Astro 4PT

Lecture Notes Set 2

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References

Basics: Kolb & Turner, Early Universe

Perturbations: Liddle & Lyth, Primordial Density Perturbation

Covariant Evolution: Mukhanov, Feldman, Brandenberger (1992), Phys. Rep., 215, 203

Particle Physics Models: Lyth, Riotto (1999), Phys. Rep., 314, 1

String Inspired Models: Burgess arXiv:0708.2865

Nongaussianity: Bartolo, Komatsu, Matarrese, Riotto (2004), Phys. Rep., 402, 103

Isocurvature & Nongaussianity: Malik & Wands (2009), Phys. Rep., 475, 1

Effective Field Theory: Chueng et al (2008) JCAP 0802 021 [0709.0295]

Scalar Fields

- A canonical scalar field is described by the action

$$S_\phi = \int d^4x \sqrt{-g}[-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)]$$

- Varying the action with respect to the metric

$$T_{\mu\nu} = \frac{1}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\sqrt{-g}\mathcal{L}_\phi$$

gives the stress-energy tensor of a scalar field

$$T^\mu{}_\nu = \nabla^\mu\phi\,\nabla_\nu\phi - \frac{1}{2}(\nabla^\alpha\phi\,\nabla_\alpha\phi + 2V)\delta^\mu{}_\nu\,.$$

- Equations of motion $\nabla^\mu T_{\mu\nu} = 0$ with closure relations for $p(\phi, \partial_\mu\phi)$, $\Pi(\phi, \partial_\mu\phi)$ or field equation $\nabla_\mu\nabla^\mu\phi = V'$ (vary with respect to ϕ)

Scalar Fields

- For the background $\langle\phi\rangle \equiv \phi_0$ (a^{-2} from conformal time)

$$\rho_\phi = \frac{1}{2}a^{-2}\dot{\phi}_0^2 + V, \quad p_\phi = \frac{1}{2}a^{-2}\dot{\phi}_0^2 - V$$

- So for kinetic dominated $w_\phi = p_\phi/\rho_\phi \to 1$
- And potential dominated $w_\phi = p_\phi/\rho_\phi \to -1$
- A slowly rolling (potential dominated) scalar field accelerates expansion
- Can use general equations of motion of dictated by stress energy conservation

$$\dot{\rho}_\phi = -3(\rho_\phi + p_\phi)\frac{\dot{a}}{a}\,,$$

to obtain the equation of motion of the background field ϕ

$$\ddot{\phi}_0 + 2\frac{\dot{a}}{a}\dot{\phi}_0 + a^2V' = 0\,,$$

Equation of Motion

- In terms of time instead of conformal time

$$\frac{d^2\phi_0}{dt^2} + 3H\frac{d\phi_0}{dt} + V' = 0$$

- Field rolls down potential hill but experiences "Hubble friction" to create slow roll. In slow roll $3H d\phi_0/dt \approx -V'$ and so kinetic energy is determined by field position $\rightarrow$ adiabatic – both kinetic and potential energy determined by single degree of freedom ϕ_0

Equation of Motion

- Likewise for the perturbations $\phi = \phi_0 + \phi_1$

$$\begin{aligned}
\delta\rho_\phi &= a^{-2}(\dot\phi_0\dot\phi_1 - \dot\phi_0^2 A) + V'\phi_1\,, \\
\delta p_\phi &= a^{-2}(\dot\phi_0\dot\phi_1 - \dot\phi_0^2 A) - V'\phi_1\,, \\
(\rho_\phi + p_\phi)(v_\phi - B) &= a^{-2}k\dot\phi_0\phi_1\,, \\
p_\phi\pi_\phi &= 0\,,
\end{aligned}$$

- For comoving slicing where $v_\phi = B$

$$\phi_1 = 0$$

and the field is spatially unperturbed - so all the dynamics are in the metric

Sound Speed

- In this slicing $\delta p_\phi = \delta\rho_\phi$ so the sound speed is $\delta p_\phi/\delta\rho_\phi = 1$.
- More generally the sound speed of the inflation is defined as the speed at which field fluctuations propagate - i.e. the kinetic piece to the energy density rather than the $V'\phi_1$ potential piece - much like in the background the $+1$ and -1 pieces of w.
- Non canonical kinetic terms– k-essence, DBI inflation – can generate $c_s \neq 1$ as do terms in the effective theory of inflation

Equation of Motion

- Scalar field fluctuations are stable inside the horizon and are a good candidate for the smooth dark energy
- Equivalently, conservation equations imply

$$\ddot{\phi}_1 = -2\frac{\dot{a}}{a}\dot{\phi}_1 - (k^2 + a^2 V'')\phi_1 + (\dot{A} - 3\dot{H}_L - kB)\dot{\phi}_0 - 2Aa^2V' \,.$$

- Alternately this follows from perturbing the Klein Gordon equation $\nabla_\mu \nabla^\mu \phi = V'$

Inflationary Perturbations

- Classical equations of motion for scalar field inflaton determine the evolution of scalar field fluctuations generated by quantum fluctuations
- Since the curvature $\mathcal{R}$ on comoving slicing is conserved in the absence of stress fluctuations (i.e. outside the apparent horizon, calculate this and we're done no matter what happens in between inflation and the late universe (reheating etc.)
- But in the comoving slicing $\phi_1 = 0$! no scalar-field perturbation
- Solution: solve the scalar field equation in the dual gauge where the curvature $H_L + H_T/3 = 0$ ("spatially flat" slicing) and transform the result to comoving slicing

Transformation to Comoving Slicing

- Scalar field transforms as scalar field

$$\tilde{\phi}_1 = \phi_1 - \dot{\phi}_0 T$$

- To get to comoving slicing $\tilde{\phi}_1 = 0$, $T = \phi_1/\dot{\phi}_0$, and $\tilde{H}_T = H_T + kL$ so

$$\mathcal{R} = H_L + \frac{H_T}{3} - \frac{\dot{a}}{a}\frac{\phi_1}{\dot{\phi}_0}$$

- Transformation particularly simple from a spatially flat slicing where $H_L + H_T/3 = 0$, i.e. spatially unperturbed metric

$$\mathcal{R} = -\frac{\dot{a}}{a}\frac{\phi_1}{\dot{\phi}_0}$$

Spatially Flat Gauge

- Spatially Flat (flat slicing, isotropic threading):

$$\begin{aligned}\tilde{H}_L + \tilde{H}_T/3 &= \tilde{H}_T = 0 \\ A_f &= \tilde{A}\,, B_f = \tilde{B} \\ T &= \left(\frac{\dot{a}}{a}\right)^{-1}\left(H_L + \frac{1}{3}H_T\right) \\ L &= -H_T/k\end{aligned}$$

