Astro 408

Set 2: Inflationary Perturbations

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Outline

- Review of canonical single-field slow roll inflation
  - Scalar field
  - Comoving curvature
  - Gravitational waves
- EFT of inflation
  - Application to $P(X,\phi)$
  - Non-Gaussianity

Horizon Problem

- The horizon in a decelerating universe scales as $\eta \propto a^{(1+3w)/2}$, $w > -1/3$. For example in a matter dominated universe

$$\eta \propto a^{1/2}$$

- CMB decoupled at $a_* = 10^{-3}$ so subtends an angle on the sky

$$\frac{\eta_*}{\eta_0} = a_*^{1/2} \approx 0.03 \approx 2^\circ$$

- So why is the CMB sky isotropic to 10^{-5} in temperature if it is composed of $\sim 10^4$ causally disconnected regions
- If smooth by fiat, why are there 10^{-5} fluctuations correlated on superhorizon scales

Flatness & Relic Problems

- Flatness problem: why is the radius of curvature larger than the observable universe $|\Omega_K| \ll \Omega_m$ (so in inflationary calculations below we set $K = 0$ for simplicity).
- Also phrased as a coincidence problem: since $\rho_K \propto a^{-2}$ and $\rho_m \propto a^{-3}$, why would they be comparable today – modern version is dark energy coincidence $\rho_\Lambda =$ const.
- Relic problem – why don't relics like monopoles dominate the energy density
- Inflation is a theory that solves all three problems at once and also supplies a source for density perturbations

Accelerating Expansion

- In a matter or radiation dominated universe, the horizon grows as a power law in a so that there is no way to establish causal contact on a scale longer than the inverse Hubble length ($1/aH$, comoving coordinates) at any given time: general for a decelerating universe

$$\eta = \int d\ln a \frac{1}{aH(a)}$$

- $H^2 \propto \rho \propto a^{-3(1+w)}$, $aH \propto a^{-(1+3w)/2}$, critical value of $w = -1/3$, the division between acceleration and deceleration
- In an accelerating universe, the Hubble length shrinks in comoving coordinates and so the horizon gets its contribution at the earliest times, e.g. in a cosmological constant universe, the horizon saturates to a constant value

Causal Contact

- Note change in nomenclature: the true horizon always grows meaning that one always sees more and more of the universe. The comoving Hubble length decreases: distance light propagates in an efolding of a decreases. Regions that were in causal contact, leave causal contact.
- During inflation, the Hubble length describes the distance that a photon can travel from the given epoch to the *end* of inflation
- Horizon problem solved if the universe was in an acceleration phase up to η_i and the conformal time since then is shorter than the total conformal age

$$\eta_0 \gg \eta_0 - \eta_i$$

$$\text{total distance} \gg \text{distance traveled since inflation}$$

$$\text{apparent horizon}$$

Flatness & Relic

- Comoving radius of curvature is constant and can even be small compared to the full horizon $R \ll \eta_0$ yet still $\eta_0 \gg R \gg \eta_0 - \eta_i$
- In physical coordinates, the rapid expansion of the universe makes the current observable universe much smaller than the curvature scale
- Likewise, the number density of relics formed before the accelerating (or inflationary) epoch is diluted to make them rare in the current observable volume
- Common to reference time to the end of inflation $\tilde{\eta} \equiv \eta - \eta_i$. Here conformal time is negative during inflation and its value (as a difference in conformal time) reflects the comoving Hubble length - defines leaving the horizon as $k|\tilde{\eta}| = 1$

Exponential Expansion

- If the accelerating component has equation of state $w = -1$, $\rho =$ const., $H = H_i$ const. so that $a \propto \exp(Ht)$

$$\tilde{\eta} = -\int_a^{a_i} d\ln a \frac{1}{aH} = \frac{1}{aH_i}\Big|_a^{a_i}$$
$$\approx -\frac{1}{aH_i} \quad (a_i \gg a)$$

- In particular, the current horizon scale $H_0\tilde{\eta}_0 \approx 1$ exited the horizon during inflation at

$$\tilde{\eta}_0 \approx H_0^{-1} = \frac{1}{a_H H_i}$$
$$a_H = \frac{H_0}{H_i}$$

Sufficient Inflation

- Current horizon scale must have exited the horizon during inflation so that the start of inflation could not be after a_H. How long before the end of inflation must it have began?

$$\frac{a_H}{a_i} = \frac{H_0}{H_i a_i}$$

$$\frac{H_0}{H_i} = \sqrt{\frac{\rho_c}{\rho_i}}, \qquad a_i = \frac{T_{\rm CMB}}{T_i}$$

- $\rho_c^{1/4} = 3 \times 10^{-12}$ GeV, $T_{\rm CMB} = 3 \times 10^{-13}$ GeV

$$\frac{a_H}{a_i} = 3 \times 10^{-29} \left(\frac{\rho_i^{1/4}}{10^{14}{\rm GeV}} \right)^{-2} \left(\frac{T_i}{10^{10}{\rm GeV}} \right)$$

$$\ln \frac{a_i}{a_H} = 65 + 2 \ln \left(\frac{\rho_i^{1/4}}{10^{14}{\rm GeV}} \right) - \ln \left(\frac{T_i}{10^{10}{\rm GeV}} \right)$$

Canonical Scalar Fields

- A canonical scalar field can drive inflation by supplying potential energy that doesn't change at a fixed field value as the universe expands

$$S_\phi = \int d^4x \sqrt{-g}[-\frac{1}{2} g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)]$$

- Varying the action with respect to the metric

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} \sqrt{-g}\mathcal{L}_\phi$$

gives the stress-energy tensor of a scalar field

$$T^\mu{}_\nu = \nabla^\mu\phi\,\nabla_\nu\phi - \frac{1}{2}(\nabla^\alpha\phi\,\nabla_\alpha\phi + 2V)\delta^\mu{}_\nu \,.$$

- Equations of motion $\nabla^\mu T_{\mu\nu} = 0$ with closure relations for $p(\phi, \partial_\mu\phi)$, $\Pi(\phi, \partial_\mu\phi)$ or field equation $\nabla_\mu\nabla^\mu\phi = V'$ (vary with respect to ϕ)

