Astro 4PT

Lecture Notes Set 3

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References

- General References

  Basics: Kolb & Turner, Early Universe

  Radiation: Dodelson, Modern Cosmology

  Matter: Binney & Tremaine, Galactic Dynamics

Boltzmann Equation

- Particle distribution is generally described by the phase space distribution function for each polarization (spin) state $f_a(\mathbf{x}, \mathbf{q}, \eta)$, where $\mathbf{x}$ is the comoving position and $\mathbf{q}$ is the particle momentum
- Spatial number density is the integral over all momentum states $(\hbar = 1)$

$$n_a = \int \frac{d^3 q}{(2\pi)^3} f_a$$

- Boltzmann equation describes the evolution of the distribution function under gravity and collisions
- Zeroth, first moments of the Boltzmann equation are simply the covariant conservation equations
- Higher moments provide the closure condition to the conservation law (specification of stress tensor)

Boltzmann Equation

- Second moment: anisotropic stress tensor

  Radiation - the quadrupole anisotropy of the distribution

  Matter - the kinetic energy tensor (shear, velocity dispersion tensor)

- For radiation higher moments mainly describe the simple geometry of source projection
- For matter higher moments (including second) small due to low thermal and bulk velocities - same behavior as perfect fluid
- Differences with perfect fluid appear in the second moment since collisionless particles can have multiple streams in the same spatial position

Liouville Equation

- In absence of scattering, the phase space distribution of photons is conserved along the propagation path
- Rewrite variables in terms of the particle propagation direction $\mathbf{q} = q\hat{\mathbf{n}}$, so $f_a(\mathbf{x}, \hat{\mathbf{n}}, q, \eta)$ and

$$\frac{d}{d\eta} f_a(\mathbf{x}, \hat{\mathbf{n}}, q, \eta) = 0$$

$$= \left(\frac{\partial}{\partial \eta} + \frac{d\mathbf{x}}{d\eta} \cdot \frac{\partial}{\partial \mathbf{x}} + \frac{d\hat{\mathbf{n}}}{d\eta} \cdot \frac{\partial}{\partial \hat{\mathbf{n}}} + \frac{dq}{d\eta} \cdot \frac{\partial}{\partial q} \right) f_a$$

- For simplicity, assume spatially flat universe $K = 0$ then $d\hat{\mathbf{n}}/d\eta = 0$ and $d\mathbf{x} = \hat{\mathbf{n}} d\eta$

$$\dot{f}_a + \hat{\mathbf{n}} \cdot \nabla f_a + \dot{q} \frac{\partial}{\partial q} f_a = 0$$

Correspondence to Einstein Eqn.

- Geodesic equation gives the redshifting term

$$\frac{\dot{q}}{q} = -\frac{\dot{a}}{a} - \frac{1}{2} n^i n^j \dot{H}_{Tij} - \dot{H}_L + n^i \dot{B}_i - \hat{\mathbf{n}} \cdot \nabla A$$

  which is incorporated in the conservation and gauge transformation equations

- Stress energy tensor involves integrals over the distribution function summed over polarization (spin) states

$$T^{\mu\nu} = \sum_a \int \frac{d^3q}{(2\pi)^3} \frac{q^\mu q^\nu}{E} f_a$$

- Components are simply the low order angular moments of the distribution function

Perfect Fluid

- Compare with the stress tensor of a perfect fluid

$$T^{\mu}{}_{\nu} = (\rho + p)U^{\mu}U_{\nu} + p\delta^{\mu}{}_{\nu}$$

- Anisotropic stress tensor of perfect fluid also related to the tensor constructed out of the 4-momentum
- However now there is no phase space distribution that allows multiple streams at a given point in space - only a single bulk velocity
- Differences in the density evolution will appear as soon as particle trajectories cross - fluid shocks and collisonless particles pass through each other

Scalar Field

- Scalar field (with arbitrary kinetic term) can be recast as a perfect fluid
- Given a Lagrangian $\mathcal{L}$ with a kinetic term

$$X = -\frac{1}{2}\nabla_\mu \phi \nabla^\mu \phi$$

the stress tensor is

$$T^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial X}\nabla^\mu \phi \nabla_\nu \phi + \mathcal{L}\delta^\mu{}_\nu$$

and we can associate a pressure $p = \mathcal{L}$ and a fluid velocity

$$U_\mu = \frac{\nabla_\mu Q}{(2X)^{1/2}}$$

Scalar Field

- Canonical scalar field cannot work as dark matter given its large sound speed (in comoving slicing) $\delta p/\delta\rho = 1$. Here

$$\mathcal{L} = X - V$$

- Attempts to unify dark matter and dark energy through a single scalar field must overcome the fact that such matter behaves as a perfect fluid, e.g.