- Einstein Poisson and Momentum

$$\begin{aligned}-3(\frac{\dot{a}}{a})^2 A_f + \frac{\dot{a}}{a} k B_f &= 4\pi G a^2 \delta\rho_\phi\,, \\ \frac{\dot{a}}{a} A_f - \frac{K}{k^2}(k B_f) &= 4\pi G a^2 (\rho_\phi + p_\phi)(v_\phi - B_f)/k\,,\end{aligned}$$

- Conservation

$$\ddot{\phi}_1 = -2\frac{\dot{a}}{a}\dot{\phi}_1 - (k^2 + a^2 V'')\phi_1 + (\dot{A}_f - k B_f)\dot{\phi}_0 - 2 A_f a^2 V'\,.$$

Spatially Flat Gauge

- For modes where $|k^2/K| \gg 1$ we obtain

$$
\begin{aligned}
\frac{\dot a}{a} A_f &= 4\pi G \dot\phi_0 \phi_1 \,, \\
\frac{\dot a}{a} k B_f &= 4\pi G [\dot\phi_0 \dot\phi_1 - \dot\phi_0^2 A_f + a^2 V' \phi_1 + 3\frac{\dot a}{a}\dot\phi_0 \phi_1]
\end{aligned}
$$

so combining $\dot A_f - k B_f$ eliminates the $\dot\phi_1$ term

- The metric source to the scalar field equation can be reexpressed in terms of the field perturbation and background quantities

$$
(\dot A_f - k B_f)\dot\phi_0 - 2A_f a^2 V' - a^2 V'' \phi_1 = f(\eta)\phi_1
$$

- Single closed form 2nd order ODE for ϕ_1

Mukhanov Equation

- Equation resembles a damped oscillator equation with a particular dispersion relation

$$\ddot{\phi}_1 + 2\frac{\dot{a}}{a}\dot{\phi}_1 + [k^2 + f(\eta)]\phi_1$$

- $f(\eta)$ involves terms with $\dot{\phi}_0$, V', V'' implying that for a sufficiently flat potential $f(\eta)$ represents a small correction
- Transform out the background expansion $u \equiv a\phi_1$

$$\dot{u} = \dot{a}\phi + a\dot{\phi}$$

$$\ddot{u} = \ddot{a}\phi_1 + 2\dot{a}\dot{\phi}_1 + a\ddot{\phi}_1$$

$$\ddot{u} + [k^2 - \frac{\ddot{a}}{a} + f(\eta)]u = 0$$

- Note Friedmann equations say if $p = -\rho$, $\ddot{a}/a = 2(\dot{a}/a)^2$

Mukhanov Equation

- Using the background Einstein and scalar field equations, this source term can be expressed in a surprisingly compact form

$$\ddot{u} + [k^2 - \frac{\ddot{z}}{z}]u = 0$$

- and

$$z \equiv \frac{a\dot{\phi}_0}{\dot{a}/a}$$

- This equation is sometimes called the "Mukhanov Equation" and is both exact (in linear theory) and compact
- For large k (subhorizon), this looks like a free oscillator equation which can be quantized
- Let's examine the relationship between z and the slow roll parameters

Slow Roll Parameters

- Rewrite equations of motion in terms of slow roll parameters but do not require them to be small or constant.
- Deviation from de Sitter expansion

$$\epsilon \equiv \frac{3}{2}(1 + w_\phi)$$
$$= \frac{\frac{3}{2}(d\phi_0/dt)^2/V}{1 + \frac{1}{2}(d\phi_0/dt)^2/V}$$

- Deviation from overdamped limit of $d^2\phi_0/dt^2 = 0$

$$\delta \equiv \frac{d^2\phi_0/dt^2}{H d\phi_0/dt}(= -\eta_H)$$
$$= \frac{\ddot{\phi}_0}{\dot{\phi}_0}\left(\frac{\dot{a}}{a}\right)^{-1} - 1$$

Slow Roll Parameters

- Friedmann equations:

$$\left(\frac{\dot{a}}{a}\right)^2 = 4\pi G \dot{\phi}_0^2 \epsilon^{-1}$$

$$\frac{d}{d\eta}\left(\frac{\dot{a}}{a}\right) = \frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 = \left(\frac{\dot{a}}{a}\right)^2 (1-\epsilon)$$

Take derivative of first equation, divide through by $(\dot{a}/a)^2$

$$2\frac{\dot{a}}{a}(1-\epsilon) = 2\frac{\ddot{\phi}_0}{\dot{\phi}_0} - \frac{\dot{\epsilon}}{\epsilon}$$

- Replace $\ddot{\phi}_0$ with δ

$$\dot{\epsilon} = 2\epsilon(\delta + \epsilon)\frac{\dot{a}}{a}$$

- Evolution of ϵ is second order in parameters

Slow Roll parameters

- Returning to the Mukhanov equation

$$\ddot{u} + [k^2 + g(\eta)]u = 0$$

where

$$\begin{aligned}
g(\eta) &\equiv f(\eta) + \epsilon - 2 \\
&= -\left(\frac{\dot{a}}{a}\right)^2 [2 + 3\delta + 2\epsilon + (\delta + \epsilon)(\delta + 2\epsilon)] - \frac{\dot{a}}{a}\dot{\delta} \\
&= -\frac{\ddot{z}}{z}
\end{aligned}$$

and recall

$$z \equiv a \left(\frac{\dot{a}}{a}\right)^{-1} \dot{\phi}_0$$

Slow Roll Limit

- Slow roll $\epsilon \ll 1$, $\delta \ll 1$, $\dot\delta \ll \frac{\dot a}{a}$

$$\ddot u + [k^2 - 2\left(\frac{\dot a}{a}\right)^2]u = 0$$

or for conformal time measured from the end of inflation

$$\tilde\eta = \eta - \eta_{\rm end}$$
$$\tilde\eta = \int_{a_{\rm end}}^{a} \frac{da}{Ha^2} \approx -\frac{1}{aH}$$

- Compact, slow-roll equation:

$$\ddot u + [k^2 - \frac{2}{\tilde\eta^2}]u = 0$$

Quantum Fluctuations

- Simple harmonic oscillator $\ll$ Hubble length

$$\ddot{u} + k^2 u = 0$$

- Quantize the simple harmonic oscillator

$$\hat{u} = u(k, \tilde{\eta})\hat{a} + u^*(k, \tilde{\eta})\hat{a}^\dagger$$

where $u(k, \tilde{\eta})$ satisfies classical equation of motion and the creation and annihilation operators satisfy

$$[a, a^\dagger] = 1, \qquad a|0\rangle = 0$$

- Normalize wavefunction $[\hat{u}, d\hat{u}/d\tilde{\eta}] = i$

$$u(k, \eta) = \frac{1}{\sqrt{2k}} e^{-ik\tilde{\eta}}$$

Quantum Fluctuations

- Zero point fluctuations of ground state

$$
\begin{aligned}
\langle u^2 \rangle &= \langle 0 | u^\dagger u | 0 \rangle \\
&= \langle 0 | (u^* \hat{a}^\dagger + u\hat{a})(u\hat{a} + u^* \hat{a}^\dagger) | 0 \rangle \\
&= \langle 0 | \hat{a}\hat{a}^\dagger | 0 \rangle |u(k, \tilde{\eta})|^2 \\
&= \langle 0 | [\hat{a}, \hat{a}^\dagger] + \hat{a}^\dagger \hat{a} | 0 \rangle |u(k, \tilde{\eta})|^2 \\
&= |u(k, \tilde{\eta})|^2 = \frac{1}{2k}
\end{aligned}
$$

- Classical equation of motion take this quantum fluctuation outside horizon where it freezes in.

