

Astro 4PT

Lecture Notes Set 1

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References

- Relativistic Cosmological Perturbation Theory
- Inflation
- Dark Energy
- Modified Gravity
- Cosmic Microwave Background
- Large Scale Structure
- Bardeen (1980), PRD 22 1882
- Kodama & Sasaki (1984), Prog. Th. Phys. Supp., 78, 1
- Mukhanov, Feldman, Brandenberger (1992), Phys. Reports, 215, 203
- Malik & Wands (2009), Phys. Reports, 475, 1

Covariant Perturbation Theory

- Covariant = takes same form in all coordinate systems
- Invariant = takes the same value in all coordinate systems
- Fundamental equations: Einstein equations, covariant conservation of stress-energy tensor:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$
$$\nabla_{\mu} T^{\mu\nu} = 0$$

- Preserve general covariance by keeping all free variables: 10 for each symmetric 4×4 tensor

1	2	3	4
	5	6	7
		8	9
			10

Metric Tensor

- Useful to think in a $3 + 1$ language since there are preferred spatial surfaces where the stress tensor is nearly homogeneous
- In general this is an Arnowitt-Deser-Misner (ADM) split
- Specialize to the case of a nearly **FRW metric**

$$g_{00} = -a^2, \quad g_{ij} = a^2 \gamma_{ij} .$$

where the “0” component is **conformal time** $\eta = dt/a$ and γ_{ij} is a **spatial metric of constant curvature** $K = H_0^2(\Omega_{\text{tot}} - 1)$.

$${}^{(3)}R = \frac{6K}{a^2}$$

Metric Tensor

- First define the slicing (lapse function A , shift function B^i)

$$g^{00} = -a^{-2}(1 - 2A),$$

$$g^{0i} = -a^{-2}B^i,$$

A defines the lapse of proper time between 3-surfaces whereas B^i defines the threading or relationship between the 3-coordinates of the surfaces

- This absorbs $1+3=4$ free variables in the metric, remaining 6 is in the spatial surfaces which we parameterize as

$$g^{ij} = a^{-2}(\gamma^{ij} - 2H_L\gamma^{ij} - 2H_T^{ij}).$$

here (1) H_L a perturbation to the spatial curvature; (5) H_T^{ij} a trace-free distortion to spatial metric (which also can perturb the curvature)

Curvature Perturbation

- Curvature perturbation on the 3D slice

$$\delta^{(3)}R = -\frac{4}{a^2} (\nabla^2 + 3K) H_L + \frac{2}{a^2} \nabla_i \nabla_j H_T^{ij}$$

- Note that we will often loosely refer to H_L as the “curvature perturbation”
- We will see that many representations have $H_T = 0$
- It is easier to work with a dimensionless quantity
- First example of a 3-scalar - SVT decomposition

Matter Tensor

- Likewise expand the matter stress energy tensor around a homogeneous density ρ and pressure p :

$$T^0_0 = -\rho - \delta\rho,$$

$$T^0_i = (\rho + p)(v_i - B_i),$$

$$T_0^i = -(\rho + p)v^i,$$

$$T^i_j = (p + \delta p)\delta^i_j + p\Pi^i_j,$$

- (1) $\delta\rho$ a density perturbation; (3) v_i a vector velocity, (1) δp a pressure perturbation; (5) Π_{ij} an anisotropic stress perturbation
- So far this is fully general and applies to any type of matter or coordinate choice including non-linearities in the matter, e.g. scalar fields, cosmological defects, exotic dark energy.

Counting Variables

20	Variables (10 metric; 10 matter)
-10	Einstein equations
-4	Conservation equations
+4	Bianchi identities
-4	Gauge (coordinate choice 1 time, 3 space)
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6	Free Variables

- Without loss of generality these can be taken to be the 6 components of the matter stress tensor
- For the background, specify $p(a)$ or equivalently $w(a) \equiv p(a)/\rho(a)$ the equation of state parameter.

Homogeneous Einstein Equations

- Einstein (Friedmann) equations:

$$\left(\frac{1}{a} \frac{da}{dt}\right)^2 = -\frac{K}{a^2} + \frac{8\pi G}{3} \rho \quad \left[= \left(\frac{1}{a} \frac{\dot{a}}{a}\right)^2\right]$$
$$\frac{1}{a} \frac{d^2 a}{dt^2} = -\frac{4\pi G}{3} (\rho + 3p) \quad \left[= \frac{1}{a^2} \frac{d}{d\eta} \frac{\dot{a}}{a}\right]$$

so that $w \equiv p/\rho < -1/3$ for acceleration

- Conservation equation $\nabla^\mu T_{\mu\nu} = 0$ implies

$$\frac{\dot{\rho}}{\rho} = -3(1 + w) \frac{\dot{a}}{a}$$

overdots are conformal time but equally true with coordinate time

Homogeneous Einstein Equations

- Counting exercise:

20	Variables (10 metric; 10 matter)
−17	Homogeneity and Isotropy
−2	Einstein equations
−1	Conservation equations
+1	Bianchi identities
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1	Free Variables

without loss of generality choose ratio of homogeneous & isotropic component of the **stress tensor** to the density $w(a) = p(a)/\rho(a)$.

