

FRW Cosmology

Qualitative Review

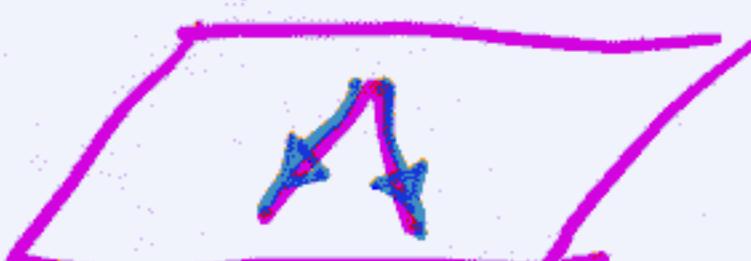
A homogeneous & isotropic cosmology is described by its geometry & expansion rate

1. Spatial Geometry



$$K>0$$

closed



$$K=0$$

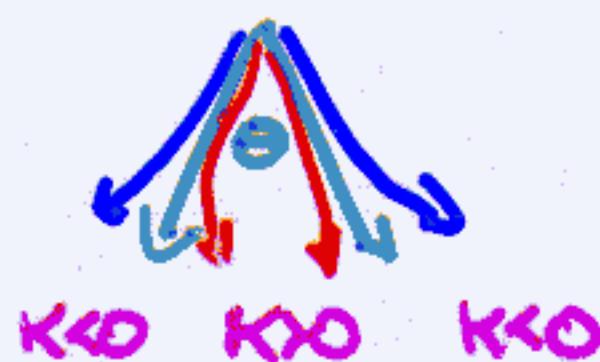
flat



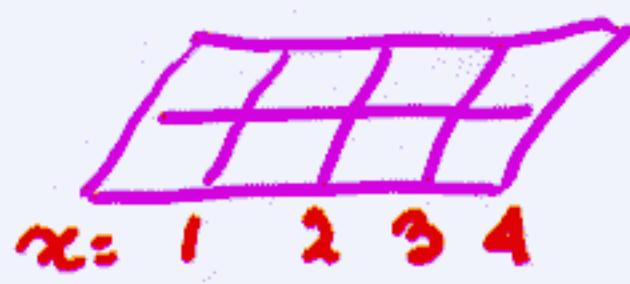
$$K<0$$

open

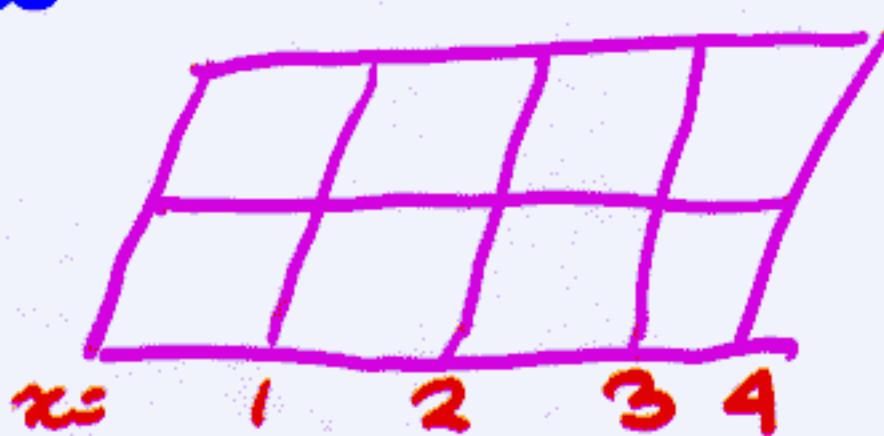
angular diameter distance: $\Theta \neq \frac{r}{d}$ if $K \neq 0$



2. Expansion stretches the scale of unbound objects



$$\rightarrow a(t)$$



so use comoving coordinates

$$d = a\chi$$

\Rightarrow conservation laws simple, but scattering processes etc revert to physical coordinates: e.g. densities, ϵ

