Appendix B

Useful Quantities and Relations

B.1 FRW Parameters

The expansion rate is given by the Hubble parameter

$$\begin{aligned} H^2 &\equiv \left(\frac{1}{a}\frac{da}{dt}\right)^2 = \left(\frac{\dot{a}}{a}\frac{a_0}{a}\right)^2 \\ &= \left(\frac{a_0}{a}\right)^4 \frac{a_{eq}+a}{a_{eq}+a_0}\Omega_0 H_0^2 - \left(\frac{a_0}{a}\right)^2 K + \Omega_\Lambda H_0^2, \end{aligned} \tag{B.1}$$

where the curvature is $K = -H_0^2(1-\Omega_0-\Omega_\Lambda)$. The value of the Hubble parameter today, for different choices of the fundamental units (see Tab. B.1), is expressed as

$$\begin{aligned} H_0 &= 100h\,\mathrm{kms^{-1}Mpc^{-1}} \\ &= 2.1331\times10^{-42}h\,\mathrm{GeV} \\ &= (2997.9)^{-1}h\,\mathrm{Mpc^{-1}} \\ &= (3.0857\times10^{17})^{-1}h\,\mathrm{s^{-1}} \\ &= (9.7778)^{-1}h\,\mathrm{Gyr^{-1}}. \end{aligned} \tag{B.2}$$

Present day densities in a given particle species X are measured in units of the critical density $\rho_X(a_0) = \Omega_X\rho_{\rm crit}$, where

$$\begin{aligned} \rho_{\rm crit} = 3H_0^2/8\pi G &= 1.8788\times10^{-29}h^2\,\mathrm{g\,cm^{-3}} \\ &= 8.0980\times10^{-47}h^2\,\mathrm{GeV^4} \\ &= 1.0539\times10^{4}h^2\,\mathrm{eV\,cm^{-3}} \\ &= 1.1233\times10^{-5}h^2\mathrm{protons\,cm^{-3}} \\ &= 2.7754\times10^{11}h^2\mathrm{M_\odot\,Mpc^{-3}}. \end{aligned} \tag{B.3}$$

For the CMB,

$$n_{\gamma0} = 399.3\Theta_{2.7}^3\,\mathrm{cm^{-3}},$$

1 s	$= 9.7157\times10^{-15}$ Mpc
1 yr	$= 3.1558\times10^{7}$ s
1 Mpc	$= 3.0856\times10^{24}$ cm
1 AU	$= 1.4960\times10^{13}$ cm
1 K	$= 8.6170\times10^{-5}$ eV
1 $M_\odot$	$= 1.989\times10^{33}$ g
1 GeV	$= 1.6022\times10^{-3}$ erg
	$= 1.7827\times10^{-24}$ g
	$= (1.9733\times10^{-14}\,\mathrm{cm})^{-1}$
	$= (6.5821\times10^{-25}\,\mathrm{s})^{-1}$

Planck's constant	$\hbar = 1.0546\times10^{-27}\,\mathrm{cm^2\,g\,s^{-1}}$
Speed of light	$c = 2.9979\times10^{10}\,\mathrm{cm\,s^{-1}}$
Boltzmann's constant	$k_B = 1.3807\times10^{-16}\,\mathrm{erg\,K^{-1}}$
Fine structure constant	$\alpha = 1/137.036$
Gravitational constant	$G = 6.6720\times10^{-8}\,\mathrm{cm^3\,g^{-1}\,s^{-2}}$
Stefan-Boltzmann constant	$\sigma = ac/4 = \pi^2k_B^4/60\hbar^3c^2$
	$a = 7.5646\times10^{-15}\,\mathrm{erg\,cm^{-3}K^{-4}}$
Thomson cross section	$\sigma_T = 8\pi\alpha^2/3m_e^2 = 6.6524\times10^{-25}\,\mathrm{cm^2}$
Electron mass	$m_e = 0.5110$ MeV
Neutron mass	$m_n = 939.566$ MeV
Proton mass	$m_p = 938.272$ MeV