$$\mathcal{L} = C(X - X_{\rm min})^2$$

- Exception: the axion where ϕ varies as e^{imt} and multiple stream info can be stored in the phase

$$\mathcal{L} = X - m^2\phi - V_0$$

Radiation Angular Moments

- Define the angularly dependent temperature perturbation

$$\Theta(\mathbf{x}, \hat{\mathbf{n}}, \eta) = \frac{1}{4\rho_\gamma} \int \frac{q^3 dq}{2\pi^2} (f_a + f_b) - 1$$

  and likewise for the linear polarization states Q and U

- Decompose into normal modes: plane waves for spatial part and spherical harmonics for angular part

$$G_\ell^m(\mathbf{k}, \mathbf{x}, \hat{\mathbf{n}}) \equiv (-i)^\ell \sqrt{\frac{4\pi}{2\ell+1}} Y_\ell^m(\hat{\mathbf{n}}) \exp(i\mathbf{k}\cdot\mathbf{x})$$

$$_{\pm 2}G_\ell^m(\mathbf{k}, \mathbf{x}, \hat{\mathbf{n}}) \equiv (-i)^\ell \sqrt{\frac{4\pi}{2\ell+1}}\, {}_{\pm 2}Y_\ell^m(\hat{\mathbf{n}}) \exp(i\mathbf{k}\cdot\mathbf{x})$$

- In a spatially curved universe generalize the plane wave part

Normal Modes

- Temperature and polarization fields

$$\Theta(\mathbf{x}, \hat{\mathbf{n}}, \eta) = \int \frac{d^3k}{(2\pi)^3} \sum_{\ell m} \Theta_\ell^{(m)} G_\ell^m$$

$$[Q \pm iU](\mathbf{x}, \hat{\mathbf{n}}, \eta) = \int \frac{d^3k}{(2\pi)^3} \sum_{\ell m} [E_\ell^{(m)} \pm iB_\ell^{(m)}]_{\pm 2} G_\ell^m$$

- For each $\mathbf{k}$ mode, work in coordinates where $\mathbf{k} \parallel \mathbf{z}$ and so $m = 0$ represents scalar modes, $m = \pm 1$ vector modes, $m = \pm 2$ tensor modes, $|m| > 2$ vanishes. Since modes add incoherently and $Q \pm iU$ is invariant up to a phase, rotation back to a fixed coordinate system is trivial.

Scalar, Vector, Tensor

- Normalization of modes is chosen so that the lowest angular mode for scalars, vectors and tensors are normalized in the same way as the mode function

$$G_0^0 = Q^{(0)} \quad G_1^0 = n^i Q_i^{(0)} \quad G_2^0 \propto n^i n^j Q_{ij}^{(0)}$$
$$G_1^{\pm 1} = n^i Q_i^{(\pm 1)} \quad G_2^{\pm 1} \propto n^i n^j Q_{ij}^{(\pm 1)}$$
$$G_2^{\pm 2} = n^i n^j Q_{ij}^{(\pm 2)}$$

where recall

$$\begin{aligned}
Q^{(0)} &= \exp(i\mathbf{k}\cdot\mathbf{x}) \\
Q_i^{(\pm 1)} &= \frac{-i}{\sqrt{2}}(\hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2)_i \exp(i\mathbf{k}\cdot\mathbf{x}) \\
Q_{ij}^{(\pm 2)} &= -\sqrt{\frac{3}{8}}(\hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2)_i(\hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2)_j \exp(i\mathbf{k}\cdot\mathbf{x})
\end{aligned}$$

Geometrical Projection

- Main content of Liouville equation is purely geometrical and describes the projection of inhomogeneities into anisotropies
- Spatial gradient term hits plane wave:

$$\hat{\mathbf{n}} \cdot \nabla e^{i\mathbf{k}\cdot\mathbf{x}} = i\hat{\mathbf{n}} \cdot \mathbf{k} e^{i\mathbf{k}\cdot\mathbf{x}} = i\sqrt{\frac{4\pi}{3}} k Y_1^0(\hat{\mathbf{n}}) e^{i\mathbf{k}\cdot\mathbf{x}}$$

- Dipole term adds to angular dependence through the addition of angular momentum

$$\sqrt{\frac{4\pi}{3}} Y_1^0 Y_\ell^m = \frac{\kappa_\ell^m}{\sqrt{(2\ell+1)(2\ell-1)}} Y_{\ell-1}^m + \frac{\kappa_{\ell+1}^m}{\sqrt{(2\ell+1)(2\ell+3)}} Y_{\ell+1}^m$$

where $\kappa_\ell^m = \sqrt{\ell^2 - m^2}$ is given by Clebsch-Gordon coefficients.