Scalar Fields

- For the background $\langle\phi\rangle \equiv \phi(\eta)$ (a^{-2} from conformal time)

$$\rho_\phi = \frac{1}{2}a^{-2}\dot\phi^2 + V, \quad p_\phi = \frac{1}{2}a^{-2}\dot\phi^2 - V$$

- So for kinetic dominated $w_\phi = p_\phi/\rho_\phi \to 1$
- And potential dominated $w_\phi = p_\phi/\rho_\phi \to -1$
- Can use general equations of motion of dictated by stress energy conservation

$$\dot\rho_\phi = -3(\rho_\phi + p_\phi)\frac{\dot a}{a}\,,$$

to obtain the equation of motion of the background field ϕ

$$\ddot\phi + 2\frac{\dot a}{a}\dot\phi + a^2 V' = 0\,,$$

$$\frac{d^2\phi}{dt^2} + 3H\frac{d\phi}{dt} + V' = 0$$

Slow Roll Parameters

- Rewrite equations of motion in terms of slow roll parameters but do not require them to be small or constant.
- Deviation from de Sitter expansion

$$\epsilon_H \equiv -\frac{d\ln H}{d\ln a} = \frac{3}{2}(1+w_\phi) = \frac{\frac{3}{2}(d\phi/dt)^2/V}{1+\frac{1}{2}(d\phi/dt)^2/V}$$

must be small, of order the inverse of 60 efolds remaining - $\epsilon_H > 1$ or $w_\phi > -1/3$ is a decelerating expansion

- Fractional evolution of ϵ_H

$$\eta_H = -\delta_1 = \epsilon_H - \frac{1}{2}\frac{d\ln\epsilon_H}{d\ln a}$$

not necessarily small instantaneously but determines doubling rate of ϵ_H – so for featureless inflation should also be small

Slow Roll Hierarchy

- Can continue this hierarchy

$$\delta_{p+1} = \frac{d\delta_p}{dN} + \delta_p(\delta_1 - p\epsilon_H)$$

- Each term in the series is suppressed by an additional power of the base slow roll parameters δ_1, ϵ_H
- Phenomenologically describes running of the tilt, running of the running etc, forming essentially a Taylor expansion in the $N \sim 60$ efolds to end of inflation
- Specific form is motivated by association with derivatives of ϕ for canonical inflation (next slide) but is used as a definition beyond

Slow Roll Parameters

- The slow roll parameter δ_1 also relates the field acceleration to Hubble friction (for canonical field)
- Rewrite the Friedman equations as

$$H^2 = 4\pi G \left(\frac{d\phi}{dt}\right)^2 \epsilon_H^{-1}$$

$$\frac{1}{a}\frac{d^2a}{dt^2} = -\frac{4\pi G}{3}(\rho_\phi + 3p_\phi) = -\frac{H^2}{2}(1+3w_\phi)$$

$$= H^2(1-\epsilon_H) = \frac{dH}{dt} + H^2$$

or $dH/dt = -\epsilon_H H^2$. So take derivative of first equation

$$-2H\epsilon_H = 2\frac{d^2\phi/dt^2}{d\phi/dt} - \frac{d\ln\epsilon_H}{dt}$$

Slow Roll Parameters

- Finally using the relation for δ we obtain

$$\delta_1 = \frac{d^2\phi/dt^2}{H d\phi/dt}$$

- More generally

$$\delta_p = \frac{1}{H^p d\phi/dt} \left(\frac{d}{dt}\right)^{p+1} \phi$$

- Therefore if $\delta_1 \ll 1$, the scalar field equation of motion is Hubble friction dominated

$$3H\frac{d\phi}{dt} + V' = 0$$

- The background roll is an attractor: given a field position on the potential the field velocity is also determined, independent of its initial value

Field Perturbations

- Perturbations of the field $\phi = \phi + \delta\phi$ evolve under the continuity and Euler equations

$$
\begin{aligned}
\delta\rho_\phi &= a^{-2}(\dot{\phi}\dot{\delta\phi} - \dot{\phi}^2 A) + V'\delta\phi\,, \\
\delta p_\phi &= a^{-2}(\dot{\phi}\dot{\delta\phi} - \dot{\phi}^2 A) - V'\delta\phi\,, \\
(\rho_\phi + p_\phi)(v_\phi - B) &= a^{-2}k\dot{\phi}\delta\phi\,, \\
p_\phi \pi_\phi &= 0\,,
\end{aligned}
$$

- For comoving slicing where $v_\phi = B$

$$\delta\phi = 0$$

and the field is spatially unperturbed - so all the dynamics are in the metric (unitary gauge ADM)

Sound Speed

- In this slicing the perturbation to the energy density and pressure due to a change in field position on the potential is absent, so they reflect the kinetic piece while the background is dominated by the potential piece

$$\delta p_\phi = \delta \rho_\phi$$

  so the sound speed is $\delta p_\phi / \delta \rho_\phi = 1$.

- More generally the sound speed of the inflation is defined as the speed at which field fluctuations propagate - i.e. the kinetic piece to the energy density rather than the $V'\delta\phi$ potential piece - much like in the background the $+1$ and -1 pieces of w.

- Non canonical kinetic terms EFT– k-essence, DBI inflation, Horndeski – can generate $c_s \neq 1$ as do terms in the effective theory of inflation

Equation of Motion

- Scalar field fluctuations are stable inside the horizon and are a good candidate for the smooth dark energy
- Equivalently, conservation equations imply

$$\ddot{\delta\phi} = -2\frac{\dot a}{a}\dot{\delta\phi} - (k^2 + a^2 V'')\delta\phi + (\dot A - 3\dot H_L - kB)\dot\phi - 2Aa^2V' \, .$$

- Alternately this follows from perturbing the Klein Gordon equation $\nabla_\mu \nabla^\mu \phi = V'$

Inflationary Perturbations

- Classical equations of motion for scalar field inflaton determine the evolution of scalar field fluctuations generated by quantum fluctuations
- Since the curvature $\mathcal{R}$ on comoving slicing is conserved in the absence of stress fluctuations (i.e. outside the apparent horizon, calculate this and we're done no matter what happens in between inflation and the late universe (reheating etc.)
- But in the comoving slicing $\delta\phi = 0$! no scalar-field perturbation
- Solution: solve the scalar field equation in the dual gauge where the curvature $H_L + H_T/3 = 0$ ("spatially flat" slicing) and transform the result to comoving slicing