Slow Roll Limit

- Classical equation of motion then has the exact solution

$$u = \frac{1}{\sqrt{2k}} \left(1 - \frac{i}{k\tilde\eta} \right) e^{-ik\tilde\eta}$$

- For $|k\tilde\eta| \ll 1$ (late times, $\gg$ Hubble length) fluctuation freezes in

$$\lim_{|k\tilde\eta| \to 0} u = -\frac{1}{\sqrt{2k}} \frac{i}{k\tilde\eta} \approx \frac{iHa}{\sqrt{2k^3}}$$

$$\phi_1 = \frac{iH}{\sqrt{2k^3}}$$

- Power spectrum of field fluctuations

$$\Delta^2_{\phi_1} = \frac{k^3 |\phi_1|^2}{2\pi^2} = \frac{H^2}{(2\pi)^2}$$

Slow Roll Limit

- Recall $\mathcal{R} = -(\dot{a}/a)\phi_1/\dot{\phi}_0$ and slow roll says

$$\left(\frac{\dot{a}}{a}\right)^2 \frac{1}{\dot{\phi}_0^2} = \frac{8\pi G a^2 V}{3} \frac{3}{2a^2 V \epsilon} = \frac{4\pi G}{\epsilon}$$

Thus the curvature power spectrum

$$\Delta_{\mathcal{R}}^2 = \frac{8\pi G}{2} \frac{H^2}{(2\pi)^2 \epsilon}$$

Tilt

- Curvature power spectrum is scale invariant to the extent that H is constant
- Scalar spectral index

$$\begin{aligned}\frac{d\ln\Delta^2_{\mathcal{R}}}{d\ln k} &\equiv n_S - 1\\ &= 2\frac{d\ln H}{d\ln k} - \frac{d\ln\epsilon}{d\ln k}\end{aligned}$$

- Evaluate at horizon crossing where fluctuation freezes

$$\begin{aligned}\frac{d\ln H}{d\ln k}\Big|_{-k\tilde\eta=1} &= \frac{k}{H}\frac{dH}{d\tilde\eta}\Big|_{-k\tilde\eta=1}\frac{d\tilde\eta}{dk}\Big|_{-k\tilde\eta=1}\\ &= \frac{k}{H}(-aH^2\epsilon)\Big|_{-k\tilde\eta=1}\frac{1}{k^2} = -\epsilon\end{aligned}$$

where $aH = -1/\tilde\eta = k$

Tilt

- Evolution of ϵ

$$\frac{d\ln\epsilon}{d\ln k} = -\frac{d\ln\epsilon}{d\ln\tilde{\eta}} = -2(\delta+\epsilon)\frac{\dot{a}}{a}\tilde{\eta} = 2(\delta+\epsilon)$$

- Tilt in the slow-roll approximation

$$n_S = 1 - 4\epsilon - 2\delta$$

Relationship to Potential

- To leading order in slow roll parameters

$$
\begin{aligned}
\epsilon &= \frac{\frac{3}{2}\dot\phi_0^2/a^2V}{1+\frac{1}{2}\dot\phi_0^2/a^2V} \\
&\approx \frac{3}{2}\dot\phi_0^2/a^2V \\
&\approx \frac{3}{2a^2V}\frac{a^4V'^2}{9(\dot a/a)^2}, \qquad (3\dot\phi_0\frac{\dot a}{a} = -a^2V') \\
&\approx \frac{1}{6}\frac{3}{8\pi G}\left(\frac{V'}{V}\right)^2, \qquad \left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}a^2V \\
&\approx \frac{1}{16\pi G}\left(\frac{V'}{V}\right)^2
\end{aligned}
$$

so $\epsilon \ll 1$ is related to the first derivative of potential being small

Relationship to Potential

- And

$$\delta = \frac{\ddot{\phi}_0}{\dot{\phi}_0}\left(\frac{\dot{a}}{a}\right)^{-1} - 1$$

$$(\dot{\phi}_0 \approx -a^2\left(\frac{\dot{a}}{a}\right)^{-1}\frac{V'}{3})$$

$$(\ddot{\phi}_0 \approx -\frac{a^2V'}{3}(1+\epsilon) + a^4\left(\frac{\dot{a}}{a}\right)^{-2}\frac{V'V''}{9})$$

$$\approx -\frac{1}{a^2V'/3}\left(-\frac{a^2V'}{3}(1+\epsilon) + \frac{a^2}{9}\frac{3}{8\pi G}\frac{V'V''}{V}\right) - 1 \quad \approx \epsilon - \frac{1}{8\pi G}\frac{V''}{V}$$

so δ is related to second derivative of potential being small. Very flat potential.

Relationship to Potential

- Exact relations

$$\frac{1}{8\pi G}\left(\frac{V'}{V}\right)^2 = 2\epsilon\frac{(1+\delta/3)^2}{(1-\epsilon/3)^2}$$

$$\frac{1}{8\pi G}\frac{V''}{V} = \frac{\epsilon - \delta - [\delta^2 - \epsilon\delta - (a/\dot{a})\dot{\delta}]/3}{1-\epsilon_H/3}$$

agree in the limit ϵ, $|\delta| \ll 1$ and $|(a/\dot{a})\dot{\delta}| \ll \epsilon$, $|\delta|$

- Like the Mukhanov to slow roll simplification, identification with potential requires a constancy of δ assumption

Gravitational Waves

- Gravitational wave amplitude satisfies Klein-Gordon equation $(K = 0)$, same as scalar field

$$\ddot{H}_T^{(\pm 2)} + 2\frac{\dot{a}}{a}\dot{H}_T^{(\pm 2)} + k^2 H_T^{(\pm 2)} = 0\,.$$

- Acquires quantum fluctuations in same manner as ϕ. Lagrangian sets the normalization

$$\phi_1 \to H_T^{(\pm 2)}\sqrt{\frac{3}{16\pi G}}$$

- Scale-invariant gravitational wave amplitude converted back to $+$ and $\times$ states $H_T^{(\pm 2)} = -(h_+ \mp i h_\times)/\sqrt{6})$

$$\Delta^2_{+,\times} = 16\pi G \Delta^2_{\phi_1} = 16\pi G \frac{H^2}{(2\pi)^2}$$