Table B.1: Physical Constants and Conversion factors

$$\begin{aligned} \rho_{\gamma0} &= 4.4738\times10^{-34}\Theta_{2.7}^4\,\mathrm{g\,cm^{-3}}, \\ \Omega_\gamma &= 2.3812\times10^{-5}h^{-2}\Theta_{2.7}^4, \end{aligned} \tag{B.4}$$

and for the neutrinos

$$\begin{aligned} \rho_{\nu0} &= [(1-f_\nu)^{-1}-1]\rho_{\gamma0}, \\ \Omega_\nu &= [(1-f_\nu)^{-1}-1]\Omega_\gamma, \end{aligned} \tag{B.5}$$

with $(1-f_\nu)^{-1} = 1.68$ for the standard model, or for the total radiation

$$\begin{aligned} \rho_{r0} &= (1-f_\nu)^{-1}\rho_{\gamma0}, \\ \Omega_r &= (1-f_\nu)^{-1}\Omega_\gamma. \end{aligned} \tag{B.6}$$

B.2 Time Variables

Throughout the text we use four time variables interchangeably, they are a the scale factor, z the redshift, η the conformal time, and t the coordinate time. In addition, three dimensionless time parameterizations are useful to consider: χ the development angle in an open universe, D the relative amplitude of pressureless matter fluctuations, and τ the optical depth to Compton scattering.

B.2.1 Scale Factor and Redshift

The scale factor $a(t)$ describes the state of expansion and is the fundamental measure of time in the Hubble equation (B.1) since it controls the energy density of the universe. In this Appendix, we leave the normalization of a free to preserve generality. However, the normalization applied in §4, §5, §6, and Appendix A is $a_{eq} = 1$. The conversion factor between the more commonly employed normalization $a_0 = 1$ is

$$\begin{aligned} \frac{a_{eq}}{a_0} &= \frac{\Omega_r}{\Omega_0 - \Omega_r} \\ &= 2.38 \times 10^{-5} (\Omega_0 h^2)^{-1} (1 - f_\nu)^{-1} \Theta_{2.7}^4 . \end{aligned} \tag{B.7}$$

The redshift z is defined by $(1 + z) = a_0/a$ and serves the same role as the scale factor normalized to the present. We give the scale factor normalized to 3/4 at baryon-photon equality a special symbol R given the frequency of its appearance in equations related to Compton scattering. More explicitly,

$$\begin{aligned} R &= \frac{3}{4}\frac{\rho_b}{\rho_\gamma} = (1 - f_\nu)^{-1} \frac{3}{4} \frac{\Omega_b}{\Omega_0} \frac{a}{a_{eq}} \\ &= 31.5 \Omega_b h^2 \Theta_{2.7}^{-4} (z/10^3)^{-1} . \end{aligned} \tag{B.8}$$

Epochs of interest for the CMB are listed in Tab. B.2 by their redshifts.

B.2.2 Conformal Time

By definition, the conformal time $\eta = \int dt/a$ is related to the scale factor as

$$\eta = \int \frac{da}{a} \frac{1}{H} \frac{a_0}{a} . \tag{B.9}$$

Note that in these $c = 1$ units, the conformal time doubles as the comoving size of the horizon. In an open universe, it is also related to the development angle by

$$\chi = \sqrt{-K} \eta . \tag{B.10}$$

Asymptotic relations are often useful for converting values. Before curvature or Λ domination, the conformal time

$$\begin{aligned} \eta &= \frac{2\sqrt{2}}{k_{eq}} \left(\sqrt{1 + a/a_{eq}} - 1 \right) \\ &= 2 (\Omega_0 H_0^2)^{-1/2} (a_{eq}/a_0)^{1/2} \left(\sqrt{1 + a/a_{eq}} - 1 \right) , \end{aligned} \tag{B.11}$$