- Hubble's Law

$$d = ax$$



$$v = \dot{d} = \dot{a}x$$

$$= \frac{\dot{a}}{a} ax = \frac{\dot{a}}{a} d \equiv H_0 d$$

$$= CZ$$

NB $\bullet = \frac{d}{dt}$ not conform time

- Expansion defined by $a(t) \Rightarrow$ metric

Einstein: metric \Leftrightarrow matter

$$H \equiv \frac{1}{a} \frac{da}{dt} \Rightarrow t(a) = \int \frac{da}{a} \frac{1}{H}$$

Friedman Eqn:

$$\frac{H^2(t)}{H^2(t_0)} = \frac{H^2(t)}{H_0^2}$$

$$= \frac{\rho_{\text{TOT}}(t)}{\rho_{\text{TOT}}(t_0)}$$

$$\left[\eta(a) = \int \frac{dt}{a} = \int \frac{da}{a^2} \frac{1}{H} \right]$$

"conformal time" comoving equiv $t \leftrightarrow \eta$
 $d \leftrightarrow x$

\therefore defined by how ρ scales with a

nr. matter particle number conservation $n \propto a^{-3}$
 $\rho = mn \propto a^{-3}$

radiation particle number conservation $n \propto a^{-3}$
Energy/momentum redshift $E \propto a^{-1} \Rightarrow \rho \propto a^{-4}$

difference? equation of state $w = p/\rho$

Pressure

Energy conservation/first Law of Thermodynamics

$$d\rho a^3 + \rho da^3 = 0$$

$$\dot{\rho} a^3 + 3\frac{\dot{a}}{a}\rho a^3 + 3\frac{\dot{a}}{a}\rho a^3 = 0$$

$$\frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a}$$

$$\rho \propto a^{-3(1+w)}$$

$w=0$	$\rho \propto a^{-3}$
$w=1/3$	$\rho \propto a^{-1}$
$w=-1$	$\rho = \text{const}$

Acceleration

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{H_0^2}{\rho_0} \rho$$

$$2\left(\frac{\dot{a}}{a}\right)\left[\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2\right] = -\frac{H_0^2}{\rho_0} 3(1+w)\frac{\dot{a}}{a} \rho$$

$$2\frac{\ddot{a}}{a} = -\frac{H_0^2}{\rho_0} (1+3w) \rho$$

$\frac{\ddot{a}}{a} = -\frac{1}{2} \frac{H_0^2}{\rho_0} (1+3w) \rho$
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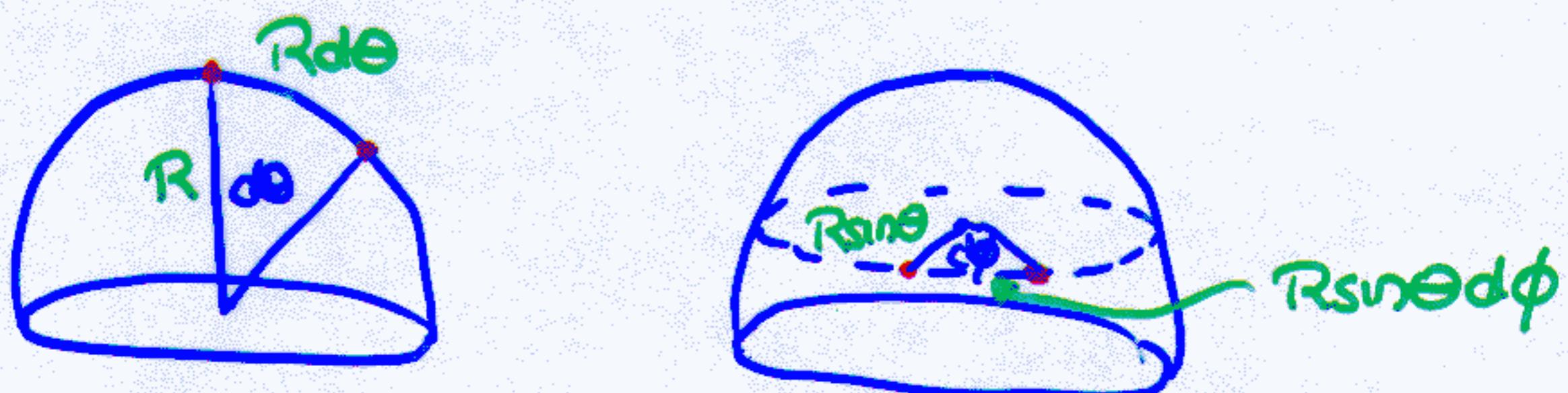
$w > -1/3$ deceleration
 $w < -1/3$ acceleration