Gravitational Waves

- Gravitational wave power $\propto H^2 \propto V \propto E_i^4$ where E_i is the energy scale of inflation
- Tensor-scalar ratio - various definitions - WMAP standard is

$$r \equiv 4\frac{\Delta_+^2}{\Delta_{\mathcal{R}}^2} = 16\epsilon$$

- Tensor tilt:

$$\frac{d\ln\Delta_H^2}{d\ln k} \equiv n_T = 2\frac{d\ln H}{d\ln k} = -2\epsilon$$

Gravitational Waves

- Consistency relation between tensor-scalar ratio and tensor tilt

$$r = 16\epsilon = -8n_T$$

- Measurement of scalar tilt and gravitational wave amplitude constrains inflationary model in the slow roll context
- Comparision of tensor-scalar ratio and tensor tilt tests the idea of slow roll itself

Gravitational Wave Phenomenology

- Equation of motion

$$\ddot{H}_T^{(\pm 2)} + 2\frac{\dot{a}}{a}\dot{H}_T^{(\pm 2)} + k^2 H_T^{(\pm 2)} = 0\,.$$

- has solutions

$$H_T^{(\pm 2)} = C_1 H_1(k\eta) + C_2 H_2(k\eta)$$

$$H_1 \propto x^{-m} j_m(x)$$

$$H_2 \propto x^{-m} n_m(x)$$

where $m = (1 - 3w)/(1 + 3w)$

- If $w > -1/3$ then gravity wave is constant above horizon $x \ll 1$ and then oscillates and damps
- If $w < -1/3$ then gravity wave oscillates and freezes into some value, just like scalar field

Gravitational Wave Phenomenology

- A gravitational wave makes a quadrupolar (transverse-traceless) distortion to metric
- Just like the scale factor or spatial curvature, a temporal variation in its amplitude leaves a residual temperature variation in CMB photons – here anisotropic
- Before recombination, anisotropic variation is eliminated by scattering
- Gravitational wave temperature effect drops sharply at the horizon scale at recombination

Gravitational Wave Phenomenology

- Source to polarization goes as $\dot\tau/\dot H_T$ and peaks at the horizon not damping scale
- More distinct signature in the polarization since symmetry of plane wave is broken by the transverse nature of gravity wave polarization
- B modes formed as photons propagate – the spatial variation in the plane waves modulate the signal: described by Boltzmann eqn.

$$\Delta B_{\rm peak} \approx 0.024 \left(\frac{E_i}{10^{16}\mathrm{GeV}} \right)^2 \mu\mathrm{K}$$

Large Field Models

- For detectable gravitational waves, scalar field must roll by order $M_{\rm pl} = (8\pi G)^{-1/2}$

$$\frac{d\phi_0}{dN} = \frac{d\phi_0}{d\ln a} = \frac{d\phi_0}{dt}\frac{1}{H}$$

- The larger ϵ is the more the field rolls in an e-fold

$$\epsilon = \frac{r}{16} = \frac{3}{2V}\left(H\frac{d\phi_0}{dN}\right)^2 = \frac{8\pi G}{2}\left(\frac{d\phi_0}{dN}\right)^2$$

- Observable scales span $\Delta N \sim 5$ so

$$\Delta\phi_0 \approx 5\frac{d\phi}{dN} = 5(r/8)^{1/2}M_{\rm pl} \approx 0.6(r/0.1)^{1/2}M_{\rm pl}$$

- Does this make sense as an effective field theory? Lyth (1997)

Large Field Models

- Large field models include monomial potentials $V(\phi) = A\phi^n$

$$\epsilon \approx \frac{n^2}{16\pi G\phi^2}$$

$$\delta \approx \epsilon - \frac{n(n-1)}{8\pi G\phi^2}$$

- Slow roll requires large field values of $\phi > M_{\rm pl}$
- Thus $\epsilon \sim |\delta|$ and a finite tilt indicates finite ϵ
- Given WMAP tilt, potentially observable gravitational waves

Small Field Models

- If the field is near an maximum of the potential

$$V(\phi) = V_0 - \frac{1}{2}\mu^2\phi^2$$

- Inflation occurs if the V_0 term dominates

$$\epsilon \approx \frac{1}{16\pi G}\frac{\mu^4\phi^2}{V_0^2}$$

$$\delta \approx \epsilon + \frac{1}{8\pi G}\frac{\mu^2}{V_0} \to \frac{\delta}{\epsilon} = \frac{V_0}{\mu^2\phi^2} \gg 1$$

- Tilt reflects δ: $n_S \approx 1 - 2\delta$ and ϵ is much smaller
- The field does not roll significantly during inflation and gravitational waves are negligible

Hybrid Models

- If the field is rolling toward a minimum of the potential

$$V(\phi) = V_0 + \frac{1}{2}m^2\phi^2$$

- Slow roll parameters similar to small field but a real m^2

$$\epsilon \approx \frac{1}{16\pi G}\frac{m^4\phi^2}{V_0^2}$$

$$\delta \approx \epsilon - \frac{1}{8\pi G}\frac{m^2}{V_0}$$

- Then V_0 domination $\epsilon, \delta < 0$ and $n_S > 1$ - blue tilt
- For m^2 domination, monomial-like.
- Intermediate cases with intermediate predictions - can have observable gravity waves but does not require it.

Hybrid Models

- But how does inflation end? V_0 remains as field settles to minimum
- Implemented as multiple field model with V_0 supplied by second field
- Inflation ends when rolling triggers motion in the second field to the true joint minimum

Features

- The slow roll simplification carried an extra assumption that δ is not only small but also constant
- Assumption that ϵ is nearly constant is justified if $|\delta| \ll 1$
- These conditions can be violated if there are rapid, not necessarily large, changes in the potential
- What happens when we relax these conditions?
- Go back to Mukhanov equation and for convenience transform variables to $y = \sqrt{2k}u$ and $x = -k\tilde{\eta}$ and $' = d/d\ln\tilde{\eta}$

$$\frac{d^2y}{dx^2} + \left(1 - \frac{2}{x^2}\right) y = \frac{g(\ln x)}{x^2} y$$

$$g = \frac{f'' - 3f'}{f}, \quad f = 2\pi \frac{\dot{\phi} a \eta}{H}$$

Generalized Slow Roll

- Note that the LHS is the slow roll version of Mukhanov and the RHS is a source that depends on the slow roll parameters as

$$g = 2[(aH\tilde{\eta})^2 - 1] + (aH\tilde{\eta})^2[2\epsilon + 3\delta + 2\epsilon^2 + 4\delta\epsilon + \delta_2]$$

where we have eliminated $\dot{\delta}$ in favor of a third parameter

$$\delta_2 = \frac{d\delta}{d\ln a} - \epsilon\delta + \delta^2$$

- Since $(aH\tilde{\eta}) \approx -(1+\epsilon)$, g vanishes to zeroth order in slow roll
- Keeping δ_2 restores the relationship with the potential

$$g \approx \frac{1}{8\pi G}\left[\frac{9}{2}(\frac{V'}{V})^2 - 3\frac{V''}{V}\right]$$