Acceleration Implies Negative Pressure

- Role of stresses in the background cosmology
- Homogeneous Einstein equations $G_{\mu\nu} = 8\pi GT_{\mu\nu}$ imply the two Friedmann equations (flat universe, or associating curvature $\rho_K = -3K/8\pi Ga^2$)

$$\left(\frac{1}{a} \frac{da}{dt}\right)^2 = \frac{8\pi G}{3} \rho$$
$$\frac{1}{a} \frac{d^2 a}{dt^2} = -\frac{4\pi G}{3} (\rho + 3p)$$

so that the total equation of state $w \equiv p/\rho < -1/3$ for acceleration

- Conservation equation $\nabla^\mu T_{\mu\nu} = 0$ implies

$$\frac{\dot{\rho}}{\rho} = -3(1 + w) \frac{\dot{a}}{a}$$

so that ρ must scale more slowly than a^{-2}

Scalar, Vector, Tensor

- In linear perturbation theory, perturbations may be separated by their transformation properties under 3D rotation and translation.
- The eigenfunctions of the Laplacian operator form a complete set

$$\begin{aligned}\nabla^2 Q^{(0)} &= -k^2 Q^{(0)} & \mathbf{S}, \\ \nabla^2 Q_i^{(\pm 1)} &= -k^2 Q_i^{(\pm 1)} & \mathbf{V}, \\ \nabla^2 Q_{ij}^{(\pm 2)} &= -k^2 Q_{ij}^{(\pm 2)} & \mathbf{T},\end{aligned}$$

- Vector and tensor modes satisfy divergence-free and transverse-traceless conditions

$$\nabla^i Q_i^{(\pm 1)} = 0$$

$$\nabla^i Q_{ij}^{(\pm 2)} = 0$$

$$\gamma^{ij} Q_{ij}^{(\pm 2)} = 0$$

Vector and Tensor Quantities

- A scalar mode carries with it associated vector (curl-free) and tensor (longitudinal) quantities
- A vector mode carries and associated tensor (trace and divergence free) quantities
- A tensor mode has only a tensor (trace and divergence free)
- These are built from the mode basis out of covariant derivatives and the metric

$$Q_i^{(0)} = -k^{-1} \nabla_i Q^{(0)},$$

$$Q_{ij}^{(0)} = (k^{-2} \nabla_i \nabla_j + \frac{1}{3} \gamma_{ij}) Q^{(0)},$$

$$Q_{ij}^{(\pm 1)} = -\frac{1}{2k} [\nabla_i Q_j^{(\pm 1)} + \nabla_j Q_i^{(\pm 1)}],$$

Perturbation k -Modes

- For the k th eigenmode, the scalar components become

$$\begin{aligned} A(\mathbf{x}) &= A(k) Q^{(0)}, & H_L(\mathbf{x}) &= H_L(k) Q^{(0)}, \\ \delta\rho(\mathbf{x}) &= \delta\rho(k) Q^{(0)}, & \delta p(\mathbf{x}) &= \delta p(k) Q^{(0)}, \end{aligned}$$

the vectors components become

$$B_i(\mathbf{x}) = \sum_{m=-1}^1 B^{(m)}(k) Q_i^{(m)}, \quad v_i(\mathbf{x}) = \sum_{m=-1}^1 v^{(m)}(k) Q_i^{(m)},$$

and the tensors components

$$H_{Tij}(\mathbf{x}) = \sum_{m=-2}^2 H_T^{(m)}(k) Q_{ij}^{(m)}, \quad \Pi_{ij}(\mathbf{x}) = \sum_{m=-2}^2 \Pi^{(m)}(k) Q_{ij}^{(m)},$$

- Note that the curvature perturbation only involves scalars

$$\delta^{(3)}R = \frac{4}{a^2} (k^2 - 3K) (H_L^{(0)} + \frac{1}{3} H_T^{(0)}) Q^{(0)}$$

Spatially Flat Case

- For a spatially flat background metric, harmonics are related to plane waves:

$$Q^{(0)} = \exp(i\mathbf{k} \cdot \mathbf{x})$$

$$Q_i^{(\pm 1)} = \frac{-i}{\sqrt{2}} (\hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2)_i \exp(i\mathbf{k} \cdot \mathbf{x})$$

$$Q_{ij}^{(\pm 2)} = -\sqrt{\frac{3}{8}} (\hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2)_i (\hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2)_j \exp(i\mathbf{k} \cdot \mathbf{x})$$

where $\hat{\mathbf{e}}_3 \parallel \mathbf{k}$. Chosen as spin states, c.f. polarization.