Temperature Hierarchy

- Absorb recoupling of angular momentum into evolution equation for normal modes

$$\dot{\Theta}_\ell^{(m)} = k\left[\frac{\kappa_\ell^m}{2\ell+1}\Theta_{\ell-1}^{(m)} - \frac{\kappa_{\ell+1}^m}{2\ell+3}\Theta_{\ell+1}^{(m)}\right] - \dot{\tau}\Theta_\ell^{(m)} + S_\ell^{(m)}$$

  where $S_\ell^{(m)}$ are the gravitational (and later scattering sources; added scattering suppression of anisotropy)

- An originally isotropic $\ell = 0$ temperature perturbation will eventually become a high order anisotropy by "free streaming" or simple projection
- Original CMB codes solved the full hierarchy equations out to the ℓ of interest.

Integral Solution

- Hierarchy equation simply represents geometric projection, exactly as we have seen before in the projection of temperature perturbations on the last scattering surface
- In general, the solution describes the decomposition of the source $S_\ell^{(m)}$ with its local angular dependence as seen at a distance $\mathbf{x} = D\hat{\mathbf{n}}$.
- Proceed by decomposing the angular dependence of the plane wave

$$e^{i\mathbf{k}\cdot\mathbf{x}} = \sum_\ell (-i)^\ell \sqrt{4\pi(2\ell+1)} j_\ell(kD) Y_\ell^0(\hat{\mathbf{n}})$$

- Recouple to the local angular dependence of G_ℓ^m

$$G_{\ell_s}^m = \sum_\ell (-i)^\ell \sqrt{4\pi(2\ell+1)} \alpha_{\ell_s \ell}^{(m)}(kD) Y_\ell^m(\hat{\mathbf{n}})$$

Integral Solution

- Projection kernels:

$$\ell_s = 0, \quad m = 0 \qquad\qquad \alpha_{0\ell}^{(0)} \equiv j_\ell$$

$$\ell_s = 1, \quad m = 0 \qquad\qquad \alpha_{1\ell}^{(0)} \equiv j'_\ell$$

- Integral solution:

$$\frac{\Theta_\ell^{(m)}(k,\eta_0)}{2\ell+1} = \int_0^{\eta_0} d\eta e^{-\tau} \sum_{\ell_s} S_{\ell_s}^{(m)} \alpha_{\ell_s \ell}^{(m)}(k(\eta_0 - \eta))$$

- Power spectrum:

$$C_\ell = \frac{2}{\pi} \int \frac{dk}{k} \sum_m \frac{k^3 \langle \Theta_\ell^{(m)*} \Theta_\ell^{(m)} \rangle}{(2\ell+1)^2}$$

- Solving for C_ℓ reduces to solving for the behavior of a handful of sources

Polarization Hiearchy

- In the same way, the coupling of a gradient or dipole angular momentum to the spin harmonics leads to the polarization hiearchy:

$$\dot{E}_\ell^{(m)} = k\left[\frac{{}_2\kappa_\ell^m}{2\ell-1}E_{\ell-1}^{(m)} - \frac{2m}{\ell(\ell+1)}B_\ell^{(m)} - \frac{{}_2\kappa_{\ell+1}^m}{2\ell+3}\right] - \dot{\tau}E_\ell^{(m)} + \mathcal{E}_\ell^{(m)}$$

$$\dot{B}_\ell^{(m)} = k\left[\frac{{}_2\kappa_\ell^m}{2\ell-1}B_{\ell-1}^{(m)} + \frac{2m}{\ell(\ell+1)}B_\ell^{(m)} - \frac{{}_2\kappa_{\ell+1}^m}{2\ell+3}\right] - \dot{\tau}E_\ell^{(m)} + \mathcal{B}_\ell^{(m)}$$

  where ${}_2\kappa_\ell^m = \sqrt{(\ell^2-m^2)(\ell^2-4)/\ell^2}$ is given by the Clebsch-Gordon coefficients and $\mathcal{E}$, $\mathcal{B}$ are the sources (scattering only).