Technical Review

1. The Metric

measure of distance in a general coordinate system

A 2-sphere of radius R



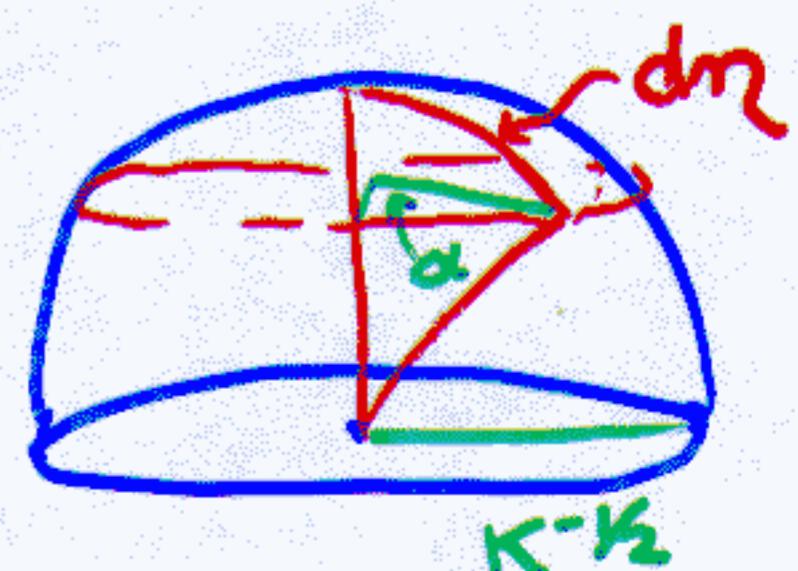
$$\therefore ds^2 = R^2(d\theta^2 + \sin^2\theta d\phi^2)$$

A 3-sphere: Closed universe of comoving radius $R = K^{-1/2}$

- First suppress one angle

Let:

$$\begin{aligned} R &\rightarrow K^{-1/2} \\ \theta &\rightarrow \eta K^{1/2} \\ \phi &\rightarrow \alpha \end{aligned}$$



$$\therefore ds^2 = d\eta^2 + \frac{1}{K} \sin^2 \eta \sqrt{K} d\alpha^2$$

- Rest of angle

$$d\theta^2 = d\Theta^2 + \sin^2\Theta d\phi^2$$

- Full comoving spatial element

$$ds_{(3)}^2 = d\eta^2 + \frac{1}{K} \sin^2 \eta \sqrt{K} [d\theta^2 + \sin^2 \theta d\phi^2]$$

$$ds_{(3)}^2 \equiv \gamma_{ij} dx^i dx^j$$

$$\gamma_{\eta\eta} = 1$$

$$\gamma_{\theta\theta} = \frac{1}{K} \sin^2 \eta \sqrt{K}$$

$$\gamma_{\phi\phi} = \sin^2 \theta$$

$$\frac{1}{K} \sin^2 \eta \sqrt{K}$$

γ_{ij} is the comoving 3-metric

for an open universe $K < 0$

$$\frac{1}{K} \sin^2 \eta \sqrt{K} \rightarrow \frac{1}{|K|} \sinh^2 \eta |K|^{1/2}$$