- For vectors, the harmonic points in a direction orthogonal to \mathbf{k} suitable for the **vortical component** of a vector

Spatially Flat Case

- Tensor harmonics are the transverse traceless gauge representation
- Tensor amplitude related to the more traditional

$$h_+ [(\mathbf{e}_1)_i (\mathbf{e}_1)_j - (\mathbf{e}_2)_i (\mathbf{e}_2)_j], \quad h_\times [(\mathbf{e}_1)_i (\mathbf{e}_2)_j + (\mathbf{e}_2)_i (\mathbf{e}_1)_j]$$

as

$$h_+ \pm i h_\times = -\sqrt{6} H_T^{(\mp 2)}$$

- $H_T^{(\pm 2)}$ proportional to the right and left circularly polarized amplitudes of gravitational waves with a normalization that is convenient to match the scalar and vector definitions

Covariant Scalar Equations

- DOF counting exercise

8 Variables (4 metric; 4 matter)

−4 Einstein equations

−2 Conservation equations

+2 Bianchi identities

−2 Gauge (coordinate choice 1 time, 1 space)

2 Free Variables

without loss of generality choose scalar components of the **stress tensor** $\delta p, \Pi$.

Covariant Scalar Equations

- Einstein equations (suppressing 0) superscripts

$$\begin{aligned} (k^2 - 3K)[H_L + \frac{1}{3}H_T] - 3\left(\frac{\dot{a}}{a}\right)^2 A + 3\frac{\dot{a}}{a}\dot{H}_L + \frac{\dot{a}}{a}kB &= \\ &= 4\pi Ga^2\delta\rho, \quad \text{00 Poisson Equation} \end{aligned}$$

$$\begin{aligned} k^2\left(A + H_L + \frac{1}{3}H_T\right) + \left(\frac{d}{d\eta} + 2\frac{\dot{a}}{a}\right)(kB - \dot{H}_T) &= \\ &= -8\pi Ga^2 p\Pi, \quad \text{ij Anisotropy Equation} \end{aligned}$$

$$\begin{aligned} \frac{\dot{a}}{a}A - \dot{H}_L - \frac{1}{3}\dot{H}_T - \frac{K}{k^2}(kB - \dot{H}_T) &= \\ &= 4\pi Ga^2(\rho + p)(v - B)/k, \quad \text{0i Momentum Equation} \end{aligned}$$

$$\begin{aligned} \left[2\frac{\ddot{a}}{a} - 2\left(\frac{\dot{a}}{a}\right)^2 + \frac{\dot{a}}{a}\frac{d}{d\eta} - \frac{k^2}{3}\right]A - \left[\frac{d}{d\eta} + \frac{\dot{a}}{a}\right](\dot{H}_L + \frac{1}{3}kB) &= \\ &= 4\pi Ga^2\left(\delta p + \frac{1}{3}\delta\rho\right), \quad \text{ii Acceleration Equation} \end{aligned}$$

Covariant Scalar Equations

- Conservation equations: continuity and Navier Stokes

$$\left[\frac{d}{d\eta} + 3\frac{\dot{a}}{a} \right] \delta\rho + 3\frac{\dot{a}}{a} \delta p = -(\rho + p)(kv + 3\dot{H}_L),$$
$$\left[\frac{d}{d\eta} + 4\frac{\dot{a}}{a} \right] \left[(\rho + p) \frac{(v - B)}{k} \right] = \delta p - \frac{2}{3} \left(1 - 3\frac{K}{k^2} \right) p\Pi + (\rho + p)A,$$

- Equations are not independent since $\nabla_\mu G^{\mu\nu} = 0$ via the Bianchi identities.
- Related to the ability to choose a coordinate system or “gauge” to represent the perturbations.

Covariant Vector Equations

- Einstein equations

$$\begin{aligned}(1 - 2K/k^2)(kB^{(\pm 1)} - \dot{H}_T^{(\pm 1)}) \\ &= 16\pi Ga^2(\rho + p)(v^{(\pm 1)} - B^{(\pm 1)})/k, \\ \left[\frac{d}{d\eta} + 2\frac{\dot{a}}{a} \right] (kB^{(\pm 1)} - \dot{H}_T^{(\pm 1)}) \\ &= -8\pi Ga^2 p \Pi^{(\pm 1)}.\end{aligned}$$

- Conservation Equations

$$\begin{aligned}\left[\frac{d}{d\eta} + 4\frac{\dot{a}}{a} \right] [(\rho + p)(v^{(\pm 1)} - B^{(\pm 1)})/k] \\ &= -\frac{1}{2}(1 - 2K/k^2)p\Pi^{(\pm 1)},\end{aligned}$$

- Gravity provides **no source** to vorticity \rightarrow **decay**

Covariant Vector Equations

- DOF counting exercise

8 Variables (4 metric; 4 matter)

−4 Einstein equations

−2 Conservation equations

+2 Bianchi identities

−2 Gauge (coordinate choice 1 time, 1 space)

2 Free Variables

without loss of generality choose vector components of the **stress tensor** $\Pi^{(\pm 1)}$.

