationship to Potential

- Exact relations

$$\frac{1}{8\pi G} \left( \frac{V'}{V} \right)^2 = 2\epsilon \frac{(1 + \delta/3)^2}{(1 - \epsilon/3)^2}$$

$$\frac{1}{8\pi G} \frac{V''}{V} = \frac{\epsilon - \delta - [\delta^2 - \epsilon\delta - (a/\dot{a})\dot{\delta}]/3}{1 - \epsilon_H/3}$$

agree in the limit  $\epsilon, |\delta| \ll 1$  and  $|(a/\dot{a})\dot{\delta}| \ll \epsilon, |\delta|$

- Like the Mukhanov to slow roll simplification, identification with potential requires a constancy of  $\delta$  assumption

# Gravitational Waves

- Gravitational wave amplitude satisfies Klein-Gordon equation ( $K = 0$ ), same as scalar field

$$\ddot{H}_T^{(\pm 2)} + 2\frac{\dot{a}}{a}\dot{H}_T^{(\pm 2)} + k^2 H_T^{(\pm 2)} = 0.$$

- Acquires quantum fluctuations in same manner as  $\phi$ . Lagrangian sets the normalization

$$\phi_1 \rightarrow H_T^{(\pm 2)} \sqrt{\frac{3}{16\pi G}}$$

- Scale-invariant gravitational wave amplitude converted back to  $+$  and  $\times$  states  $H_T^{(\pm 2)} = -(h_+ \mp ih_\times)/\sqrt{6}$

$$\Delta_{+, \times}^2 = 16\pi G \Delta_{\phi_1}^2 = 16\pi G \frac{H^2}{(2\pi)^2}$$

# Gravitational Waves

- Gravitational wave power  $\propto H^2 \propto V \propto E_i^4$  where  $E_i$  is the energy scale of inflation
- Tensor-scalar ratio - various definitions - WMAP standard is

$$r \equiv 4 \frac{\Delta_+^2}{\Delta_{\mathcal{R}}^2} = 16\epsilon$$

- Tensor tilt:

$$\frac{d \ln \Delta_H^2}{d \ln k} \equiv n_T = 2 \frac{d \ln H}{d \ln k} = -2\epsilon$$

# Gravitational Waves

- Consistency relation between tensor-scalar ratio and tensor tilt

$$r = 16\epsilon = -8n_T$$

- Measurement of scalar tilt and gravitational wave amplitude constrains inflationary model in the slow roll context
- Comparison of tensor-scalar ratio and tensor tilt tests the idea of slow roll itself

# Gravitational Wave Phenomenology

- Equation of motion

$$\ddot{H}_T^{(\pm 2)} + 2\frac{\dot{a}}{a}\dot{H}_T^{(\pm 2)} + k^2 H_T^{(\pm 2)} = 0.$$

- has solutions

$$H_T^{(\pm 2)} = C_1 H_1(k\eta) + C_2 H_2(k\eta)$$

$$H_1 \propto x^{-m} j_m(x)$$

$$H_2 \propto x^{-m} n_m(x)$$

where  $m = (1 - 3w)/(1 + 3w)$

- If  $w > -1/3$  then gravity wave is constant above horizon  $x \ll 1$  and then oscillates and damps
- If  $w < -1/3$  then gravity wave oscillates and freezes into some value, just like scalar field

# Gravitational Wave Phenomenology

- A gravitational wave makes a quadrupolar (transverse-traceless) distortion to metric
- Just like the scale factor or spatial curvature, a temporal variation in its amplitude leaves a residual temperature variation in CMB photons – here anisotropic
- Before recombination, anisotropic variation is eliminated by scattering
- Gravitational wave temperature effect drops sharply at the horizon scale at recombination

# Gravitational Wave Phenomenology

- Source to polarization goes as  $\dot{\tau}/\dot{H}_T$  and peaks at the horizon not damping scale
- More distinct signature in the polarization since symmetry of plane wave is broken by the transverse nature of gravity wave polarization
- $B$  modes formed as photons propagate – the spatial variation in the plane waves modulate the signal: described by Boltzmann eqn.

$$\Delta B_{\text{peak}} \approx 0.024 \left( \frac{E_i}{10^{16} \text{GeV}} \right)^2 \mu\text{K}$$

# Large Field Models

- For detectable gravitational waves, scalar field must roll by order

$$M_{\text{pl}} = (8\pi G)^{-1/2}$$

$$\frac{d\phi_0}{dN} = \frac{d\phi_0}{d \ln a} = \frac{d\phi_0}{dt} \frac{1}{H}$$