- Full Line element

$$ds^2 = -dt^2 + a^2 \gamma_{ij} dx^i dx^j \equiv g_{\mu\nu} dx^\mu dx^\nu$$

Consequences:

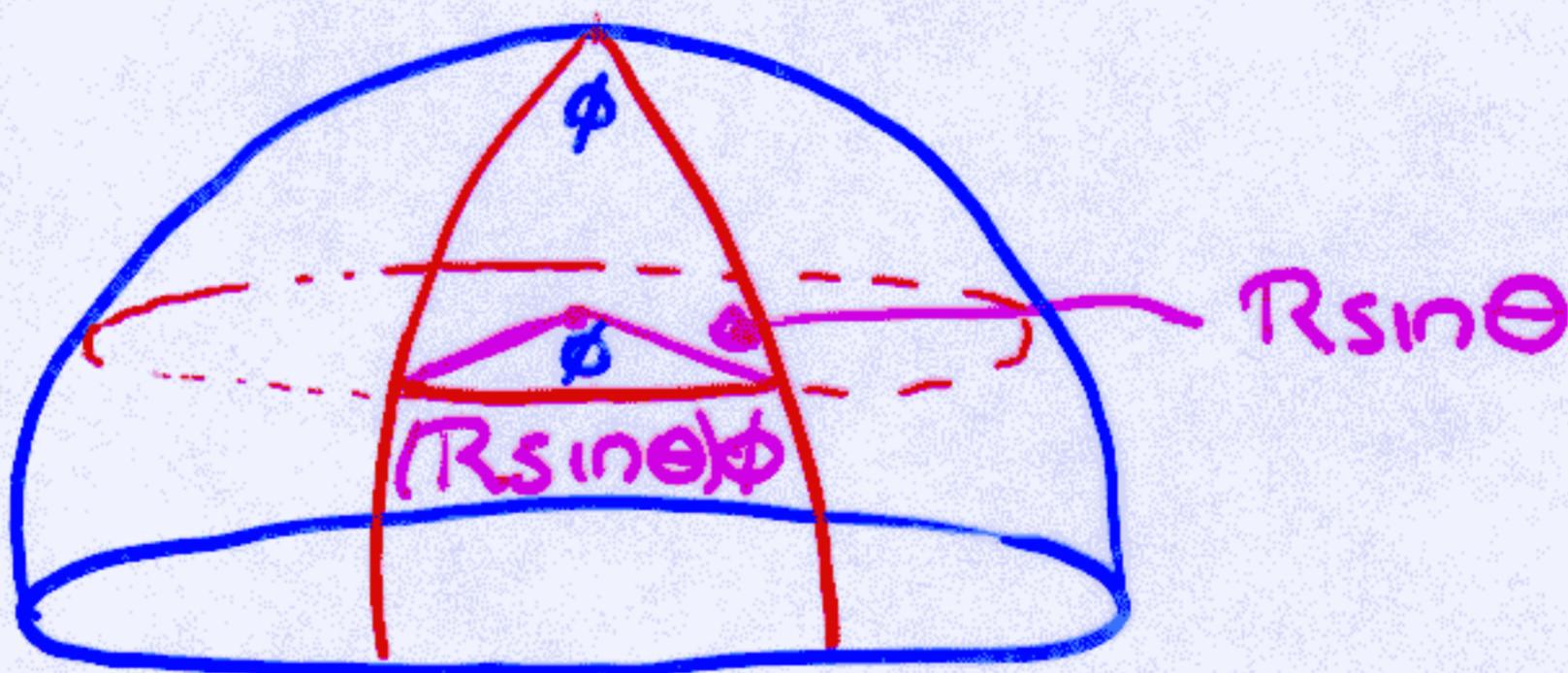
- Horizon light travels on null geodesics

$$ds^2 = 0 \rightarrow a^2 d\eta^2 = dt^2$$

$$\eta = \int_0^t \frac{dt}{a} = \text{comoving distance travelled in time } t$$

(problem set: work out η for diff expansion histories $a(t)$)

- Angular Diameter Distance (comoving)