Covariant Tensor Equation

- Einstein equation

$$\left[\frac{d^2}{d\eta^2} + 2\frac{\dot{a}}{a} \frac{d}{d\eta} + (k^2 + 2K) \right] H_T^{(\pm 2)} = 8\pi G a^2 p \Pi^{(\pm 2)}.$$

- DOF counting exercise

4 Variables (2 metric; 2 matter)

−2 Einstein equations

−0 Conservation equations

+0 Bianchi identities

−0 Gauge (coordinate choice 1 time, 1 space)

2 Free Variables

wlog choose tensor components of the **stress tensor** $\Pi^{(\pm 2)}$.

Arbitrary Dark Components

- Total stress energy tensor can be broken up into **individual pieces**
- **Dark components** interact only through gravity and so satisfy **separate conservation equations**
- Einstein equation source remains the sum of components.
- To specify an arbitrary dark component, give the behavior of the **stress tensor: 6 components: $\delta p, \Pi^{(i)}$** , where $i = -2, \dots, 2$.
- Many types of dark components (dark matter, scalar fields, massive neutrinos,..) have **simple forms** for their stress tensor in terms of the energy density, i.e. described by **equations of state**.
- An equation of state for the background $w = p/\rho$ is **not sufficient** to determine the behavior of the perturbations.

Separate Universes

- Geometry of the gauge or time slicing and spatial threading
- For perturbations larger than the horizon, a local observer should just see a different (separate) FRW universe
- Scalar equations should be equivalent to an appropriately remapped Friedmann equation
- Unit normal vector N^μ to constant time hypersurfaces
 $N_\mu dx^\mu = N_0 d\eta$, $N^\mu N_\mu = -1$, to linear order in metric

$$\begin{aligned} N_0 &= -a(1 + AQ), & N_i &= 0 \\ N^0 &= a^{-1}(1 - AQ), & N^i &= -BQ^i \end{aligned}$$

- Expansion of spatial volume per proper time is given by 4-divergence

$$\nabla_\mu N^\mu \equiv \theta = 3H(1 - AQ) + \frac{k}{a}BQ + \frac{3}{a}\dot{H}_L Q$$

Shear and Acceleration

- Other pieces of $\nabla_\nu N_\mu$ give the vorticity, shear and acceleration

$$\nabla_\nu N_\mu \equiv \omega_{\mu\nu} + \sigma_{\mu\nu} + \frac{1}{3}\theta P_{\mu\nu} - a_\mu N_\nu$$

with

$$P_{\mu\nu} = g_{\mu\nu} + N_\mu N_\nu$$

$$\omega_{\mu\nu} = P_\mu^\alpha P_\nu^\beta (\nabla_\beta N_\alpha - \nabla_\alpha N_\beta)$$

$$\sigma_{\mu\nu} = \frac{1}{2} P_\mu^\alpha P_\nu^\beta (\nabla_\beta N_\alpha + \nabla_\alpha N_\beta) - \frac{1}{3} \theta P_{\mu\nu}$$

$$a_\mu = (\nabla_\alpha N_\mu) N^\alpha$$

projection $P_{\mu\nu} N^\nu = 0$, trace free antisymmetric vorticity,
symmetric shear and acceleration

Shear and Acceleration

- Vorticity $\omega_{\mu\nu} = 0$, $\sigma_{00} = \sigma_{0i} = 0 = a_0$
- Remaining perturbed quantities are the spatial shear and acceleration

$$\begin{aligned}\sigma_{ij} &= a(\dot{H}_T - kB)Q_{ij} \\ a_i &= -kAQ_i\end{aligned}$$

- A convenient choice of coordinates might be shear free
 $\dot{H}_T - kB = 0$
- A alone is related to the perturbed acceleration

Separate Universes

- So the e-foldings of the expansion are given by $d\tau = (1 + AQ)ad\eta$

$$\begin{aligned} N &= \int d\tau \frac{1}{3}\theta \\ &= \int d\eta \left(\frac{\dot{a}}{a} + \dot{H}_L Q + \frac{1}{3}kBQ \right) \end{aligned}$$

Thus if kB can be ignored as $k \rightarrow 0$ then H_L plays the role of a local change in the scale factor, more generally B plays the role of Eulerian \rightarrow Lagrangian coordinates.

- Change in H_L between separate universes related to change in number of e-folds: so called δN approach, simplifying equations by using N as time variable to absorb local scale factor effects
- We shall see that for adiabatic perturbations $p(\rho)$ that $\dot{H}_L = 0$ outside horizon for an appropriate choice of slicing – plays an important role in simplifying calculations