- The larger  $\epsilon$  is the more the field rolls in an e-fold

$$\epsilon = \frac{r}{16} = \frac{3}{2V} \left( H \frac{d\phi_0}{dN} \right)^2 = \frac{8\pi G}{2} \left( \frac{d\phi_0}{dN} \right)^2$$

- Observable scales span  $\Delta N \sim 5$  so

$$\Delta\phi_0 \approx 5 \frac{d\phi}{dN} = 5(r/8)^{1/2} M_{\text{pl}} \approx 0.6(r/0.1)^{1/2} M_{\text{pl}}$$

- Does this make sense as an effective field theory? [Lyth \(1997\)](#)

# Large Field Models

- Large field models include monomial potentials  $V(\phi) = A\phi^n$

$$\epsilon \approx \frac{n^2}{16\pi G\phi^2}$$

$$\delta \approx \epsilon - \frac{n(n-1)}{8\pi G\phi^2}$$

- Slow roll requires large field values of  $\phi > M_{\text{pl}}$
- Thus  $\epsilon \sim |\delta|$  and a finite tilt indicates finite  $\epsilon$
- Given WMAP tilt, potentially observable gravitational waves

# Small Field Models

- If the field is near an maximum of the potential

$$V(\phi) = V_0 - \frac{1}{2}\mu^2\phi^2$$

- Inflation occurs if the  $V_0$  term dominates

$$\epsilon \approx \frac{1}{16\pi G} \frac{\mu^4\phi^2}{V_0^2}$$

$$\delta \approx \epsilon + \frac{1}{8\pi G} \frac{\mu^2}{V_0} \rightarrow \frac{\delta}{\epsilon} = \frac{V_0}{\mu^2\phi^2} \gg 1$$

- Tilt reflects  $\delta$ :  $n_S \approx 1 - 2\delta$  and  $\epsilon$  is much smaller
- The field does not roll significantly during inflation and gravitational waves are negligible

# Hybrid Models

- If the field is rolling toward a minimum of the potential

$$V(\phi) = V_0 + \frac{1}{2}m^2\phi^2$$

- Slow roll parameters similar to small field but a real  $m^2$

$$\epsilon \approx \frac{1}{16\pi G} \frac{m^4\phi^2}{V_0^2}$$

$$\delta \approx \epsilon - \frac{1}{8\pi G} \frac{m^2}{V_0}$$

- Then  $V_0$  domination  $\epsilon, \delta < 0$  and  $n_S > 1$  - blue tilt
- For  $m^2$  domination, monomial-like.
- Intermediate cases with intermediate predictions - can have observable gravity waves but does not require it.

# Hybrid Models

- But how does inflation end?  $V_0$  remains as field settles to minimum
- Implemented as multiple field model with  $V_0$  supplied by second field
- Inflation ends when rolling triggers motion in the second field to the true joint minimum

# Features

- The slow roll simplification carried an extra assumption that  $\delta$  is not only small but also constant
- Assumption that  $\epsilon$  is nearly constant is justified if  $|\delta| \ll 1$
- These conditions can be violated if there are rapid, not necessarily large, changes in the potential
- What happens when we relax these conditions?
- Go back to Mukhanov equation and for convenience transform variables to  $y = \sqrt{2k}u$  and  $x = -k\tilde{\eta}$  and  $' = d/d \ln \tilde{\eta}$

$$\frac{d^2 y}{dx^2} + \left(1 - \frac{2}{x^2}\right) y = \frac{g(\ln x)}{x^2} y$$

$$g = \frac{f'' - 3f'}{f}, \quad f = 2\pi \frac{\dot{\phi} a \eta}{H}$$

# Generalized Slow Roll

- Note that the LHS is the slow roll version of Mukhanov and the RHS is a source that depends on the slow roll parameters as

$$g = 2[(aH\tilde{\eta})^2 - 1] + (aH\tilde{\eta})^2[2\epsilon + 3\delta + 2\epsilon^2 + 4\delta\epsilon + \delta_2]$$

where we have eliminated  $\dot{\delta}$  in favor of a third parameter

$$\delta_2 = \frac{d\delta}{d \ln a} - \epsilon\delta + \delta^2$$

- Since  $(aH\tilde{\eta}) \approx -(1 + \epsilon)$ ,  $g$  vanishes to zeroth order in slow roll
- Keeping  $\delta_2$  restores the relationship with the potential

$$g \approx \frac{1}{8\pi G} \left[ \frac{9}{2} \left( \frac{V'}{V} \right)^2 - 3 \frac{V''}{V} \right]$$