\therefore object of physical comoving size

$$\lambda \text{ subtends } \phi = \frac{\lambda}{R \sin \theta}$$

$$\text{Euclidean expects } \phi = \frac{\lambda}{d_A}$$

$$\therefore d_A = R \sin \theta$$

translates to

$$d_A = |K|^{-\frac{1}{2}} \sin \eta \sqrt{R} \quad K > 0$$

$$|K|^{-\frac{1}{2}} \sinh \eta |K|^{\frac{1}{2}} \quad K < 0$$

i.e.

$$ds_{(3)}^2 = d\eta^2 + d_A^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

• The Metric (cont)

Raising & Lowering indices

$$\chi_\mu = g_{\mu\nu} \chi^\nu$$

$$g^{\mu\nu} = g^{\alpha\beta} g^{\nu\gamma} g_{\alpha\beta} \therefore g^{\nu\beta} g_{\alpha\beta} = \delta_\alpha^\nu$$

Differentiation

$$\nabla_\nu Y^\mu = \partial_\nu Y^\mu + \Gamma_{\nu\alpha}^\mu Y^\alpha$$

for each index

Christoffel Symbol

$$\Gamma_{\alpha\beta}^\mu = \frac{g^{\mu\nu}}{2} [g_{\alpha\nu,\beta} + g_{\beta\nu,\alpha} + g_{\alpha\beta,\nu}]$$

$$\text{Shorthand } g_{\alpha\beta,\nu} \equiv \frac{\partial}{\partial x^\nu} g_{\alpha\beta}$$

• EINSTEIN Equations

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$



EINSTEIN TENSOR



STRESS-ENERGY TENSOR

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

$$R_{\mu\nu} = \Gamma_{\mu\nu,\alpha}^\alpha - \Gamma_{\mu\alpha,\nu}^\alpha + \Gamma_{\beta\alpha}^\alpha \Gamma_{\mu\nu}^\beta - \Gamma_{\beta\nu}^\alpha \Gamma_{\mu\alpha}^\beta$$

• Index conventions

$$\mu = 0 \quad \text{conformal time} \quad \eta = \int dt/a$$

$$\mu = i \quad \text{spatial index} \quad ds^2 = a^2(-d\eta^2 + \gamma_{ij}dx^i dx^j)$$

$$\text{Implicit Summation} \quad g_{\mu\nu} T^{\mu\nu} \equiv \sum_{\mu} g_{\mu\nu} T^{\mu\nu}$$

$$\text{overdots} \quad \dot{\cdot} = \frac{\partial}{\partial x^0} = \frac{\partial}{\partial \eta}$$

• FRW metric

$$g_{00} = -a^2$$

$$\Gamma_{00}^0 = \frac{\dot{a}}{a}$$

$$\Gamma_{ij}^0 = \frac{\dot{a}}{a} \gamma_{ij}$$

$$g_{ij} = a^2 \delta_{ij}$$

$$\Gamma_{0j}^i = \frac{\dot{a}}{a} \delta_j^i$$

$$\Gamma_{jk}^i = {}^{(3)}\Gamma_{jk}^i$$

$$G_0^0 = -\frac{3}{a^2} \left[\left(\frac{\dot{a}}{a} \right)^2 + K \right]$$

$$G_j^i = -\frac{1}{a^2} \left[2\ddot{a} - \left(\frac{\dot{a}}{a} \right)^2 + K \right] \delta_j^i$$