Separate Universes

- Choosing coordinates where $\dot{H}_L + kB/3 = 0$ (defines the slicing), the e-folding remains unperturbed, we get that the 00 Einstein equations at $k \rightarrow 0$ are

$$-\left(\frac{\dot{a}}{a}\right)^2 A + \frac{1}{3} \frac{k^2 - 3K}{a^2} (H_L + H_T/3) = \frac{4\pi G}{3} a^2 \delta\rho$$

which is to be compared to the Friedmann equation

$$H^2 + \frac{K}{a^2} = \frac{8\pi G}{3} \rho$$

Noting that $H = \bar{H}(1 - AQ)$ and using the perturbation to ${}^{(3)}\mathcal{R}$

$$\begin{aligned} 2\delta H \bar{H} + \frac{\delta K}{a^2} &= \frac{8\pi G}{3} \delta\rho Q \\ -2AQ\bar{H}^2 + \frac{2}{3} \frac{k^2 - 3K}{a^2} (H_L + H_T/3)Q &= \frac{8\pi G}{3} \delta\rho Q \\ -\left(\frac{\dot{a}}{a}\right)^2 A + \frac{1}{3} \frac{k^2 - 3K}{a^2} (H_L + H_T/3) &= \frac{4\pi G}{3} \delta\rho \end{aligned}$$

Separate Universes

- And the space-space piece

$$\left[2\frac{\ddot{a}}{a} - 2\left(\frac{\dot{a}}{a}\right)^2 + \frac{\dot{a}}{a}\frac{d}{d\eta} \right] A = \frac{4\pi G}{3} a^2 (\delta\rho + 3\delta p)$$

which is to be compared with the acceleration equation

$$\frac{d}{d\eta}(aH) = -\frac{4\pi G}{3} a^2 (\rho + 3p)$$

again expanding $H = \bar{H}(1 - AQ)$ and also $d\eta = (1 + AQ)d\bar{\eta}$

$$\begin{aligned} \frac{d}{d\eta}(aH) &= (1 - AQ)\frac{d}{d\bar{\eta}}(a\bar{H})[1 - AQ] \\ &\approx \frac{d}{d\bar{\eta}}(a\bar{H}) - 2AQ\frac{d}{d\bar{\eta}}\frac{\dot{a}}{a} + \frac{\dot{a}}{a}\frac{d}{d\bar{\eta}}AQ \end{aligned}$$

Separate Universes

- Finally the continuity equation (using slicing with $\dot{H}_L = -kB/3$)

$$\dot{\delta\rho} + 3\frac{\dot{a}}{a}(\delta\rho + \delta p) = -(\rho + p)k(v - B)$$

is to be compared to

$$d\rho/d\eta = -3(aH)(\rho + p)$$

which again with the substitutions becomes

$$(1 - AQ)\frac{d}{d\bar{\eta}}(\bar{\rho} + \delta\rho Q) = -3(aH)(1 - AQ)[\bar{\rho} + \bar{p}] - 3(aH)[\delta\rho + \delta p]Q$$

$$\frac{d}{d\bar{\eta}}\delta\rho = -3\frac{\dot{a}}{a}(\delta\rho + \delta p)$$

- $\delta\rho/\rho$ constant in $\dot{H}_L + kB/3 = 0$ slicing outside horizon where peculiar velocity cannot move matter (cf. Newtonian gauge below).
- Note also that $v - B$ has a special interpretation as well: setting $v = B$ gives a comoving slicing since $N^i \propto v^i$, $N_i \propto v_i - B_i = 0$

Separate Universes

- There are other possible choices what to hold fixed on constant time slices besides $N = \ln a$. While separate universe statements still hold a must be perturbed and the simplest gauge to see these identifications with the Friedmann equations changes.
- More generally the analysis of the normal to constant time surfaces has identified geometric quantities associated with the metric perturbations
- Uniform refolding: $\dot{H}_L + kB/3 = 0$
- Shear free: $\dot{H}_T - kB = 0$
- Zero acceleration, coordinate and proper time coincide: $A = 0$
- Uniform expansion: $-3HA + (3\dot{H}_L + kB) = 0$
- Comoving: $v = B$

Gauge

- Metric and matter fluctuations take on **different values** in different coordinate system
- No such thing as a “gauge invariant” density perturbation!
- General **coordinate transformation**:

$$\begin{aligned}\tilde{\eta} &= \eta + T \\ \tilde{x}^i &= x^i + L^i\end{aligned}$$

free to choose (T, L^i) to simplify equations or physics - corresponds to a choice of slicing and threading in ADM.

- Decompose these into scalar $T, L^{(0)}$ and vector harmonics $L^{(\pm 1)}$.

Gauge

- $g_{\mu\nu}$ and $T_{\mu\nu}$ transform as **tensors**, so components in different frames can be related

$$\begin{aligned}\tilde{g}_{\mu\nu}(\tilde{\eta}, \tilde{x}^i) &= \frac{\partial x^\alpha}{\partial \tilde{x}^\mu} \frac{\partial x^\beta}{\partial \tilde{x}^\nu} g_{\alpha\beta}(\eta, x^i) \\ &= \frac{\partial x^\alpha}{\partial \tilde{x}^\mu} \frac{\partial x^\beta}{\partial \tilde{x}^\nu} g_{\alpha\beta}(\tilde{\eta} - TQ, \tilde{x}^i - LQ^i)\end{aligned}$$

- Fluctuations are compared at the same coordinate positions (not same space time positions) between the two gauges
- For example with a TQ perturbation, an event labeled with $\tilde{\eta} = \text{const.}$ and $\tilde{x} = \text{const.}$ represents a different time with respect to the underlying homogeneous and isotropic background