- Note that for vectors and tensors $|m| > 0$ and B modes may be generated from E modes by projection. Cosmologically $\mathcal{B}_\ell^{(m)} = 0$

Polarization Integral Solution

- Again, we can recouple the plane wave angular momentum of the source inhomogeneity to its local angular dependence directly

$$\frac{E_\ell^{(m)}(k,\eta_0)}{2\ell+1} = \int_0^{\eta_0} d\eta e^{-\tau} \mathcal{E}_{\ell_s}^{(m)} \epsilon_{\ell_s \ell}^{(m)}(k(\eta_0-\eta))$$

$$\frac{B_\ell^{(m)}(k,\eta_0)}{2\ell+1} = \int_0^{\eta_0} d\eta e^{-\tau} \mathcal{E}_{\ell_s}^{(m)} \beta_{\ell_s \ell}^{(m)}(k(\eta_0-\eta))$$

- The only source to the polarization is from the quadrupole anisotropy so we only need $\ell_s = 2$, e.g. for scalars

$$\epsilon_{2\ell}^{(0)}(x) = \sqrt{\frac{3}{8}\frac{(\ell+2)!}{(\ell-2)!}}\frac{j_\ell(x)}{x^2} \qquad \beta_{2\ell}^{(0)} = 0$$

Truncated Hierarchy

- CMBFast uses the integral solution and relies on a fast j_ℓ generator
- However sources are not external to system and are defined through the Boltzmann hierarchy itself
- Solution: recall that we used this technique in the tight coupling regime by applying a closure condition from tight coupling
- CMBFast extends this idea by solving a truncated hierarchy of equations, e.g. out to $\ell = 25$ with non-reflecting boundary conditions

Thomson Collision Term

- Full Boltzmann equation

$$\frac{d}{d\eta} f_{a,b} = C[f_a, f_b]$$

- Collision term describes the scattering out of and into a phase space element
- Thomson collision based on differential cross section

$$\frac{d\sigma}{d\Omega} = \frac{3}{8\pi} |\hat{\mathbf{E}}' \cdot \hat{\mathbf{E}}|^2 \sigma_T \,,$$

where $\hat{\mathbf{E}}'$ and $\hat{\mathbf{E}}$ denote the incoming and outgoing directions of the electric field or polarization vector.

Scattering Calculation

- Start in the electron rest frame and in a coordinate system fixed by the scattering plane, spanned by incoming and outgoing directional vectors $-\hat{\mathbf{n}}' \cdot \hat{\mathbf{n}} = \cos\beta$, where β is the scattering angle
- $\Theta_{||}$: in-plane polarization state; $\Theta_{\perp}$: $\perp$-plane polarization state
- Transfer probability (constant set by $\dot{\tau}$)

$$\Theta_{||} \propto \cos^2\beta\, \Theta'_{||}, \qquad \Theta_{\perp} \propto \Theta'_{\perp}$$

- and with the $45°$ axes as

$$\hat{\mathbf{E}}_1 = \frac{1}{\sqrt{2}}(\hat{\mathbf{E}}_{||} + \hat{\mathbf{E}}_{\perp}), \qquad \hat{\mathbf{E}}_2 = \frac{1}{\sqrt{2}}(\hat{\mathbf{E}}_{||} - \hat{\mathbf{E}}_{\perp})$$

Stokes Parameters

- Define the temperature in this basis

$$\begin{aligned}
\Theta_1 &\propto |\hat{\mathbf{E}}_1 \cdot \hat{\mathbf{E}}_1|^2 \Theta_1' + |\hat{\mathbf{E}}_1 \cdot \hat{\mathbf{E}}_2|^2 \Theta_2' \\
&\propto \frac{1}{4}(\cos\beta + 1)^2 \Theta_1' + \frac{1}{4}(\cos\beta - 1)^2 \Theta_2' \\
\Theta_2 &\propto |\hat{\mathbf{E}}_2 \cdot \hat{\mathbf{E}}_2|^2 \Theta_2' + |\hat{\mathbf{E}}_2 \cdot \hat{\mathbf{E}}_1|^2 \Theta_1' \\
&\propto \frac{1}{4}(\cos\beta + 1)^2 \Theta_2' + \frac{1}{4}(\cos\beta - 1)^2 \Theta_1'
\end{aligned}$$

or $\Theta_1 - \Theta_2 \propto \cos\beta(\Theta_1' - \Theta_2')$

- Define Θ, Q, U in the scattering coordinates

$$\Theta \equiv \frac{1}{2}(\Theta_\parallel + \Theta_\perp), \quad Q \equiv \frac{1}{2}(\Theta_\parallel - \Theta_\perp), \quad U \equiv \frac{1}{2}(\Theta_1 - \Theta_2)$$