Transformation to Comoving Slicing

- Scalar field transforms as scalar field

$$\tilde{\delta\phi} = \delta\phi - \dot{\phi}T$$

- To get to comoving slicing $\tilde{\delta\phi} = 0$, $T = \delta\phi/\dot{\phi}$, and $\tilde{H}_T = H_T + kL$ so

$$\mathcal{R} = H_L + \frac{H_T}{3} - \frac{\dot{a}}{a}\frac{\delta\phi}{\dot{\phi}}$$

- Transformation particularly simple from a spatially flat slicing where $H_L + H_T/3 = 0$, i.e. spatially unperturbed metric

$$\mathcal{R} = -\frac{\dot{a}}{a}\frac{\delta\phi}{\dot{\phi}}\Big|_{\rm flat} = -\frac{\delta\phi}{d\phi/d\ln a}\Big|_{\rm flat} = -\delta\ln a$$

i.e. a perturbation to the scale factor at fixed scalar field

Spatially Flat Gauge

- Spatially Flat (flat slicing, isotropic threading):

$$\begin{aligned} \tilde{H}_L + \tilde{H}_T/3 &= \tilde{H}_T = 0 \\ A_f &= \tilde{A}\,, B_f = \tilde{B} \\ T &= \left(\frac{\dot{a}}{a}\right)^{-1} \left(H_L + \frac{1}{3} H_T\right) \\ L &= -H_T/k \end{aligned}$$

- Einstein Poisson and Momentum

$$-3(\frac{\dot{a}}{a})^2 A_f + \frac{\dot{a}}{a} k B_f = 4\pi G a^2 \delta\rho_\phi\,,$$

$$\frac{\dot{a}}{a} A_f - \frac{K}{k^2}(k B_f) = 4\pi G a^2 (\rho_\phi + p_\phi)(v_\phi - B_f)/k\,,$$

- Conservation

$$\ddot{\delta\phi} = -2\frac{\dot{a}}{a}\dot{\delta\phi} - (k^2 + a^2 V'')\delta\phi + (\dot{A}_f - k B_f)\dot{\phi} - 2 A_f a^2 V'\,.$$

Spatially Flat Gauge

- Lapse and shift are nondynamical and can be eliminated by the Poisson (Hamiltonian) and momentum constraints
- For modes where $|k^2/K| \gg 1$ we obtain

$$\frac{\dot a}{a} A_f = 4\pi G \dot\phi \delta\phi \,,$$
$$\frac{\dot a}{a} k B_f = 4\pi G [\dot\phi \dot{\delta\phi} - \dot\phi^2 A_f + a^2 V' \delta\phi + 3\frac{\dot a}{a}\dot\phi\delta\phi]$$

  so combining $\dot A_f - kB_f$ eliminates the $\dot{\delta\phi}$ term

- The metric source to the scalar field equation can be reexpressed in terms of the field perturbation and background quantities

$$(\dot A_f - kB_f)\dot\phi - 2A_f a^2 V' - a^2 V'' \delta\phi = f(\eta)\delta\phi$$

- Single closed form 2nd order ODE for $\delta\phi$

Mukhanov Equation

- Equation resembles a damped oscillator equation with a particular dispersion relation

$$\ddot{\delta\phi} + 2\frac{\dot a}{a}\dot{\delta\phi} + [k^2 + f(\eta)]\delta\phi$$

- $f(\eta)$ involves terms with $\dot\phi$, V', V'' implying that for a sufficiently flat potential $f(\eta)$ represents a small correction
- Transform out the background expansion $u \equiv a\delta\phi$

$$\dot u = \dot a\delta\phi + a\dot{\delta\phi}$$

$$\ddot u = \ddot a\delta\phi + 2\dot a\dot{\delta\phi} + a\ddot{\delta\phi}$$

$$\ddot u + [k^2 - \frac{\ddot a}{a} + f(\eta)]u = 0$$

- Note Friedmann equations say if $p = -\rho$, $\ddot a/a = 2(\dot a/a)^2$

Mukhanov Equation

- Using the background Einstein and scalar field equations, this source term can be expressed in a surprisingly compact form

$$\ddot{u} + [k^2 - \frac{\ddot{z}}{z}]u = 0$$

- and

$$z \equiv \frac{a\dot{\phi}}{\dot{a}/a}$$

- This equation is sometimes called the "Mukhanov Equation" and is both exact (in linear theory) and compact
- For large k (subhorizon), this looks like a free oscillator equation which can be quantized
- Let's examine the relationship between z and the slow roll parameters

Slow Roll parameters

- Returning to the Mukhanov equation

$$\ddot{u} + [k^2 + g(\eta)]u = 0$$

where

$$\begin{aligned} g(\eta) &\equiv f(\eta) + \epsilon_H - 2 \\ &= -\left(\frac{\dot{a}}{a}\right)^2 [2 + 3\delta_1 + 2\epsilon_H + (\delta_1 + \epsilon_H)(\delta_1 + 2\epsilon_H)] - \frac{\dot{a}}{a}\dot{\delta}_1 \\ &= -\frac{\ddot{z}}{z} \end{aligned}$$

and recall

$$z \equiv a \left(\frac{\dot{a}}{a}\right)^{-1} \dot{\phi}$$

Slow Roll Limit

- Slow roll $\epsilon_H \ll 1$, $\delta_1 \ll 1$, $\dot{\delta}_1 \ll \frac{\dot{a}}{a}$

$$\ddot{u} + [k^2 - 2\left(\frac{\dot{a}}{a}\right)^2]u = 0$$

or for conformal time measured from the end of inflation

$$\tilde{\eta} = \eta - \eta_{\rm end}$$

$$\tilde{\eta} = \int_{a_{\rm end}}^{a} \frac{da}{Ha^2} \approx -\frac{1}{aH}$$

- Compact, slow-roll equation:

$$\ddot{u} + [k^2 - \frac{2}{\tilde{\eta}^2}]u = 0$$

Quantum Fluctuations

- Simple harmonic oscillator $\ll$ Hubble length

$$\ddot{u} + k^2 u = 0$$

- Quantize the simple harmonic oscillator

$$\hat{u} = u(k, \tilde{\eta})\hat{a} + u^*(k, \tilde{\eta})\hat{a}^\dagger$$

where $u(k, \tilde{\eta})$ satisfies classical equation of motion and the creation and annihilation operators satisfy