Generalized Slow Roll

- Generalized slow roll approximation exploits this fact by taking an iterative Green function approach
- LHS "homogeneous" equation is solved by

$$y_0(x) = \left(1 + \frac{i}{x}\right) e^{ix}$$

so in the approximation that $y \approx y_0$ on the RHS

$$y(x) = y_0(x) - \int_x^\infty \frac{du}{u^2} g(\ln u) y_0(u) \mathrm{Im}[y_0^*(u) y_0(x)]$$

which is an integral equation for the field an curvature spectrum given some deviation from slow roll as quantified by V'/V and V''/V

Generalized Slow Roll

- For example, by assuming $|\delta| \ll 1$, $\epsilon \approx$ const. and

$$\Delta^2_{\mathcal{R}}(k) = \frac{V(8\pi G)^2}{12\pi^2}\left(\frac{V}{V'}\right)^2 \Big\{1 + (3\alpha - \frac{1}{6})\frac{1}{8\pi G}(\frac{V'}{V})^2|_{k=aH}$$
$$- \frac{2}{8\pi G}\int_0^\infty \frac{du}{u} W_\theta(u)\frac{V''}{V}\Big\},$$

where $\alpha \approx 0.73$ and

$$W_\theta(u) = \frac{3\sin(2u)}{2u^3} - \frac{3\cos 2u}{u^2} - \frac{3\sin(2u)}{2u} - \theta(1-u)$$

- $u \ll 1$ goes $2u^2/5$ and $u \gg 1$ oscillates as $-3\sin(2u)/2u$
- Therefore changes in V''/V around horizon crossing for mode produce a ringing response in the power spectrum – WMAP glitches?
- Can extend to $|\delta| \sim \mathcal{O}(1)$, and can be iterated to higher order

Non-Gaussianity

- In single field slow roll inflation, the inflaton is nearly free field - modes don't interact - fluctuations are Gaussian to a high degree.
- Non-gaussianities are at best second order effects and with 10^{-5} fluctuations, this is a 10^{-10} effect!
- Coupling of long wavelength curvature to short wavelength (field) must satisfy consistency relation and hence related to $n_s - 1$ or ϵ, δ.

$$\mathcal{R}^{(2)} = [\mathcal{R}^{(1)}]^2 \mathcal{O}(\epsilon, \delta)$$

- Second order effects generate larger correction. Curvature on constant density slicing ζ still conserved, transform back to Newtonian gauge and extract fluctuation returns a form

$$\Phi(\mathbf{x}) = \Phi^{(1)}(\mathbf{x}) + f_{\rm NL}[(\Phi^{(1)}(\mathbf{x}))^2 - \langle \Phi^{(1)}(\mathbf{x}) \rangle^2]$$

called a local non-Gaussianity - generic prediction is $f_{\rm NL} = \mathcal{O}(1)$

Non-Gaussianity

- Decompose in harmonics (assume $k^2 \gg |K|$, nearly flat)

$$\Phi(\mathbf{k}) = \int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}}\Phi(\mathbf{x})$$

$$= \Phi^{(1)}(\mathbf{k}) + f_{\rm NL}\int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}}\int \frac{d^3k_1}{(2\pi)^3}e^{i\mathbf{k}_1\cdot\mathbf{x}}\Phi^{(1)}(\mathbf{k}_1)\int \frac{d^3k_2}{(2\pi)^3}e^{i\mathbf{k}_2\cdot\mathbf{x}}\Phi^{(1)}(\mathbf{k}_2)$$

with $\int d^3x e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}} = (2\pi)^3\delta(\mathbf{k}-\mathbf{k}')$ yields

$$\Phi(\mathbf{k}) = \Phi^{(1)}(\mathbf{k}) + f_{\rm NL}\int \frac{d^3k_1}{(2\pi)^3}\Phi^{(1)}(\mathbf{k}_1)\Phi^{(1)}(\mathbf{k}-\mathbf{k}_1)$$

- Given that the first order term is a Gaussian field represented by the power spectrum

$$\langle\Phi^{(1)}(\mathbf{k})\Phi^{(1)}(\mathbf{k}')\rangle = (2\pi)^3\delta(\mathbf{k}+\mathbf{k}')P_\Phi(k)$$

we get a bispectrum contribution

Non-Gaussianity

- The bispectrum is proportional to the product of power spectra

$$\langle \Phi(\mathbf{k}_1)\Phi(\mathbf{k}_2)\Phi(\mathbf{k}_3)\rangle = (2\pi)^3\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)[2f_{\rm NL}P_\Phi(k_1)P_\Phi(k_2) + \text{perm}]$$

- Bispectrum contains most of the information on local non-Gaussianity
- Higher point correlations also exist and can be measured with good S/N but their sample variance is high and so don't help in constraining $f_{\rm NL}$

Non-Gaussianity Measurements

- Measure in 2D analogue in the Planck CMB data $f_{\rm NL} = 2.7 \pm 5.8$ (68% CL)
- Counterintuitively, this level can also be reached by measuring power spectrum of rare objects at low k (large scales)
- Bispectrum for $\mathbf{k_1} \sim -\mathbf{k_2}$ and $|k_3| \ll |k_1|, k_2$ looks like a correlation of high k power with low k amplitude
- Enhanced power allows for formation of rare objects in essentially the same way as the peak-background split picture for bias
- Relative effect increases at low k where the intrinsic power in the density spectrum falls
- Strong scale dependence of bias (Dalal et al 2007)

Curvaton

- Beyond single field slow roll inflation, larger $f_{\rm NL}$ can be generated
- Suppose there is a second light field σ called the curvaton during inflation, it also has quantum fluctuations

$$\Delta_\sigma^2 \approx \left(\frac{H}{2\pi}\right)^2$$

- After inflation the curvaton oscillates around its minimum and decays leaving its field fluctuations as density fluctuations

$$V_\sigma = \frac{1}{2} m_\sigma^2 \sigma^2$$

- Energy density oscillates from potential to kinetic, $w = \pm 1$ with average of $w = 0$ - redshifts like matter - can dominate the energy density of the universe