Gauge Transformation

- Scalar Metric:

$$\tilde{A} = A - \dot{T} - \frac{\dot{a}}{a}T,$$

$$\tilde{B} = B + \dot{L} + kT,$$

$$\tilde{H}_L = H_L - \frac{k}{3}L - \frac{\dot{a}}{a}T,$$

$$\tilde{H}_T = H_T + kL, \quad \tilde{H}_L + \frac{1}{3}\tilde{H}_T = H_L + \frac{1}{3}H_T - \frac{\dot{a}}{a}T$$

curvature perturbation depends on slicing not threading

- Scalar Matter (J th component):

$$\delta\tilde{\rho}_J = \delta\rho_J - \dot{\rho}_J T,$$

$$\delta\tilde{p}_J = \delta p_J - \dot{p}_J T,$$

$$\tilde{v}_J = v_J + \dot{L},$$

density and pressure likewise depend on slicing only

Gauge Transformation

- Vector:

$$\begin{aligned}\tilde{B}^{(\pm 1)} &= B^{(\pm 1)} + \dot{L}^{(\pm 1)}, \\ \tilde{H}_T^{(\pm 1)} &= H_T^{(\pm 1)} + kL^{(\pm 1)}, \\ \tilde{v}_J^{(\pm 1)} &= v_J^{(\pm 1)} + \dot{L}^{(\pm 1)},\end{aligned}$$

- Spatial vector has no background component hence no dependence on slicing at first order

Tensor: no dependence on slicing or threading at first order

- Gauge transformations and covariant representation can be extended to higher orders
- A coordinate system is **fully specified** if there is an explicit prescription for (T, L^i) or for scalars (T, L)

Slicing

Common choices for slicing T : set something geometric to zero

- Proper time slicing $A = 0$: proper time between slices corresponds to coordinate time – T allows c/a freedom
- Comoving (velocity orthogonal) slicing: $v - B = 0$, matter 4 velocity is related to N^ν and orthogonal to slicing - T fixed
- Newtonian (shear free) slicing: $\dot{H}_T - kB = 0$, expansion rate is isotropic, shear free, T fixed
- Uniform expansion slicing: $-(\dot{a}/a)A + \dot{H}_L + kB/3 = 0$, perturbation to the volume expansion rate θ vanishes, T fixed
- Flat (constant curvature) slicing, $\delta^{(3)}R = 0$, $(H_L + H_T/3 = 0)$, T fixed
- Constant density slicing, $\delta\rho_I = 0$, T fixed

Threading

- Threading specifies the relationship between constant spatial coordinates between slices and is determined by L

Typically involves a condition on v , B , H_T

- Orthogonal threading $B = 0$, constant spatial coordinates orthogonal to slicing (zero shift), allows $\delta L = c$ translational freedom
- Comoving threading $v = 0$, allows $\delta L = c$ translational freedom.
- Isotropic threading $H_T = 0$, fully fixes L

Newtonian (Longitudinal) Gauge

- Newtonian (shear free slicing, isotropic threading):

$$\tilde{B} = \tilde{H}_T = 0$$

$$\Psi \equiv \tilde{A} \quad (\text{Newtonian potential})$$

$$\Phi \equiv \tilde{H}_L \quad (\text{Newtonian curvature})$$

$$L = -H_T/k$$

$$T = -B/k + \dot{H}_T/k^2$$

Good: intuitive Newtonian like gravity; matter and metric algebraically related; commonly chosen for **analytic CMB** and **lensing** work

Bad: numerically **unstable**

Newtonian (Longitudinal) Gauge

- Newtonian (shear free) slicing, isotropic threading $B = H_T = 0$:

$$(k^2 - 3K)\Phi = 4\pi G a^2 \left[\delta\rho + 3\frac{\dot{a}}{a}(\rho + p)v/k \right] \quad \text{Poisson + Momentum}$$

$$k^2(\Psi + \Phi) = -8\pi G a^2 p \Pi \quad \text{Anisotropy}$$

so $\Psi = -\Phi$ if anisotropic stress $\Pi = 0$ and

$$\left[\frac{d}{d\eta} + 3\frac{\dot{a}}{a} \right] \delta\rho + 3\frac{\dot{a}}{a} \delta p = -(\rho + p)(kv + 3\dot{\Phi}),$$

$$\left[\frac{d}{d\eta} + 4\frac{\dot{a}}{a} \right] (\rho + p)v = k\delta p - \frac{2}{3}\left(1 - 3\frac{K}{k^2}\right)p k\Pi + (\rho + p) k\Psi,$$

- Newtonian competition between **stress** (pressure and viscosity) and **potential** gradients
- Note: Poisson source is the density perturbation on comoving slicing