$$[a, a^\dagger] = 1, \qquad a|0\rangle = 0$$

- Normalize wavefunction $[\hat{u}, d\hat{u}/d\tilde{\eta}] = i$

$$u(k, \eta) = \frac{1}{\sqrt{2k}} e^{-ik\tilde{\eta}}$$

Quantum Fluctuations

- Zero point fluctuations of ground state

$$
\begin{aligned}
\langle u^2 \rangle &= \langle 0 | u^\dagger u | 0 \rangle \\
&= \langle 0 | (u^* \hat{a}^\dagger + u \hat{a})(u \hat{a} + u^* \hat{a}^\dagger) | 0 \rangle \\
&= \langle 0 | \hat{a} \hat{a}^\dagger | 0 \rangle |u(k, \tilde{\eta})|^2 \\
&= \langle 0 | [\hat{a}, \hat{a}^\dagger] + \hat{a}^\dagger \hat{a} | 0 \rangle |u(k, \tilde{\eta})|^2 \\
&= |u(k, \tilde{\eta})|^2 = \frac{1}{2k}
\end{aligned}
$$

- Classical equation of motion take this quantum fluctuation outside horizon where it freezes in.

Slow Roll Limit

- Classical equation of motion then has the exact solution

$$u = \frac{1}{\sqrt{2k}} \left(1 - \frac{i}{k\tilde\eta} \right) e^{-ik\tilde\eta}$$

- For $|k\tilde\eta| \ll 1$ (late times, $\gg$ Hubble length) fluctuation freezes in

$$\lim_{|k\tilde\eta| \to 0} u = -\frac{1}{\sqrt{2k}} \frac{i}{k\tilde\eta} \approx \frac{iHa}{\sqrt{2k^3}}$$

$$\delta\phi = \frac{iH}{\sqrt{2k^3}}$$

- Power spectrum of field fluctuations

$$\Delta^2_{\delta\phi} = \frac{k^3 |\delta\phi|^2}{2\pi^2} = \frac{H^2}{(2\pi)^2}$$

Slow Roll Limit

- Recall $\mathcal{R} = -(\dot{a}/a)\delta\phi/\dot{\phi}$ and slow roll says

$$\left(\frac{\dot{a}}{a}\right)^2 \frac{1}{\dot{\phi}^2} = \frac{8\pi G a^2 V}{3} \frac{3}{2a^2 V \epsilon_H} = \frac{4\pi G}{\epsilon_H}$$

Thus the curvature power spectrum

$$\Delta^2_{\mathcal{R}} = \frac{8\pi G}{2} \frac{H^2}{(2\pi)^2 \epsilon_H}$$

- Curvature power spectrum is scale invariant to the extent that H is constant
- Scalar spectral index

$$\begin{aligned}\frac{d \ln \Delta^2_{\mathcal{R}}}{d \ln k} &\equiv n_S - 1 \\ &= 2\frac{d \ln H}{d \ln k} - \frac{d \ln \epsilon_H}{d \ln k}\end{aligned}$$

Tilt

- Evaluate at horizon crossing where fluctuation freezes

$$\frac{d\ln H}{d\ln k}\Big|_{-k\tilde\eta=1} = \frac{k}{H}\frac{dH}{d\tilde\eta}\Big|_{-k\tilde\eta=1}\frac{d\tilde\eta}{dk}\Big|_{-k\tilde\eta=1}$$
$$= \frac{k}{H}(-aH^2\epsilon_H)\Big|_{-k\tilde\eta=1}\frac{1}{k^2} = -\epsilon_H$$

where $aH = -1/\tilde\eta = k$

- Evolution of ϵ_H

$$\frac{d\ln\epsilon_H}{d\ln k} = -\frac{d\ln\epsilon_H}{d\ln\tilde\eta} = -2(\delta_1+\epsilon_H)\frac{\dot a}{a}\tilde\eta = 2(\delta_1+\epsilon_H)$$

- Tilt in the slow-roll approximation

$$n_S = 1 - 4\epsilon_H - 2\delta_1$$

Relationship to Potential

- Exact relations

$$\frac{1}{8\pi G}(\frac{V'}{V})^2 = 2\epsilon_H \frac{(1+\delta_1/3)^2}{(1-\epsilon_H/3)^2}$$

$$\frac{1}{8\pi G}\frac{V''}{V} = \frac{\epsilon_H - \delta_1 - [\delta_1^2 - \epsilon_H\delta_1 - (a/\dot{a})\dot{\delta}_1]/3}{1-\epsilon_H/3}$$

agree in the limit $\epsilon_H, |\delta_1| \ll 1$ and $|(a/\dot{a})\dot{\delta}_1| \ll \epsilon_H, |\delta_1|$

- Like the Mukhanov to slow roll simplification, identification with potential requires a constancy of δ_1 assumption

Gravitational Waves

- Gravitational wave amplitude satisfies Klein-Gordon equation $(K = 0)$, same as scalar field

$$\ddot{H}_T^{(\pm 2)} + 2\frac{\dot{a}}{a}\dot{H}_T^{(\pm 2)} + k^2 H_T^{(\pm 2)} = 0\,.$$

- Acquires quantum fluctuations in same manner as ϕ. Lagrangian sets the normalization

$$\delta\phi \to H_T^{(\pm 2)}\sqrt{\frac{3}{16\pi G}}$$

- Scale-invariant gravitational wave amplitude converted back to $+$ and $\times$ states $H_T^{(\pm 2)} = -(h_+ \mp i h_\times)/\sqrt{6})$

$$\Delta^2_{+,\times} = 16\pi G \Delta^2_{\delta\phi} = 16\pi G \frac{H^2}{(2\pi)^2}$$

Gravitational Waves

- Gravitational wave power $\propto H^2 \propto V \propto E_i^4$ where E_i is the energy scale of inflation
- Tensor-scalar ratio - various definitions - WMAP standard is

$$r \equiv 4\frac{\Delta_+^2}{\Delta_{\mathcal{R}}^2} = 16\epsilon_H$$

- Tensor tilt:

$$\frac{d\ln\Delta_H^2}{d\ln k} \equiv n_T = 2\frac{d\ln H}{d\ln k} = -2\epsilon_H$$