3 space
of const
curvature

• Stress Tensor

$$T_0^0 = -\rho$$

$$T_j^i = p \delta_j^i$$

- Time-Time Piece

$$-\frac{3}{a^2} \left[\left(\frac{\dot{a}}{a} \right)^2 + K \right] = -8\pi G \rho$$

$$\therefore \left[\left(\frac{\dot{a}}{a} \right)^2 + K \right] = \frac{8\pi G}{3} a^2 \rho$$

$$\ddot{a} = \frac{da}{d\eta} = a \frac{da}{dt} = a^2 H$$

$$\Rightarrow H^2 = \frac{8\pi G}{3} \rho + \frac{K}{a^2}$$

curvature acts as an energy density $\rho_K a a^{-2}$

$$\Rightarrow H^2 = \frac{8\pi G}{3} \rho_{\text{TOT}}$$

- Space-Space Piece

$$-\frac{1}{a^2} \left[2 \frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a} \right)^2 + K \right] = 8\pi G \rho$$

$$-\frac{1}{a^2} \left[2 \frac{\ddot{a}}{a} - 2 \left(\frac{\dot{a}}{a} \right)^2 \right] = \frac{8\pi G}{3} (\rho + 3p)$$

$$\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a} \right)^2 = -\frac{4\pi G}{3} a^2 (\rho + 3p)$$

$$\frac{1}{a} \frac{d^2 a}{dt^2} = -\frac{4\pi G}{3} (1+3w) \rho$$

$$\begin{aligned} \frac{d^2 a}{dt^2} &= \frac{1}{a} \frac{d}{d\eta} \frac{\dot{a}}{a} \\ &= \frac{1}{a} \left(\ddot{a} - \left(\frac{\dot{a}}{a} \right)^2 \right) \end{aligned}$$

A universe with $w \equiv P/\rho < -\frac{1}{3}$

has an accelerating expansion

Covariant conservation of energy-momentum

$$\nabla_\mu T^{\mu\nu} = 0 \quad \left\{ \begin{array}{l} \nu = 0 \text{ energy conservation} \\ \nu = i \text{ momentum conservation} \end{array} \right.$$

$$\partial_0 T^{00} + \partial_i T^{i0} + \Gamma_{0\beta}^0 T^{0\beta} + \Gamma_{\alpha\beta}^0 T^{0\beta} = 0$$

$$T^{i0} = 0$$

$$\therefore -\partial_0 T^{00} = \Gamma_{0\beta}^0 T^{0\beta} + \Gamma_{\alpha 0}^0 T^{00}$$

$$T^{\mu\nu} = g^{\mu\alpha} T_\alpha^\nu$$

$$T^{00} = g^{00} T_0^0 = a^{-2} \rho$$

$$T^{ij} = g^{ik} T_k^j = a^{-2} \delta^{ik} T_k^j = a^{-2} \rho \delta^{ij}$$

$$-\partial_0 [a^{-2} \rho] = \frac{\dot{a}}{a} a^{-2} \rho \delta^{ij} \delta_{ij} + 5 \frac{\dot{a}}{a} a^{-2} \rho$$

$$-a^{-2} \dot{\rho} + 2 \frac{\dot{a}}{a} a^{-2} \rho = 3 \frac{\dot{a}}{a} a^{-2} \rho + 5 \frac{\dot{a}}{a} a^{-2} \rho$$

$$\dot{\rho} = -3(1+w) \rho \frac{\dot{a}}{a} \Rightarrow$$

$$\rho \propto a^{-3(1+w)}$$

Redundant

$$\frac{d}{dt} \left[\left(\frac{\dot{a}}{a}\right)^2 + K \right] = \frac{8\pi G}{3} a^2 \rho$$

$$2 \left(\frac{\ddot{a}}{a}\right) \left[\frac{\dot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2\right] = \frac{8\pi G}{3} \left[a^2 \dot{\rho} + 2 \frac{\dot{a}}{a} a^2 \rho\right]$$

$$= \frac{8\pi G}{3} a^2 \rho [-3(1+w) + 2] \frac{\dot{a}}{a}$$

$$\left[\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2\right] = -\frac{4\pi G}{3} a^2 (1+3w) \rho$$

Δ	matter	$\omega=0$	$\rho \propto a^{-3}$
	radiation	$\omega=1/3$	$\rho \propto a^{-4}$
		$\omega=-1$	$\rho = \text{const}$

Bianchi Identity $\nabla_\mu G^{\mu\nu} = 0$