Total Matter Gauge

- Total matter: (comoving slicing, isotropic threading)

$$\tilde{B} = \tilde{v} \quad (T_i^0 = 0)$$

$$H_T = 0$$

$$\xi = \tilde{A}$$

$$\mathcal{R} = \tilde{H}_L \quad (\text{comoving curvature})$$

$$\Delta = \tilde{\delta} \quad (\text{total density pert})$$

$$T = (v - B)/k$$

$$L = -H_T/k$$

Good: Algebraic relations between matter and metric;
comoving curvature perturbation obeys conservation law

Bad: Non-intuitive threading involving v

Total Matter Gauge

- Euler equation becomes an algebraic relation between stress and potential

$$(\rho + p)\xi = -\delta p + \frac{2}{3} \left(1 - \frac{3K}{k^2}\right) p\Pi$$

- Einstein equation lacks momentum density source

$$\frac{\dot{a}}{a}\xi - \dot{\mathcal{R}} - \frac{K}{k^2}kv = 0$$

Combine: \mathcal{R} is conserved if stress fluctuations negligible, e.g. above the horizon if $|K| \ll H^2$

$$\dot{\mathcal{R}} + Kv/k = \frac{\dot{a}}{a} \left[-\frac{\delta p}{\rho + p} + \frac{2}{3} \left(1 - \frac{3K}{k^2}\right) \frac{p}{\rho + p} \Pi \right] \rightarrow 0$$

“Gauge Invariant” Approach

- Gauge transformation rules allow variables which take on a geometric meaning in one choice of slicing and threading to be accessed from variables on another choice
- Functional form of the relationship between the variables is gauge invariant (*not* the variable values themselves! – i.e. equation is *covariant*)
- E.g. comoving curvature and density perturbations

$$\mathcal{R} = H_L + \frac{1}{3}H_T - \frac{\dot{a}}{a}(v - B)/k$$
$$\Delta\rho = \delta\rho + 3(\rho + p)\frac{\dot{a}}{a}(v - B)/k$$

Newtonian-Total Matter Hybrid

- With the gauge in(*or co*)variant approach, express variables of **one gauge** in terms of those in **another** – allows a mixture in the equations of motion
- **Example:** Newtonian curvature and comoving density

$$(k^2 - 3K)\Phi = 4\pi G a^2 \rho \Delta$$

ordinary Poisson equation then implies Φ approximately constant if stresses negligible.

- **Example:** Exact Newtonian curvature above the horizon derived through comoving curvature conservation

Gauge transformation

$$\Phi = \mathcal{R} + \frac{\dot{a}}{a} \frac{v}{k}$$

Hybrid “Gauge Invariant” Approach

Einstein equation to eliminate velocity

$$\frac{\dot{a}}{a}\Psi - \dot{\Phi} = 4\pi G a^2 (\rho + p)v/k$$

Friedmann equation with no spatial curvature

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} a^2 \rho$$

With $\dot{\Phi} = 0$ and $\Psi \approx -\Phi$

$$\frac{\dot{a} v}{a k} = -\frac{2}{3(1+w)}\Phi$$

Newtonian-Total Matter Hybrid

Combining gauge transformation with velocity relation

$$\Phi = \frac{3 + 3w}{5 + 3w} \mathcal{R}$$

Usage: calculate \mathcal{R} from inflation determines Φ for any choice of matter content or causal evolution.

- **Example:** Scalar field (“quintessence” dark energy) equations in total matter gauge imply a **sound speed** $\delta p / \delta \rho = 1$ independent of potential $V(\phi)$. Solve in synchronous gauge.

Synchronous Gauge

- Synchronous: (proper time slicing, orthogonal threading)

$$\begin{aligned}\tilde{A} &= \tilde{B} = 0 \\ \eta_T &\equiv -\tilde{H}_L - \frac{1}{3}\tilde{H}_T \\ h_L &\equiv 6H_L \\ T &= a^{-1} \int d\eta a A + c_1 a^{-1} \\ L &= - \int d\eta (B + kT) + c_2\end{aligned}$$

Good: stable, the choice of numerical codes

Bad: residual **gauge freedom** in constants c_1, c_2 must be specified as an initial condition, intrinsically relativistic, threading conditions breaks down beyond linear regime if c_1 is fixed to CDM comoving.

Synchronous Gauge

- The Einstein equations give

$$\dot{\eta}_T - \frac{K}{2k^2}(\dot{h}_L + 6\dot{\eta}_T) = 4\pi G a^2 (\rho + p) \frac{v}{k},$$

$$\ddot{h}_L + \frac{\dot{a}}{a} \dot{h}_L = -8\pi G a^2 (\delta\rho + 3\delta p),$$

$$-(k^2 - 3K)\eta_T + \frac{1}{2} \frac{\dot{a}}{a} \dot{h}_L = 4\pi G a^2 \delta\rho$$

[choose (1 & 2) or (1 & 3)] while the conservation equations give

$$\left[\frac{d}{d\eta} + 3\frac{\dot{a}}{a} \right] \delta\rho_J + 3\frac{\dot{a}}{a} \delta p_J = -(\rho_J + p_J) (k v_J + \frac{1}{2} \dot{h}_L),$$

$$\left[\frac{d}{d\eta} + 4\frac{\dot{a}}{a} \right] (\rho_J + p_J) \frac{v_J}{k} = \delta p_J - \frac{2}{3} \left(1 - 3\frac{K}{k^2} \right) p_J \Pi_J.$$

Synchronous Gauge

- Lack of a lapse A implies no gravitational forces in Navier-Stokes equation. Hence for stress free matter like cold dark matter, zero velocity initially implies zero velocity always.
- Choosing the momentum and acceleration Einstein equations is good since for CDM domination, curvature η_T is conserved and \dot{h}_L is simple to solve for.
- Choosing the momentum and Poisson equations is good when the equation of state of the matter is complicated since δp is not involved. This is the choice of CAMB.