Gravitational Waves

- Consistency relation between tensor-scalar ratio and tensor tilt

$$r = 16\epsilon_H = -8n_T$$

- Measurement of scalar tilt and gravitational wave amplitude constrains inflationary potential
- Comparision of tensor-scalar ratio and tensor tilt tests the idea of canonical single field slow roll inflation itself

Gravitational Wave Phenomenology

- Equation of motion

$$\ddot{H}_T^{(\pm 2)} + 2\frac{\dot{a}}{a}\dot{H}_T^{(\pm 2)} + k^2 H_T^{(\pm 2)} = 0\,.$$

- has solutions

$$H_T^{(\pm 2)} = C_1 H_1(k\eta) + C_2 H_2(k\eta)$$

$$H_1 \propto x^{-m} j_m(x)$$

$$H_2 \propto x^{-m} n_m(x)$$

where $m = (1-3w)/(1+3w)$

- If $w > -1/3$ then gravity wave is constant above horizon $x \ll 1$ and then oscillates and damps
- If $w < -1/3$ then gravity wave oscillates and freezes into some value, just like scalar field

Gravitational Wave Phenomenology

- A gravitational wave makes a quadrupolar (transverse-traceless) distortion to metric
- Just like the scale factor or spatial curvature, a temporal variation in its amplitude leaves a residual temperature variation in CMB photons – here anisotropic
- Before recombination, anisotropic variation is eliminated by scattering
- Gravitational wave temperature effect drops sharply at the horizon scale at recombination

Gravitational Wave Phenomenology

- Source to polarization goes as $\dot{\tau}/\dot{H}_T$ and peaks at the horizon not damping scale
- More distinct signature in the polarization since symmetry of plane wave is broken by the transverse nature of gravity wave polarization
- B modes formed as photons propagate – the spatial variation in the plane waves modulate the signal: described by Boltzmann eqn.

$$\Delta B_{\rm peak} \approx 0.024 \left(\frac{E_i}{10^{16}{\rm GeV}} \right)^2 \mu{\rm K}$$

Large Field Models

- For detectable gravitational waves, ϵ_H cannot be too small
- A large ϵ_H means that the field rolls a substantial distance over the efolds of inlfation

$$\frac{d\phi}{dN} = \frac{d\phi}{d\ln a} = \frac{d\phi}{dt}\frac{1}{H}$$

- The larger ϵ_H is the more the field rolls in an e-fold

$$\epsilon_H = \frac{r}{16} = \frac{3}{2V}\left(H\frac{d\phi}{dN}\right)^2 = \frac{8\pi G}{2}\left(\frac{d\phi}{dN}\right)^2$$

- Observable scales span $\Delta N \sim 5$ so

$$\Delta\phi \approx 5\frac{d\phi}{dN} = 5(r/8)^{1/2} M_{\rm pl} \approx 0.6(r/0.1)^{1/2} M_{\rm pl}$$

- Difficult to keep the potential flat out to the Planck scale, or more generally out to the cutoff of an effective field theory Lyth (1997)

Large Field Models

- Large field models include monomial potentials $V(\phi) = A\phi^n$

$$\epsilon_H = \frac{r}{16} \approx \frac{n^2}{16\pi G\phi^2}$$

$$\delta_1 \approx \epsilon_H - \frac{n(n-1)}{8\pi G\phi^2}$$

$$n_s - 1 = -\frac{n(n+2)}{8\pi G\phi^2} = -\frac{2+n}{8n}r$$

- Thus $\epsilon_H \sim |\delta_1|$ and a observed finite tilt $n_s - 1 \sim -0.04$ indicates finite ϵ_H and observable gravitational waves
- Inflation rolls on specific trajectories of $n_s - 1, r$ plane depending on index n
- CMB-large scale structure efolds $\sim 50 - 60$ efolds select portions of these curves, which are tested by upper limits on r

Small Field Models

- If the field is near an maximum of the potential

$$V(\phi) = V_0 - \frac{1}{2}\mu^2\phi^2$$

- Inflation occurs if the V_0 term dominates

$$\epsilon_H \approx \frac{1}{16\pi G}\frac{\mu^4\phi^2}{V_0^2}$$

$$\delta_1 \approx \epsilon_H + \frac{1}{8\pi G}\frac{\mu^2}{V_0} \rightarrow \frac{\delta_1}{\epsilon_H} = \frac{V_0}{\mu^2\phi^2} \gg 1$$

- Tilt reflects δ_1: $n_S \approx 1 - 2\delta_1$ and ϵ_H is much smaller
- The field does not roll significantly during inflation and gravitational waves are negligible

Non-Gaussianity

- In single field slow roll inflation, the inflaton is nearly free field - modes don't interact - fluctuations are Gaussian to a high degree.
- Non-gaussianities are at best second order effects and with 10^{-5} fluctuations, this is a 10^{-10} effect!
- Local form follows from considering a generic second order effect that is where potential gains a piece that is quadratic in the linear Gaussian fluctuation

$$\Phi(\mathbf{x}) = \Phi^{(1)}(\mathbf{x}) + f_{\rm NL}[(\Phi^{(1)}(\mathbf{x}))^2 - \langle\Phi^{(1)}(\mathbf{x})\rangle^2]$$

called a local non-Gaussianity - generic prediction is $f_{\rm NL} = \mathcal{O}(1)$

Non-Gaussianity

- Decompose in harmonics (assume $k^2 \gg |K|$, nearly flat)

$$\Phi(\mathbf{k}) = \int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}}\Phi(\mathbf{x})$$

$$= \Phi^{(1)}(\mathbf{k}) + f_{\rm NL}\int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}} \int \frac{d^3k_1}{(2\pi)^3} e^{i\mathbf{k}_1\cdot\mathbf{x}}\Phi^{(1)}(\mathbf{k}_1) \int \frac{d^3k_2}{(2\pi)^3} e^{i\mathbf{k}_2\cdot\mathbf{x}}\Phi^{(1)}(\mathbf{k}_2)$$

with $\int d^3x e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}} = (2\pi)^3\delta(\mathbf{k}-\mathbf{k}')$ yields

$$\Phi(\mathbf{k}) = \Phi^{(1)}(\mathbf{k}) + f_{\rm NL}\int \frac{d^3k_1}{(2\pi)^3}\Phi^{(1)}(\mathbf{k}_1)\Phi^{(1)}(\mathbf{k}-\mathbf{k}_1)$$