Scattering Matrix

- Transfer of Stokes states, e.g.

$$\Theta = \frac{1}{2}(\Theta_{\parallel} + \Theta_{\perp}) \propto \frac{1}{4}(\cos^2\beta + 1)\Theta' + \frac{1}{4}(\cos^2\beta - 1)Q'$$

- Transfer matrix of Stokes state $\mathbf{T} \equiv (\Theta, Q + iU, Q - iU)$

$$\mathbf{T} \propto \mathbf{S}(\beta)\mathbf{T}'$$

$$\mathbf{S}(\beta) = \frac{3}{4}\begin{pmatrix} \cos^2\beta + 1 & -\frac{1}{2}\sin^2\beta & -\frac{1}{2}\sin^2\beta \\ -\frac{1}{2}\sin^2\beta & \frac{1}{2}(\cos\beta + 1)^2 & \frac{1}{2}(\cos\beta - 1)^2 \\ -\frac{1}{2}\sin^2\beta & \frac{1}{2}(\cos\beta - 1)^2 & \frac{1}{2}(\cos\beta + 1)^2 \end{pmatrix}$$

normalization factor of 3 is set by photon conservation in scattering

Scattering Matrix

- Transform to a fixed basis, by a rotation of the incoming and outgoing states $\mathbf{T} = \mathbf{R}(\psi)\mathbf{T}$ where

$$\mathbf{R}(\psi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-2i\psi} & 0 \\ 0 & 0 & e^{2i\psi} \end{pmatrix}$$

giving the scattering matrix

$$\mathbf{R}(-\gamma)\mathbf{S}(\beta)\mathbf{R}(\alpha) =$$

$$\frac{1}{2}\sqrt{\frac{4\pi}{5}} \begin{pmatrix} Y_2^0(\beta,\alpha) + 2\sqrt{5}Y_0^0(\beta,\alpha) & -\sqrt{\frac{3}{2}}Y_2^{-2}(\beta,\alpha) & -\sqrt{\frac{3}{2}}Y_2^2(\beta,\alpha) \\ -\sqrt{6}\,{}_2Y_2^0(\beta,\alpha)e^{2i\gamma} & 3\,{}_2Y_2^{-2}(\beta,\alpha)e^{2i\gamma} & 3\,{}_2Y_2^2(\beta,\alpha)e^{2i\gamma} \\ -\sqrt{6}\,{}_{-2}Y_2^0(\beta,\alpha)e^{-2i\gamma} & 3\,{}_{-2}Y_2^{-2}(\beta,\alpha)e^{-2i\gamma} & 3\,{}_{-2}Y_2^2(\beta,\alpha)e^{-2i\gamma} \end{pmatrix}$$

Addition Theorem for Spin Harmonics

- Spin harmonics are related to rotation matrices as

$$
{}_sY_\ell^m(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}}\mathcal{D}^\ell_{-ms}(\phi,\theta,0)
$$

Note: for explicit evaluation sign convention differs from usual (e.g. Jackson) by $(-1)^m$

- Multiplication of rotations

$$
\sum_{m''}\mathcal{D}^\ell_{mm''}(\alpha_2,\beta_2,\gamma_2)\mathcal{D}^\ell_{m''m}(\alpha_1,\beta_1,\gamma_1) = \mathcal{D}^\ell_{mm'}(\alpha,\beta,\gamma)
$$

- Implies

$$
\sum_m {}_{s_1}Y_\ell^{m*}(\theta',\phi')\ {}_{s_2}Y_\ell^m(\theta,\phi) = (-1)^{s_1-s_2}\sqrt{\frac{2\ell+1}{4\pi}}\ {}_{s_2}Y_\ell^{-s_1}(\beta,\alpha)e^{is_2\gamma}
$$