Epoch	Definition
$z_* = 10^3 \Omega_b^{-0.027/(1+0.11 \ln \Omega_b)} \quad \Omega_0 = 1$	Last scattering (recomb.)
$\quad = 10^2 (\Omega_0 h^2/0.25)^{1/3} (x_e \Omega_b h^2/0.0125)^{-2/3}$	Last scattering (reion.)
$z_d = 160 (\Omega_0 h^2)^{1/5} x_e^{-2/5}$	Drag epoch
$z_{eq} = 4.20 \times 10^4 \Omega_0 h^2 (1 - f_\nu) \Theta_{2.7}^{-4}$	Matter-radiation equality
$z_{b\gamma} = 3.17 \times 10^4 \Omega_b h^2 \Theta_{2.7}^{-4}$	Baryon-photon equality
$z_H = (1 + z_{eq}) \{ 4 (k/k_{eq})^2 / [1 + (1 + 8 (k/k_{eq})^2)^{1/2}] \} - 1$	Hubble length crossing
$z = (1 - \Omega_0 - \Omega_\Lambda)/\Omega_0 - 1$	Matter-curvature equality
$z = (\Omega_\Lambda/\Omega_0)^{1/3} - 1$	Matter-Λ equality
$z = [\Omega_\Lambda/(1 - \Omega_0 - \Omega_\Lambda)]^{1/2} - 1$	Curvature-Λ equality
$z_{cool} = 9.08 \Theta_{2.7}^{-16/5} f_{cool}^{2/5} (\Omega_0 h^2)^{1/5} - 1$	Compton cooling era
$z > 4\sqrt{2} z_K$	Bose-Einstein era
$z < z_K/8$	Compton-y era
$z_K = 7.09 \times 10^3 (1 - Y_p/2)^{-1/2} (x_e \Omega_b h^2)^{-1/2} \Theta_{2.7}^{1/2}$	Comptonization epoch
$z_{\mu,dc} = 4.09 \times 10^5 (1 - Y_p/2)^{-2/5} (x_e \Omega_b h^2)^{-2/5} \Theta_{2.7}^{1/5}$	Dbl. Compton therm. epoch
$z_{\mu,br} = 5.60 \times 10^4 (1 - Y_p/2)^{-4/5} (x_e \Omega_b h^2)^{-6/5} \Theta_{2.7}^{13/5}$	Bremss. therm. epoch
$\Theta_{2.7} = T_0/2.7\text{K} \simeq 1.01$	Temperature Scaling
$Y_p = 4 n_{He}/n_b \simeq 0.23$	Helium mass fraction
$(1 - f_\nu)^{-1} = 1 + \rho_\nu/\rho_\gamma \to 1.68132$	Neutrino density correction
$k_{eq} = (2 \Omega_0 H_0^2 a_0/a_{eq})^{1/2}$	Equality Hubble wavenumber
$\quad = 9.67 \times 10^{-2} \Omega_0 h^2 (1 - f_\nu)^{1/2} \Theta_{2.7}^{-2} \text{Mpc}^{-1}$	
$f_{cool} = x_e^{-1} [(1 + x_e)/2 - (3 + 2x_e) Y_p/8] (1 - Y_p/2)^{-1}$	Cooling correction factor

Table B.2: Critical Redshifts

Critical epochs are also denoted as the corresponding value in the coordinate time t, scale factor a, and conformal time η. The neutrino fraction f_ν is given for three families of massless neutrinos and the standard thermal history. The Hubble crossing redshift z_H is given for the matter and radiation dominated epochs.

and reduces to

$$\eta = \begin{cases} (\Omega_r H_0^2)^{-1/2} a/a_0 & \text{RD} \\ 2(\Omega_0 H_0^2)^{-1/2}(a/a_0)^{1/2}, & \text{MD} \end{cases} \tag{B.12}$$

where $\Omega_r/\Omega_0 \simeq a_{eq}/a_0$. In a $\Lambda = 0$ universe, it also has an asymptotic solution for $a \gg a_{eq}$

$$\begin{aligned} \eta &= \frac{1}{\sqrt{-K}} \cosh^{-1}\left[1 + \frac{2(1-\Omega_0)}{\Omega_0}\frac{a}{a_0}\right] \qquad \text{MD/CD} \\ \lim_{\Omega_0 \to 0} \eta_0 &\to (-K)^{-1/2} \ln(4/\Omega_0), \end{aligned} \tag{B.13}$$

and thus the horizon scale is larger than the curvature scale $(-K)^{-1/2}$ for low Ω_0 universes. In a flat universe,

$$\eta_0 \simeq 2(\Omega_0 H_0^2)^{-1/2}(1 + \ln \Omega_0^{0.085}), \qquad \Omega_0 + \Omega_\Lambda = 1 \tag{B.14}$$

and the horizon goes to a constant $\eta = 2.8\ H_0^{-1}\Omega_0^{-1/3}(1-\Omega_0)^{-1/6}$ as $a/a_0 \to \infty$.