# Generalized Slow Roll

- Generalized slow roll approximation exploits this fact by taking an iterative Green function approach
- LHS “homogeneous” equation is solved by

$$y_0(x) = \left(1 + \frac{i}{x}\right) e^{ix}$$

so in the approximation that  $y \approx y_0$  on the RHS

$$y(x) = y_0(x) - \int_x^\infty \frac{du}{u^2} g(\ln u) y_0(u) \text{Im}[y_0^*(u) y_0(x)]$$

which is an integral equation for the field an curvature spectrum given some deviation from slow roll as quantified by  $V'/V$  and  $V''/V$

# Generalized Slow Roll

- For example, by assuming  $|\delta| \ll 1$ ,  $\epsilon \approx \text{const.}$  and

$$\Delta_{\mathcal{R}}^2(k) = \frac{V(8\pi G)^2}{12\pi^2} \left( \frac{V}{V'} \right)^2 \left\{ 1 + \left( 3\alpha - \frac{1}{6} \right) \frac{1}{8\pi G} \left( \frac{V'}{V} \right)^2 \Big|_{k=aH} - \frac{2}{8\pi G} \int_0^\infty \frac{du}{u} W_\theta(u) \frac{V''}{V} \right\},$$

where  $\alpha \approx 0.73$  and

$$W_\theta(u) = \frac{3 \sin(2u)}{2u^3} - \frac{3 \cos 2u}{u^2} - \frac{3 \sin(2u)}{2u} - \theta(1 - u)$$

- $u \ll 1$  goes  $2u^2/5$  and  $u \gg 1$  oscillates as  $-3 \sin(2u)/2u$
- Therefore changes in  $V''/V$  around horizon crossing for mode produce a ringing response in the power spectrum – WMAP glitches?
- Can extend to  $|\delta| \sim \mathcal{O}(1)$ , and can be iterated to higher order

# Non-Gaussianity

- In single field slow roll inflation, the inflaton is nearly a free field - modes don't interact - and its fluctuations are Gaussian to a high degree.
- Non-gaussianities are at best second order effects and with  $10^{-5}$  fluctuations, this is a  $10^{-10}$  effect!
- Fluctuations coming directly from self-coupling terms in the field are further reduced by order  $\epsilon, \delta$ .

$$\mathcal{R}^{(2)} = [\mathcal{R}^{(1)}]^2 \mathcal{O}(\epsilon, \delta)$$

- Second order effects generate a larger correction. Curvature on constant density slicing  $\zeta$  still conserved, transform back to Newtonian gauge and extract fluctuation returns a form

$$\Phi(\mathbf{x}) = \Phi^{(1)}(\mathbf{x}) + f_{\text{NL}} [(\Phi^{(1)}(\mathbf{x}))^2 - \langle \Phi^{(1)}(\mathbf{x}) \rangle^2]$$

called a local non-Gaussianity - generic prediction is  $f_{\text{NL}} = \mathcal{O}(1)$

# Non-Gaussianity

- Decompose in harmonics (assume  $k^2 \gg |K|$ , nearly flat)

$$\begin{aligned}\Phi(\mathbf{k}) &= \int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}} \Phi(\mathbf{x}) \\ &= \Phi^{(1)}(\mathbf{k}) + f_{\text{NL}} \int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}} \int \frac{d^3k_1}{(2\pi)^3} e^{i\mathbf{k}_1\cdot\mathbf{x}} \Phi^{(1)}(\mathbf{k}_1) \int \frac{d^3k_2}{(2\pi)^3} e^{i\mathbf{k}_2\cdot\mathbf{x}} \Phi^{(1)}(\mathbf{k}_2)\end{aligned}$$

with  $\int d^3x e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}} = (2\pi)^3 \delta(\mathbf{k} - \mathbf{k}')$  yields

$$\Phi(\mathbf{k}) = \Phi^{(1)}(\mathbf{k}) + f_{\text{NL}} \int \frac{d^3k_1}{(2\pi)^3} \Phi^{(1)}(\mathbf{k}_1) \Phi^{(1)}(\mathbf{k} - \mathbf{k}_1)$$