Sky Basis

- Scattering into the state (rest frame)

$$C_{\rm in}[\mathbf{T}] = \dot\tau \int \frac{d\hat{\mathbf{n}}'}{4\pi} \mathbf{R}(-\gamma)\mathbf{S}(\beta)\mathbf{R}(\alpha)\mathbf{T}(\hat{\mathbf{n}}') \,,$$

$$= \dot\tau \int \frac{d\hat{\mathbf{n}}'}{4\pi} (\Theta', 0, 0) + \frac{1}{10}\dot\tau \int d\hat{\mathbf{n}}' \sum_{m=-2}^{2} \mathbf{P}^{(m)}(\hat{\mathbf{n}},\hat{\mathbf{n}}')\mathbf{T}(\hat{\mathbf{n}}') \,.$$

where the quadrupole coupling term is $\mathbf{P}^{(m)}(\hat{\mathbf{n}},\hat{\mathbf{n}}') =$

$$\left(\begin{array}{ccc} Y_2^{m*}(\hat{\mathbf{n}}')\, Y_2^m(\hat{\mathbf{n}}) & -\sqrt{\frac{3}{2}}\, {}_2Y_2^{m*}(\hat{\mathbf{n}}')\, Y_2^m(\hat{\mathbf{n}}) & -\sqrt{\frac{3}{2}}\, {}_{-2}Y_2^{m*}(\hat{\mathbf{n}}')\, Y_2^m(\hat{\mathbf{n}}) \\ -\sqrt{6} Y_2^{m*}(\hat{\mathbf{n}}')\, {}_2Y_2^m(\hat{\mathbf{n}}) & 3\, {}_2Y_2^{m*}(\hat{\mathbf{n}}')\, {}_2Y_2^m(\hat{\mathbf{n}}) & 3\, {}_{-2}Y_2^{m*}(\hat{\mathbf{n}}')\, {}_2Y_2^m(\hat{\mathbf{n}}) \\ -\sqrt{6} Y_2^{m*}(\hat{\mathbf{n}}')\, {}_{-2}Y_2^m(\hat{\mathbf{n}}) & 3\, {}_2Y_2^{m*}(\hat{\mathbf{n}}')\, {}_{-2}Y_2^m(\hat{\mathbf{n}}) & 3\, {}_{-2}Y_2^{m*}(\hat{\mathbf{n}}')\, {}_{-2}Y_2^m(\hat{\mathbf{n}}) \end{array} \right),$$

expression uses angle addition relation above. We call this term C_Q.

Scattering Matrix

- Full scattering matrix involves difference of scattering into and out of state

$$C[\mathbf{T}] = C_{\rm in}[\mathbf{T}] - C_{\rm out}[\mathbf{T}]$$

- In the electron rest frame

$$C[\mathbf{T}] = \dot\tau \int \frac{d\hat{\mathbf{n}}'}{4\pi}(\Theta', 0, 0) - \dot\tau \mathbf{T} + C_Q[\mathbf{T}]$$

which describes isotropization in the rest frame. All moments have $e^{-\tau}$ suppression except for isotropic temperature Θ_0. Transformation into the background frame simply induces a dipole term

$$C[\mathbf{T}] = \dot\tau \left(\hat{\mathbf{n}} \cdot \mathbf{v}_b + \int \frac{d\hat{\mathbf{n}}'}{4\pi}\Theta', 0, 0 \right) - \dot\tau \mathbf{T} + C_Q[\mathbf{T}]$$

Source Terms

- Temperature source terms $S_l^{(m)}$ (rows $\pm|m|$; flat assumption

$$\left(\begin{array}{lll} \dot\tau\Theta_0^{(0)} - \dot H_L^{(0)} & \dot\tau v_b^{(0)} + \dot B^{(0)} & \dot\tau P^{(0)} - \frac{2}{3}\dot H_T^{(0)} \\ 0 & \dot\tau v_b^{(\pm 1)} + \dot B^{(\pm 1)} & \dot\tau P^{(\pm 1)} - \frac{\sqrt{3}}{3}\dot H_T^{(\pm 1)} \\ 0 & 0 & \dot\tau P^{(\pm 2)} - \dot H_T^{(\pm 2)} \end{array}\right)$$

where

$$P^{(m)} \equiv \frac{1}{10}(\Theta_2^{(m)} - \sqrt{6}E_2^{(m)})$$

- Polarization source term

$$\mathcal{E}_\ell^{(m)} = -\dot\tau\sqrt{6}P^{(m)}\delta_{\ell,2}$$

$$\mathcal{B}_\ell^{(m)} = 0$$