Curvaton

- Its energy density is given by the *square* of the field amplitude. To leading order

$$\rho_\sigma \approx \frac{1}{2} m_\sigma^2 \sigma^2$$

- The fractional density fluctuation is related to the fractional field fluctuation

$$\frac{\delta\rho_\sigma}{\rho_\sigma} \approx 2\frac{\delta\sigma}{\sigma}$$

- Power spectrum enhanced by $1/\sigma_*^2$ (evaluated at horizon crossing)
- Via change in dominant species (or more generally change in equation of state of total), once negligible $\delta\rho_\sigma \approx \delta\rho$
- From density fluctuation in the spatially flat gauge move to a curvature fluctuation on the constant density gauge ζ

Curvaton

- Via the gauge transformation from spatially flat gauge to constant density gauge

$$\begin{aligned}\zeta &= H_L + \frac{H_T}{3} - \frac{\dot{a}}{a}\frac{\delta\rho}{\dot{\rho}} \\ &= -\frac{\dot{a}}{a}\frac{\delta\rho_\sigma}{\dot{\rho}_\sigma} = \frac{1}{3}\frac{\delta\rho_\sigma}{\rho_\sigma} = \frac{2}{3}\frac{\delta\sigma}{\sigma_*}\end{aligned}$$

- So if the curvaton dominates the energy density before decaying into particles then the curvature fluctuation

$$\Delta_\zeta^2 = \frac{4}{9}\left(\frac{H_*}{2\pi\sigma_*}\right)^2$$

- If the curvaton decays before it fully dominates the energy density then its density fluctuations are diluted by the ratio of curvaton to total energy density

Curvaton

- More generally, curvaton decays when $H = \Gamma$ the decay rate and H is controlled by total energy density and Friedmann equation
- So if both the radiation and curvaton contribute significantly to the energy density, we go to the surface of constant total density and define the decay time t_d

$$H^2(t_d) = 8\pi G\rho(t_d)/3 = \Gamma^2$$

$$\rho(\mathbf{x}, t_d) = \rho(t_d)|_{\delta=0} = \rho_r(\mathbf{x}, t_d) + \rho_\sigma(\mathbf{x}, t_d)$$

where note that the r and σ contributions are inhomogeneous on this surface in general

$$(1+\delta_r)\rho_r + (1+\delta_\sigma)\rho_\sigma = \rho$$

$$(1+\delta_r)\left(1 - \frac{\rho_\sigma}{\rho}\right) + (1+\delta_\sigma)\frac{\rho_\sigma}{\rho} = 1$$

Curvaton

- The surfaces for which they are individually homogeneous are reached by the gauge transformation

$$\zeta_r = \zeta - \frac{\dot a}{a}\frac{\delta\rho_r}{\dot\rho_r} = \zeta + \frac{1}{4}\delta_r$$

$$\zeta_\sigma = \zeta - \frac{\dot a}{a}\frac{\delta\rho_\sigma}{\dot\rho_\sigma} = \zeta + \frac{1}{3}\delta_\sigma$$

- So we can eliminate $\rho_r(\mathbf{x}, t_d)$ and $\rho_\sigma(\mathbf{x}, t_d)$ in favor of ζ's

$$[1 + 4(\zeta_r - \zeta)]\left(1 - \frac{\rho_\sigma}{\rho}\right) + [1 + 3(\zeta_\sigma - \zeta)]\frac{\rho_\sigma}{\rho} = 1$$

- Given ζ_σ, ζ_r, and r this defines the curvature fluctuation ζ
- ζ_r is the curvature perturbation associated with the inflaton decaying to radiation

Curvaton

- Take $\zeta_r \ll \zeta_\sigma$

$$[1-4\zeta]\left(1-\frac{\rho_\sigma}{\rho}\right)+[1+3(\zeta_\sigma-\zeta)]\frac{\rho_\sigma}{\rho}=1$$

or

$$\zeta=\frac{3\rho_\sigma}{4\rho-\rho_\sigma}\zeta_\sigma=\frac{3\rho_\sigma}{4\rho_r+3\rho_\sigma}\zeta_\sigma=r\zeta_\sigma$$

where r is the ratio of $\rho+p$

$$r=\frac{\dot\rho_\sigma}{\dot\rho}=\frac{3\rho_\sigma}{4\rho_r+3\rho_\sigma}$$

- Thus the curvature spectrum becomes

$$\Delta_\zeta^2=\frac{4}{9}r^2\left(\frac{H_*}{2\pi\sigma_*}\right)^2$$

Curvaton

- Moreover since the density is the square of the field, there is a local non-Gaussianity that can be much larger

$$\frac{\delta\rho_\sigma}{\rho_\sigma} \approx 2\frac{\delta\sigma}{\sigma} + \frac{(\delta\sigma)^2}{\sigma^2}$$

- So with $\Phi = 3\zeta/5$ for $r = 1$

$$\Phi = \frac{1}{5}\left[2\left(\frac{\delta\sigma}{\sigma}\right) + \left(\frac{\delta\sigma}{\sigma}\right)^2\right], \qquad f_{\rm NL} = \frac{5}{4}$$

- For $r < 1$ it is enhanced by $1/r$ with the detailed relation requiring expanding the decay condition to second order via the relations

$$\rho_r(\mathbf{x}, t_d) = \bar\rho_r(t_d)e^{4(\zeta_r - \zeta)}$$

$$\rho_\sigma(\mathbf{x}, t_d) = \bar\rho_\sigma(t_d)e^{4(\zeta_r - \zeta)}$$

Curvaton

- Generalizing beyond just curvaton and radiation to multiple species like CDM, baryons etc entails keeping a ζ_i for each
- Difference between $\zeta_i - \zeta_j$ represents a fluctuation in the relative number density of various particle species
- If for example, the CDM was created before curvaton decay (no longer in equilibrium with plasma), its curvature fluctuation comes from the inflation $\zeta_{\rm CDM} = \zeta_r$ whereas the other particles share $\zeta_j = \zeta_\sigma$
- This gives a number density fluctuation between e.g. the photons and the CDM

Variable Decay

- Suppose the decay rate of the inflation during reheating were controlled by some light scalar $\Gamma \propto \chi$
- χ gets quantum fluctuations $\delta\chi \approx H/2\pi$ as any light scalar
- Fluctuations in the field cause fluctuations in the decay rate
- In a region where decay is more rapid, the inflaton converts its energy to radiation earlier, thereafter redshifting like radiation
- Net result is a density fluctuation of order the decay rate fluctuation $\delta\rho/\rho \propto \delta\Gamma/\Gamma$
- In the constant density gauge this is a curvature fluctuation

$$\zeta = -\frac{1}{6}\frac{\delta\Gamma}{\Gamma}$$

Effective Field Theory

- Choosing a gauge where scalar field is unperturbed (comoving gauge to leading order) is a preferred hypersurface
- Generalize to arbitrary single field model: field is the only clock
- Suggests that we can move beyond canonical fields and linear order by exploiting this concept
- Choose the unitary gauge as constant field gauge to all orders

$$\phi(\mathbf{x}, t_u) = \phi_0(t_u)$$

- Given that by assumption the universe is dominated by this scalar field and it is homogenous in this frame, the only thing that the action can be built out of is terms that depend on t_u
- In the EFT language, write down all possible terms that is consistent with unbroken spatial diffeomorphism invariance in this slicing