Caution: since the curvature η_T appears and it has zero CDM source, subtle effects like dark energy perturbations are important everywhere

Spatially Flat Gauge

- Spatially Flat (flat slicing, isotropic threading):

$$\tilde{H}_L = \tilde{H}_T = 0$$

$$L = -H_T/k$$

$$\tilde{A}, \tilde{B} = \text{metric perturbations}$$

$$T = \left(\frac{\dot{a}}{a}\right)^{-1} \left(H_L + \frac{1}{3}H_T\right)$$

Good: eliminates spatial metric in evolution equations; useful in inflationary calculations ([Mukhanov et al](#))

Bad: non-intuitive slicing (no curvature!) and threading

- **Caution:** perturbation evolution is governed by the behavior of stress fluctuations and an isotropic stress fluctuation δp is gauge dependent.

Uniform Density Gauge

- Uniform density: (constant density slicing, isotropic threading)

$$H_T = 0,$$

$$\zeta_I \equiv H_L$$

$$B_I \equiv B$$

$$A_I \equiv A$$

$$T = \frac{\delta\rho_I}{\dot{\rho}_I}$$

$$L = -H_T/k$$

Good: Curvature conserved involves only stress energy conservation; simplifies isocurvature treatment

Bad: non intuitive slicing (no density pert! problems beyond linear regime) and threading

Uniform Density Gauge

- Einstein equations with I as the total or dominant species

$$(k^2 - 3K)\zeta_I - 3 \left(\frac{\dot{a}}{a} \right)^2 A_I + 3 \frac{\dot{a}}{a} \dot{\zeta}_I + \frac{\dot{a}}{a} k B_I = 0 ,$$

$$\frac{\dot{a}}{a} A_I - \dot{\zeta}_I - \frac{K}{k} B_I = 4\pi G a^2 (\rho + p) \frac{v - B_I}{k} ,$$

- The conservation equations (if $J = I$ then $\delta\rho_J = 0$)

$$\left[\frac{d}{d\eta} + 3 \frac{\dot{a}}{a} \right] \delta\rho_J + 3 \frac{\dot{a}}{a} \delta p_J = -(\rho_J + p_J)(k v_J + 3 \dot{\zeta}_I) ,$$

$$\left[\frac{d}{d\eta} + 4 \frac{\dot{a}}{a} \right] (\rho_J + p_J) \frac{v_J - B_I}{k} = \delta p_J - \frac{2}{3} \left(1 - 3 \frac{K}{k^2} \right) p_J \Pi_J + (\rho_J + p_J) A_I .$$

Uniform Density Gauge

- Conservation of curvature - single component I

$$\dot{\zeta}_I = -\frac{\dot{a}}{a} \frac{\delta p_I}{\rho_I + p_I} - \frac{1}{3} k v_I .$$

- Since $\delta\rho_I = 0$, δp_I is the non-adiabatic stress and curvature is constant as $k \rightarrow 0$ for internally adiabatic stresses $p_I(\rho_I)$.
- Note that this conservation law does not involve the Einstein equations at all: just local energy momentum conservation so it is valid for alternate theories of gravity
- Curvature on comoving slices \mathcal{R} and ζ_I related by

$$\zeta_I = \mathcal{R} + \frac{1}{3} \frac{\rho_I \Delta_I}{(\rho_I + p_I)} \Big|_{\text{comoving}} .$$

and coincide above the horizon for adiabatic fluctuations

Uniform Density Gauge

- Simple relationship to density fluctuations in the spatially flat gauge

$$\zeta_I = \frac{1}{3} \frac{\delta \tilde{\rho}_I}{(\rho_I + p_I)} \Big|_{\text{flat}}.$$

- For each particle species $\delta\rho/(\rho + p) = \delta n/n$, the number density fluctuation
- Multiple ζ_J carry information about number density fluctuations between species
- ζ_J constant component by component outside horizon if each component is adiabatic $p_J(\rho_J)$.

Vector Gauges

- Vector gauge depends only on threading L
- Poisson gauge: orthogonal threading $B^{(\pm 1)} = 0$, leaves constant L translational freedom
- Isotropic gauge: isotropic threading $H_T^{(\pm 1)} = 0$, fixes L
- To first order scalar and vector gauge conditions can be chosen separately
- More care required for second and higher order where scalars and vectors mix