- Given that the first order term is a Gaussian field represented by the power spectrum

$$\langle \Phi^{(1)}(\mathbf{k})\Phi^{(1)}(\mathbf{k}')\rangle = (2\pi)^3\delta(\mathbf{k}+\mathbf{k}')P_\Phi(k)$$

we get a bispectrum contribution

Non-Gaussianity

- The bispectrum is proportional to the product of power spectra

$$\langle \Phi(\mathbf{k}_1)\Phi(\mathbf{k}_2)\Phi(\mathbf{k}_3)\rangle = (2\pi)^3\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)[2f_{\rm NL}P_\Phi(k_1)P_\Phi(k_2) + {\rm perm}]$$

- Bispectrum contains most of the information on local non-Gaussianity
- Higher point correlations also exist and can be measured with good S/N but their sample variance is high and so don't help in constraining $f_{\rm NL}$

Non-Gaussianity

- For canonical single field slow roll inflation $f_{\rm NL}$ is of order $n_s - 1$, simply understood from separate universe perspective
- In the squeezed limit where $k_1 \ll k_2 \sim k_3$, the short wavelength modes see separate universe with perturbed scale factor - i.e. the local background field is slightly perturbed from global value
- Change in the power spectrum given a change in efolds is given by $n_s - 1$ - so short wavelength power is correlated with the long wavelength scale factor perturbation or curvature

$$f_{\rm NL} = 5(1 - n_s)/12$$

- To get a larger non-Gaussianity requires multifield or non-canonical inflation
- Conversely, observation of primordial non Gaussianity at the $|f_{\rm NL}| > 1$ level rules out canonical single field, slow roll inflation

Effective Field Theory

- Beyond canonical scalar fields, we can think of the scalar as simply a clock which decides when inflation ends
- Choose the unitary gauge as constant field gauge to all orders

$$\phi(\mathbf{x}, t_u) = \phi(t_u)$$

- Given that by assumption the universe is dominated by this scalar field and it is homogenous in this frame, the only thing that the action can be built out of is terms that depend on t_u
- In the EFT language, write down all possible terms that is consistent with unbroken spatial diffeomorphism invariance in this slicing
- EFT is an organizational principle that highlights the generality of certain relations, e.g. between non-Gaussianity and sound speed

Effective Field Theory

- Recall that in unitary gauge, we can build the Lagrangian from just the metric or ADM objects in 3+1: lapse N, extrinsic curvature K_{ij}, intrinsic curvature R_{ij}, and covariant 3-derivatives with respect to the spatial metric h_{ij}

$$\mathcal{L}(N, K^i{}_j, R^i{}_j, \nabla^i; t)$$

  where the function is constructed out of spatial scalars.

- This encompasses single field models with one extra DOF and second order (in time) EOMs except for special degenerate cases

- To make considerations concrete focus on the freedom associated with GR + $\mathcal{L}(N; t)$

Effective Field Theory

- This simplest class of EFT operators corresponds to scalar field Lagrangians of $\mathcal{L} = P(X, \phi)$ class where the kinetic term

$$X = g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$$

  becomes $X = -\dot{\phi}^2/N^2$ and $\phi = \phi(t)$ in the background

- For a canonical kinetic term $P(X, \phi) = -X/2 + V(\phi)$
- Unitary gauge and comoving gauge coincide this class since $T^0_{\ i} \propto \delta\phi$ (but not all Horndeski models)
- To stay close to the inflationary literature, 0 will represent coordinate rather than conformal time
- Two equivalent ways of proceeding: stay in unitary gauge and write the action for the metric, Stuckelberg to restore gauge invariance and work in gauge where the (Stuckelberg field) ϕ carries the dynamics

Unitary Gauge Action

- Expand the action

$$S = \int d^4x \sqrt{-g}\mathcal{L} = \int d^4x N\sqrt{h}\mathcal{L}$$

  in the ADM metric fluctuations around FRW: N, shift N^i and spatial metric

$$h_{ij} = a^2 e^{2\mathcal{R}}(\gamma_{ij} + h_{+ij} + h_{\times ij})$$

  Note that ADM applies beyond linear theory so this completely defines the scalar and tensor metric in unitary gauge and coincides with the linear metric fluctuations to leading order

- For simplicity we will assume that the FRW curvature is negligible during inflation

Unitary Gauge Action

- Varying the action leads to background EOMs for linear action (lapse and spatial metric a)

$$\mathcal{L} + \mathcal{L}_{,N} + 6H^2 = 0$$
$$\mathcal{L} - 2\epsilon_H H^2 + 6H^2 = 0$$

  where $\epsilon_H = -d\ln H/d\ln a$ expresses the deviation from a deSitter expansion.

- For the EH part of the action $\mathcal{L}_{\rm EH} = -3H^2$
- For the additional contributions, $\mathcal{L}$ defines a pressure and $-\mathcal{L}_{,N} - \mathcal{L}$ defines the energy density so these are just the Friedmann equations as expected

Curvature Fluctuations

- Similarly the EOMs for the linear fluctuations $A, B, \mathcal{R}, h_+, h_\times$. for the quadratic action
- Action does not involve time derivatives of the lapse or shift so at each order their EOMs produce constraints
- For the scalars, this leaves $\mathcal{R}$ as the dynamical variable

$$S_2 = \int d^4x \frac{a^3 \epsilon_H}{c_s^2} (\dot{\mathcal{R}}^2 + \frac{c_s^2 k^2}{a^2} \mathcal{R}^2)$$

where and the scalar sound speed is

$$c_s^2 = -\frac{L_{,N}}{L_{,NN} + 2L_{,N}}$$

Curvature Fluctuations

- Covariant formulation the lapse dependence comes from $P(X, \phi)$ where $X = -\dot{\phi}^2(t)/N^2$, so transforming $\partial \ln X = -2\partial \ln N$, we have

$$c_s^2 = \frac{P_{,X}}{2XP_{,XX} + P_{,X}}$$

which recovers the well known k-essence result.