Effective Field Theory

- In unitary gauge, there is only the metric to work with. In general it transforms as a tensor

$$\tilde{g}^{\mu\nu}(\tilde{t}, \tilde{x}^i) \quad = \quad \frac{\partial \tilde{x}^\mu}{\partial x^\alpha} \frac{\partial \tilde{x}^\nu}{\partial x^\beta} g^{\alpha\beta}(t, x^i)$$

- Note: to stay close to the inflationary literature, 0 will represent coordinate rather than conformal time
- Consider the restricted set of gauge transformations that change only the spatial coordinates

$$\tilde{x}^i = x^i + L^i; \quad \tilde{t} = t$$

- Only component that is left invariant under this transformation is g^{00}; g_{00} is not invariant if L^i depends on t.
- So the most general action is the most general function of g^{00}

Effective Field Theory

- Now consider that $g^{00}+1$ is a small metric perturbation. A general function may be expanded around this value in a Taylor series

$$S = \int d^4x\sqrt{-g}\Big[\frac{1}{2}M_{\rm Pl}^2 R + \sum_{n=0}^{\infty}\frac{1}{n!}M_n^4(t_u)(g_u^{00}+1)^n\Big],$$

- Varying action with respect to $g^{\mu\nu}$ we get the Einstein equations
- Constant term gives a cosmological constant whereas the $n=1$ term gives the effective stress tensor of the field in the background

$$H^2 = -\frac{1}{3M_{\rm Pl}^2}\left[M_0^4 + 2M_1^4\right]$$

$$\dot{H} + H^2 = -\frac{1}{3M_{\rm Pl}^2}\left[M_0^4 - M_1^4\right]$$

Effective Field Theory

- Friedmann equation can thus eliminate $n = 0, 1$

$$\begin{aligned} M_0^4 &= -(3H^2 + 2\dot{H})M_{\rm Pl}^2 \\ M_1^4 &= \dot{H} M_{\rm Pl}^2 \end{aligned}$$

- Now we an restore time slicing invariance or temporal diffs allowing for a general change in the time coordinate

$$t_u = t + \pi(t, x^i)$$

- In particle physics language this is the Stuckelberg trick and π is a Stuckelberg field.
- To connect with the canonical linearized treatment $\phi_1 = \dot{\phi}_0 \pi$ so $\mathcal{R} = -H\pi$ but here defined to be the field fluctuation to all orders.

Effective Field Theory

- Transformation to arbitrary slicing is given by

$$g_u^{00} = \frac{\partial t_u}{\partial x^\mu}\frac{\partial t_u}{\partial x^\nu} g^{\mu\nu}$$

- Each $M_n^4(t_u = t + \pi)$ and hence carry extra Taylor expansion terms

- In general, transformation mixes π and metric fluctuations $\delta g^{\mu\nu}$ including terms like

$$\dot\pi \delta g^{00}, \quad \delta g \dot\pi, \quad \partial_i \pi g^{0i}, \quad \partial_i \pi \partial_j \pi \delta g^{ij}$$

  in the canonical linear theory calculation, the first three were the $\dot A$, $\dot H_L$, kB terms after integration by parts and the last is cubic order

Effective Field Theory

- Again we make use of the fact that sub horizon scales these metric terms are subdominant
- In spatially flat gauge the domain of validity extends even through the horizon if we neglect slow roll corrections
- In this case we can ignore the terms associated with the spatial pieces of the metric and replace

$$g_u^{00} = -(1+\dot\pi)^2 + \frac{(\partial_i \pi)^2}{a^2}$$

- Each $g_u^{00} + 1$ factor carries terms that are linear and quadratic in π

$$(g_u^{00} + 1)^n = (-\dot\pi)^n \sum_{i=0}^{n} \frac{2^{n-i} n!}{i!(n-i)!} \Pi^i$$

Effective Field Theory

- So each M_n^4 term contributes from π^n to π^{2n}

$$\Pi = \dot\pi\left(1 - \frac{(\partial_i\pi)^2}{a^2\dot\pi^2}\right)$$

- For example M_2

$$\begin{aligned}(g_u^{00}+1)^2 &= \dot\pi^2\left[4 + 4\dot\pi\left(1 - \frac{(\partial_i\pi)^2}{a^2\dot\pi^2}\right) + \dot\pi^2\left(1 - \frac{(\partial_i\pi)^2}{a^2\dot\pi^2}\right)^2\right] \\ &= 4(\dot\pi^2 + \dot\pi^3 - \dot\pi\frac{(\partial_i\pi)^2}{a^2}) + \ldots\end{aligned}$$

implies both a cubic and quartic Lagrangian. To cubic order

$$S_\pi = \int d^4x\sqrt{-g}\Big[-M_{\rm Pl}^2\dot H\left(\dot\pi^2 - \frac{(\partial_i\pi)^2}{a^2}\right) + 2M_2^4(\dot\pi^2 + \dot\pi^3 - \dot\pi\frac{(\partial_i\pi)^2}{a^2}) + \ldots$$

Effective Field Theory

- Isolate the quadratic action

$$S_{\pi 2} = \int d^4x\sqrt{-g}\left[(-M_{\rm Pl}^2\dot H + 2M_2^4)\dot\pi^2 + M_{\rm Pl}^2\dot H\frac{(\partial_i\pi)^2}{a^2}\right]$$

and identify the sound speed from $\omega = (k/a)c_s$

$$c_s^{-2} = 1 - \frac{2M_2^4}{M_{\rm Pl}^2\dot H}; \quad \Pi \sim \dot\pi\left(1 - \frac{1}{c_s^2}\right)$$

using $-\dot H = \epsilon H^2$

$$\begin{aligned} S_{\pi 2} &= \int dt d^3x (a^3\epsilon H^2) M_{\rm Pl}^2 c_s^{-2}\left[\dot\pi^2 - c_s^2\frac{(\partial_i\pi)^2}{a^2}\right] \\ &= \int d\eta d^3x \frac{z^2 H^2 M_{\rm Pl}^2}{2}\left[\left(\frac{\partial\pi}{\partial\eta}\right)^2 - c_s^2(\partial_i\pi)^2\right]\end{aligned}$$

where $z^2 = 2a^2\epsilon/c_s^2$ is the generalization of Mukhanov z

Effective Field Theory

- So a field redefinition canonically normalizes the field

$$u = zH\pi M_{\rm Pl}$$

brings the EFT action to canonical form (assuming M_n^4 =const.)