- Given that the first order term is a Gaussian field represented by the power spectrum

$$\langle \Phi^{(1)}(\mathbf{k}) \Phi^{(1)}(\mathbf{k}') \rangle = (2\pi)^3 \delta(\mathbf{k} + \mathbf{k}') P_{\Phi}(k)$$

we get a bispectrum contribution

# Non-Gaussianity

- The bispectrum is proportional to the product of power spectra

$$\langle \Phi(\mathbf{k}_1)\Phi(\mathbf{k}_2)\Phi(\mathbf{k}_3) \rangle = (2\pi)^3 \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) [2f_{\text{NL}} P_{\Phi}(k_1) P_{\Phi}(k_2) + \text{perm}]$$

- Bispectrum contains most of the information on local non-Gaussianity
- Higher point correlations also exist and can be measured with good  $S/N$  but their sample variance is high and so don't help in constraining  $f_{\text{NL}}$

# Non-Gaussianity Measurements

- Measure in 2D analogue in the CMB  $f_{\text{NL}} < 80$  (95% CL)
- Planck can reach  $f_{\text{NL}} \sim \text{few}$
- Counterintuitively, measure the power spectrum of rare objects at low  $k$  (large scales)
- Bispectrum for  $\mathbf{k}_1 \sim -\mathbf{k}_2$  and  $|k_3| \ll |k_1|, k_2$  looks like a correlation of high  $k$  power with low  $k$  amplitude
- Enhanced power allows for formation of rare objects in essentially the same way as the peak-background split picture for bias
- Relative effect increases at low  $k$  where the intrinsic power in the density spectrum falls
- Strong scale dependence of bias (Dalal et al 2007)

# Curvaton

- Beyond single field slow roll inflation, larger  $f_{\text{NL}}$  can be generated
- Suppose there is a second light field  $\sigma$  called the curvaton during inflation, it also has quantum fluctuations

$$\Delta_{\sigma}^2 \approx \left( \frac{H}{2\pi} \right)^2$$

- After inflation the curvaton oscillates around its minimum and decays leaving its field fluctuations as density fluctuations

# Curvaton

- Its energy density is given by the *square* of the field amplitude

$$\rho_\sigma = m_\sigma^2 \sigma^2$$

- The fractional density fluctuation is related to the fractional field fluctuation

$$\frac{\delta\rho_\sigma}{\rho_\sigma} \approx 2\frac{\delta\sigma}{\sigma}$$

and so the power spectrum is enhanced by  $1/\sigma_*^2$  (evaluated at horizon crossing)

- Can be arranged to dominate the inflaton curvature fluctuations
- If the curvaton comes to dominate the energy density than the density fluctuation becomes a curvature fluctuation on the constant density gauge  $\zeta$

# Curvaton

- More generally if  $r$  is the ratio of curvaton to total energy density

$$\zeta = r\zeta_\sigma \approx -r \frac{\dot{a}}{a} \frac{\delta\rho_\sigma}{\dot{\rho}_\sigma} \approx \frac{r}{3} \frac{\delta\rho_\sigma}{\rho_\sigma}$$

$$\Delta_\zeta^2 = \frac{4}{9} r^2 \left( \frac{H_*}{2\pi\sigma_*} \right)^2$$

- Moreover since the density is the square of the field, there is a local non-Gaussianity that can be much larger

$$\frac{\delta\rho_\sigma}{\rho_\sigma} \approx 2 \frac{\delta\sigma}{\sigma} + \frac{(\delta\sigma)^2}{\sigma^2}$$

# Curvaton

- So with  $\Phi = 3\zeta/5$

$$\Phi = -\frac{r}{5} \left[ 2 \left( \frac{\delta\sigma}{\sigma} \right) + \left( \frac{\delta\sigma}{\sigma} \right)^2 \right]$$

$$f_{\text{NL}} = \frac{5}{4r}$$

- And if  $r \sim 10^{-2}$  can reach of order  $f_{\text{NL}} \sim 10^2$

# Variable Decay

- Suppose the decay rate of the inflation during reheating were controlled by some light scalar  $\Gamma \propto \chi$
- $\chi$  gets quantum fluctuations  $\delta\chi \approx H/2\pi$  as any light scalar
- Fluctuations in the field cause fluctuations in the decay rate
- In a region where decay is more rapid, the inflaton converts its energy to radiation earlier, thereafter redshifting like radiation
- Net result is a density fluctuation of order the decay rate fluctuation  $\delta\rho/\rho \propto \delta\Gamma/\Gamma$
- In the constant density gauge this is a curvature fluctuation

$$\zeta = -\frac{1}{6} \frac{\delta\Gamma}{\Gamma}$$