- Canonical scalar has $P(X, \phi) = -X/2 - V(\phi)$ and so $c_s^2 = 1$
- We can continue this beyond the quadratic action - leads to nonlinearity in the EOMs and non-Gaussianity in their solution
- We shall see that non-Gaussianity is enhanced if $c_s < 1$

Tensor Fluctuations

- Likewise for tensors

$$S_2 = \int d^4x \frac{a^3}{4} \left(\dot{h}^2_{+,\times} + \frac{k^2}{a^2} h^2_{+,\times} \right)$$

- Gravitational waves travel at the speed of light if $\mathcal{L} = \mathcal{L}_{\rm EH} + P(X, \phi)$
- Tensor speed can change in the general EFT Lagrangian through non EH dependence on R_{ij}
- Scalars and tensors differ in the appearance of ϵ_H in the scalar action

Stuckelberg Restoration

- $P(X,\phi)$ also provides an illustration of how the scalar is reintroduced by the so called "Stuckelberg" trick as a field that restores gauge invariance (temporal diffeomorphisms)
- Alternate to unitary gauge is to instead transform to spatially flat gauge where the ADM metric has no dynamics.
- Clarifies the origin of the ϵ_H difference in the scalar action
- EFT of inflation originally formulated in this language and with $g^{00} = -1/N^2$ so we will switch notion below.
- This language is also convenient for showing how the cubic action or non-Gaussianity of inflation

Effective Field Theory

- Now consider that $g^{00}+1$ is a small metric perturbation. A general function of the lapse may be expanded around this value in a Taylor series

$$S = \int d^4x\sqrt{-g}\Big[\frac{1}{2}M_{\rm Pl}^2 R + \sum_{n=0}^{\infty}\frac{1}{n!}M_n^4(t_u)(g_u^{00}+1)^n\Big],$$

- Varying action with respect to $g^{\mu\nu}$ we get the Einstein equations
- Constant term gives a cosmological constant whereas the $n=1$ term gives the effective stress tensor of the field in the background

$$\begin{aligned} H^2 &= -\frac{1}{3M_{\rm Pl}^2}\left[M_0^4+2M_1^4\right] \\ \dot H + H^2 &= -\frac{1}{3M_{\rm Pl}^2}\left[M_0^4-M_1^4\right] \end{aligned}$$

where recall $\dot H = -H^2\epsilon_H$

Effective Field Theory

- Friedmann equation can thus eliminate $n = 0, 1$

$$\begin{aligned} M_0^4 &= -(3-2\epsilon_H)H^2 M_{\rm Pl}^2 \\ M_1^4 &= -H^2\epsilon_H M_{\rm Pl}^2 \end{aligned}$$

- Now we an restore time slicing invariance or temporal diffs allowing for a general change in the time coordinate

$$t_u = t + \pi(t, x^i)$$

- In particle physics language this is the Stuckelberg trick and π is a Stuckelberg field.
- Transformation to arbitrary slicing is given by

$$g_u^{00} = \frac{\partial t_u}{\partial x^\mu}\frac{\partial t_u}{\partial x^\nu} g^{\mu\nu}$$

- Each $M_n^4(t_u = t + \pi)$ and hence carry extra Taylor expansion terms

Effective Field Theory

- In general, transformation mixes π and metric fluctuations $\delta g^{\mu\nu}$ including terms like

$$\dot{\pi}\delta g^{00}, \quad \delta g \dot{\pi}, \quad \partial_i \pi g^{0i}, \quad \partial_i \pi \partial_j \pi \delta g^{ij}$$

  in the canonical linear theory calculation, the first three were the $\dot{A}$, $\dot{H}_L$, kB terms after integration by parts and the last is cubic order

- The lapse and shift are non-dynamical for the class of EFT we consider including $P(X, \phi)$, (beyond) Horndeski, so the most useful transformation to consider is to spatially flat gauge to eliminate dynamics in the spatial metric

- For this case, the gauge transformation of the curvature fluctuation tells us $\pi = -\mathcal{R}/H$

- We thus expect to recover the action for $\mathcal{R}$ from the action for π

Effective Field Theory

- In fact on scales below the horizon in most gauges the field fluctuations reduce to spatially flat gauge since curvature effects are negligible
- Spatially flat gauge extends the domain of validity extends even through the horizon if we neglect slow roll corrections
- In this case we can ignore the terms associated with the spatial pieces of the metric and replace

$$g_u^{00} = -(1+\dot{\pi})^2 + \frac{(\partial_i \pi)^2}{a^2}$$

- Each $g_u^{00} + 1$ factor carries terms that are linear and quadratic in π

$$(g_u^{00}+1)^n = (-\dot{\pi})^n \sum_{i=0}^{n} \frac{2^{n-i} n!}{i!(n-i)!} \Pi^i$$

Effective Field Theory

- So each M_n^4 term contributes from π^n to π^{2n}

$$\Pi = \dot\pi\left(1 - \frac{(\partial_i \pi)^2}{a^2\dot\pi^2}\right)$$

- For example M_2

$$\begin{aligned}(g_u^{00}+1)^2 &= \dot\pi^2\left[4 + 4\dot\pi\left(1 - \frac{(\partial_i\pi)^2}{a^2\dot\pi^2}\right) + \dot\pi^2\left(1 - \frac{(\partial_i\pi)^2}{a^2\dot\pi^2}\right)^2\right] \\ &= 4(\dot\pi^2 + \dot\pi^3 - \dot\pi\frac{(\partial_i\pi)^2}{a^2}) + \ldots\end{aligned}$$

implies both a cubic and quartic Lagrangian. To cubic order

$$S_\pi = \int d^4x\sqrt{-g}\Big[M_{\rm Pl}^2\epsilon_H H^2\left(\dot\pi^2 - \frac{(\partial_i\pi)^2}{a^2}\right) + 2M_2^4(\dot\pi^2 + \dot\pi^3 - \dot\pi\frac{(\partial_i\pi)^2}{a^2}) + \ldots$$