$$\begin{aligned} S_u &= \int d\eta d^3x \left[\left(\frac{\partial u}{\partial \eta}\right)^2 - c_s^2(\partial_i u)^2 - 2u\frac{\partial u}{\partial \eta}\frac{d\ln z}{d\eta} + u^2\left(\frac{d\ln z}{d\eta}\right)^2 \right] \\ &= \int d\eta d^3x \left[\left(\frac{\partial u}{\partial \eta}\right)^2 - c_s^2(\partial_i u)^2 + \frac{u^2}{z}\frac{d^2 z}{d\eta} \right] \end{aligned}$$

which is the generalization of the u field of canonical inflation

- Quantize this field, noting that $1/\sqrt{E}$ normalization factor goes to $1/\sqrt{kc_s}$ yielding the modefunction

$$u = \frac{1}{\sqrt{2kc_s}}\left(1 - \frac{i}{kc_s\tilde\eta}\right)e^{-ikc_s\tilde\eta}$$

Effective Field Theory

- Curvature fluctuations then freezeout at $kc_s\tilde\eta = 1$ (sound horizon crossing) at a value

$$\mathcal{R} = -H\pi = \frac{c_s}{a\sqrt{2\epsilon}}\frac{1}{\sqrt{2kc_s}}\frac{i}{kc_s\tilde\eta M_{\rm Pl}} \approx \frac{-iH}{2k^{3/2}\sqrt{\epsilon c_s}M_{\rm Pl}}$$

- So

$$\Delta^2_{\mathcal{R}} = \frac{k^3|\mathcal{R}|^2}{2\pi^2} = \frac{H^2}{8\pi^2\epsilon c_s M_{\rm Pl}^2}$$

- Generalization is that the sound speed enters in two ways: boosts scalars over tensors by c_s and changes the epoch of freezeout between scalars and tensors

Effective Field Theory

- Returning to the original π action, since M_2^4 carries cubic term this requires a non-Gaussianity

$$S_\pi = \int d^4x\sqrt{-g}\left[-\frac{M_{\rm Pl}^2\dot H}{c_s^2}\left(\dot\pi^2 + c_s^2\frac{(\partial_i\pi)^2}{a^2}\right) + M_{\rm Pl}^2\dot H\left(1-\frac{1}{c_s^2}\right)\left(\dot\pi^3 - \dot\pi\frac{(\partial_i\pi)^2}{a^2}\right)\right] + \ldots$$

- For $c_s \ll 1$, spatial gradients dominate temporal derivatives

$$\partial_0 \to \omega, \partial_i \to k, \omega = kc_s/a$$

and leading order cubic term is $\dot\pi(\partial_i\pi)^2$

- Estimate the size of the non-Gaussianity by taking the ratio of cubic to quadratic at $c_s \ll 1$

$$\frac{\dot\pi(\partial_i\pi)^2}{a^2\dot\pi^2} \sim \frac{k\pi_{\rm rms}}{c_s a} \qquad \text{where} \quad \pi_{\rm rms} = \left(\frac{k^3|\pi|^2}{2\pi^2}\right)^{1/2}$$

Effective Field Theory

- Deep within the horizon $u = 1/\sqrt{2kc_s}$ and so

$$\begin{aligned}\frac{k\pi_{\rm rms}}{c_s a} &\sim \frac{k}{c_s a}\left(\frac{k^2}{2z^2H^2c_sM_{\rm Pl}^2}\right)^{1/2} \\ &\sim \left(\frac{kc_s}{aH}\right)^2\frac{H}{M_{\rm Pl}\sqrt{\epsilon c_s}}\frac{1}{c_s^2} \\ &\sim \left(\frac{kc_s}{aH}\right)^2\frac{\Delta_{\mathcal{R}}}{c_s^2} < 1\end{aligned}$$

- Since $kc_s/aH \sim \omega/H$ is a ratio of an energy scale to Hubble, the bound determines the strong coupling scale

$$\frac{\omega_{sc}}{H} \sim \frac{c_s}{\sqrt{\Delta_{\mathcal{R}}}} \sim 10^2 c_s$$

- For $c_s < 0.01$ the strong coupling scale is near the horizon and the effective theory has broken down before freezeout

Effective Field Theory

- Now consider a less extreme c_s
- Here the effective theory becomes valid at least several efolds before horizon crossing and we can make predictions within the theory
- Not surprisingly non-Gaussianity is enhanced by these self interactions and freezeout at $kc_s \sim aH$

$$
\begin{aligned}
\frac{k\pi_{\rm rms}}{c_s a} &\sim \frac{k}{c_s a}\left(\frac{1}{\epsilon c_s M_{\rm Pl}^2}\right)^{1/2} \\
&\sim \frac{kc_s}{aH}\frac{H}{\sqrt{\epsilon c_s} M_{\rm Pl}}\frac{1}{c_s^2} \\
&\sim \frac{\Delta_{\mathcal{R}}}{c_s^2}
\end{aligned}
$$

and so bispectrum is enhanced over the naive expectation by c_s^{-2}

Effective Field Theory

- More generally, each M_n^4 sets its own strong coupling scale

$$\frac{\mathcal{L}_n}{\mathcal{L}_2} \sim 1$$

These coincide if

$$\frac{M_n^4}{M_2^4} \sim \left(\frac{1}{c_s^2}\right)^{n-2}$$

which would be the natural prediction if the M_2 strong coupling scale indicated the scale of new physics and we take all allowed operators as order unity at that scale

K-inflation $P(X, \phi)$

- EFT was built to cover the case of scalar field Lagrangian that is a general function of its kinetic term and field value

$$\mathcal{L} = P(X, \phi)$$

where $2X = -g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$

- In this case

$$M_n^4 = (-X)^n \frac{\partial^n P}{\partial X^n}$$

and

$$c_s^{-2} = 1 + \frac{2XP_{,XX}}{P_{,X}}$$

DBI

- An example coming from string inspired models is the Dirac-Born-Infeld Lagrangian

$$\mathcal{L} = \left[1 - \sqrt{1 - 2X/T(\phi)}\right] T(\phi) - V(\phi),$$

  where $T(\phi)$ is the warped brane tension and ϕ denotes the position of the brane in a higher dimension

- If $X/T \ll 1$ then $\mathcal{L} = X - V$, the same as a canonical scalar field

DBI

- Here

$$c_s(\phi, X) = \sqrt{1 - 2X/T(\phi)}$$

and

$$c_n = (-1)^n \frac{M_n^4}{M_2^4} = \frac{(2n-3)!!}{2^{n-2}} \left(\frac{1}{c_s^2} - 1 \right)^{n-2}$$

satisfying the c_s scaling of the EFT prescription