Effective Field Theory

- Isolate the quadratic action

$$S_{\pi 2} = \int d^4x\sqrt{-g}\left[(M_{\rm Pl}^2 H^2\epsilon_H + 2M_2^4)\dot\pi^2 + M_{\rm Pl}^2\dot H\frac{(\partial_i\pi)^2}{a^2}\right]$$

and identify the sound speed from $\omega = (k/a)c_s$

$$c_s^{-2} = 1 + \frac{2M_2^4}{M_{\rm Pl}^2\epsilon_H H^2}; \quad \Pi \sim \dot\pi\left(1 - \frac{1}{c_s^2}\right)$$

$$\begin{aligned} S_{\pi 2} &= \int dt d^3x (a^3\epsilon_H H^2) M_{\rm Pl}^2 c_s^{-2}\left[\dot\pi^2 - c_s^2\frac{(\partial_i\pi)^2}{a^2}\right] \\ &= \int d\eta d^3x \frac{z^2 H^2 M_{\rm Pl}^2}{2}\left[\left(\frac{\partial\pi}{\partial\eta}\right)^2 - c_s^2(\partial_i\pi)^2\right] \end{aligned}$$

where $z^2 = 2a^2\epsilon_H/c_s^2$ is known as the Mukhanov variable

Effective Field Theory

- So a field redefinition canonically normalizes the field

$$u = zH\pi M_{\rm Pl}$$

brings the EFT action to canonical form (assuming M_n^4 =const.)

$$\begin{aligned} S_u &= \int d\eta d^3x \left[\left(\frac{\partial u}{\partial \eta}\right)^2 - c_s^2(\partial_i u)^2 - 2u\frac{\partial u}{\partial \eta}\frac{d\ln z}{d\eta} + u^2\left(\frac{d\ln z}{d\eta}\right)^2 \right] \\ &= \int d\eta d^3x \left[\left(\frac{\partial u}{\partial \eta}\right)^2 - c_s^2(\partial_i u)^2 + \frac{u^2}{z}\frac{d^2 z}{d\eta} \right] \end{aligned}$$

which is the generalization of the u field of canonical inflation

- Quantize this field, noting that $1/\sqrt{E}$ normalization factor goes to $1/\sqrt{kc_s}$ yielding the modefunction

$$u = \frac{1}{\sqrt{2kc_s}}\left(1 - \frac{i}{kc_s\tilde\eta}\right)e^{-ikc_s\tilde\eta}$$

Effective Field Theory

- Curvature fluctuations then freezeout at $kc_s\tilde\eta = 1$ (sound horizon crossing) at a value

$$\mathcal{R} = -H\pi = \frac{c_s}{a\sqrt{2\epsilon_H}} \frac{1}{\sqrt{2kc_s}} \frac{i}{kc_s\tilde\eta M_{\rm Pl}} \approx \frac{-iH}{2k^{3/2}\sqrt{\epsilon_H c_s} M_{\rm Pl}}$$

- So

$$\Delta^2_{\mathcal{R}} = \frac{k^3|\mathcal{R}|^2}{2\pi^2} = \frac{H^2}{8\pi^2\epsilon_H c_s M_{\rm Pl}^2}$$

- Generalization is that the sound speed enters in two ways: boosts scalars over tensors by c_s and changes the epoch of freezeout between scalars and tensors

Effective Field Theory

- Returning to the original π action, since M_2^4 carries cubic term this requires a non-Gaussianity

$$S_\pi = \int d^4x \sqrt{-g}\left[-\frac{M_{\rm Pl}^2 \dot H}{c_s^2}\left(\dot\pi^2 + c_s^2 \frac{(\partial_i \pi)^2}{a^2}\right) + M_{\rm Pl}^2 \dot H \left(1 - \frac{1}{c_s^2}\right)\left(\dot\pi^3 - \dot\pi \frac{(\partial_i\pi)^2}{a^2}\right)\right] + \ldots$$

- For $c_s \ll 1$, spatial gradients dominate temporal derivatives

$$\partial_0 \to \omega, \partial_i \to k, \omega = k c_s/a$$

and leading order cubic term is $\dot\pi(\partial_i\pi)^2$

- Estimate the size of the non-Gaussianity by taking the ratio of cubic to quadratic at $c_s \ll 1$

$$\frac{\dot\pi(\partial_i\pi)^2}{a^2\dot\pi^2} \sim \frac{k\pi_{\rm rms}}{c_s a} \qquad \text{where} \quad \pi_{\rm rms} = \left(\frac{k^3|\pi|^2}{2\pi^2}\right)^{1/2}$$

Effective Field Theory

- Deep within the horizon $u = 1/\sqrt{2kc_s}$ and so

$$
\begin{aligned}
\frac{k\pi_{\rm rms}}{c_s a} &\sim \frac{k}{c_s a}\left(\frac{k^2}{2z^2H^2c_sM_{\rm Pl}^2}\right)^{1/2} \\
&\sim \left(\frac{kc_s}{aH}\right)^2 \frac{H}{M_{\rm Pl}\sqrt{\epsilon_H c_s}}\frac{1}{c_s^2} \\
&\sim \left(\frac{kc_s}{aH}\right)^2 \frac{\Delta_{\mathcal{R}}}{c_s^2} < 1
\end{aligned}
$$

- Since $kc_s/aH \sim \omega/H$ is a ratio of an energy scale to Hubble, the bound determines the strong coupling scale

$$
\frac{\omega_{sc}}{H} \sim \frac{c_s}{\sqrt{\Delta_{\mathcal{R}}}} \sim 10^2 c_s
$$

- For $c_s < 0.01$ the strong coupling scale is near the horizon and the effective theory has broken down before freezeout

Effective Field Theory

- Now consider a less extreme c_s
- Here the effective theory becomes valid at least several efolds before horizon crossing and we can make predictions within the theory
- Not surprisingly non-Gaussianity is enhanced by these self interactions and freezeout at $kc_s \sim aH$

$$
\begin{aligned}
\frac{k\pi_{\rm rms}}{c_s a} &\sim \frac{k}{c_s a}\left(\frac{1}{\epsilon_H c_s M_{\rm Pl}^2}\right)^{1/2} \\
&\sim \frac{kc_s}{aH}\frac{H}{\sqrt{\epsilon_H c_s}M_{\rm Pl}}\frac{1}{c_s^2} \\
&\sim \frac{\Delta_{\mathcal{R}}}{c_s^2}
\end{aligned}
$$

and so bispectrum is enhanced over the naive expectation by c_s^{-2}

Effective Field Theory

- More generally, each M_n^4 sets its own strong coupling scale

$$\frac{\mathcal{L}_n}{\mathcal{L}_2} \sim 1$$

These coincide if

$$\frac{M_n^4}{M_2^4} \sim \left(\frac{1}{c_s^2}\right)^{n-2}$$

which would be the natural prediction if the M_2 strong coupling scale indicated the scale of new physics and we take all allowed operators as order unity at that scale