B.2.3 Coordinate Time

The coordinate time is defined in terms of the scale factor as,

$$t = \int \frac{da}{a}\frac{1}{H}. \tag{B.15}$$

It also takes on simple asymptotic forms, *e.g.*

$$t = \frac{2}{3}(\Omega_0 H_0^2)^{-1/2} a_0^{-3/2}[(a + a_{eq})^{1/2}(a - 2a_{eq}) + 2a_{eq}^{3/2}]. \qquad \text{RD/MD} \tag{B.16}$$

Explicitly, this becomes

$$\begin{aligned} t &= \frac{1}{2}(\Omega_0 H_0^2)^{-1/2}(a_0/a_{eq})^{1/2}(a/a_0)^2 \qquad \text{RD} \\ &= 2.4358 \times 10^{19} \Theta_{2.7}^{-2}(1+z)^{-2}\text{s}. \end{aligned} \tag{B.17}$$

and

$$\begin{aligned} t &= \frac{2}{3}(\Omega_0 H_0^2)^{-1/2}(a/a_0)^{3/2} \ \text{MD} \\ &= 2.0571 \times 10^{17}(\Omega_0 h^2)^{-1/2}(1+z)^{-3/2}\text{s}. \end{aligned} \tag{B.18}$$

The expansion time, defined as H^{-1} scales similarly

$$\begin{aligned} t_{exp} &= (\Omega_0 H_0^2)^{-1/2}(a/a_0)^2 a_0^{1/2}(a + a_{eq})^{-1/2} \\ &= 4.88 \times 10^{19}(z + z_{eq} + 2)^{-1/2}\Theta_{2.7}^{-2}(1+z)^{-3/2}\text{s}. \end{aligned} \tag{B.19}$$

For $\Lambda = 0$ universes, the coordinate time at late epochs when radiation can be neglected is given by

$$t = H_0^{-1}\left[\frac{(1+\Omega_0 z)^{1/2}}{(1-\Omega_0)(1+z)} - \frac{\Omega_0}{2(1-\Omega_0)^{3/2}}\cosh^{-1}\left(\frac{2(1-\Omega_0)}{\Omega_0(1+z)} + 1\right)\right]. \qquad \text{MD/CD} \tag{B.20}$$

In particular, the age of the universe today is

$$t_0 = H_0^{-1}(1-\Omega_0)^{-1}\left[1 - \frac{\Omega_0}{2}(1-\Omega_0)^{-1/2}\cosh(2/\Omega_0 - 1)\right], \qquad \Omega_\Lambda = 0 \tag{B.21}$$

where the factor in square brackets goes to unity as $\Omega_0 \to 0$. This should be compared with the flat $\Omega_0 + \Omega_\Lambda = 1$ result

$$t_0 = \frac{2}{3}H_0^{-1}(1-\Omega_0)^{-1/2}\ln\left[\frac{1+\sqrt{1-\Omega_0}}{\sqrt{\Omega_0}}\right], \qquad \Omega_0 + \Omega_\Lambda = 1, \tag{B.22}$$

which diverges logarithmically as $\Omega_0 \to 0$. Finally a microphysical time scale of interest for the CMB,

$$\begin{aligned} t_C &= (d\tau/dt)^{-1} = (x_e n_e \sigma_T)^{-1} \\ &= 4.4674 \times 10^{18}(1 - Y_p/2)^{-1}(x_e \Omega_b h^2)^{-1}(1+z)^{-3}\text{s}, \end{aligned} \tag{B.23}$$

is the Compton mean free time between scatterings.

B.2.4 Growth Function

The amplitude of matter fluctuations undergoing pressureless growth is another useful parameterization of time. It is given by equation (5.9) as

$$D = \frac{5}{2}\Omega_0 \frac{a_0}{a_{eq}} g(a) \int \frac{da}{a}\frac{1}{g^3(a)}\left(\frac{a_0}{a}\right)^2, \tag{B.24}$$

where the dimensionless, "pressureless" Hubble parameter is

$$g^2(a) = \left(\frac{a_0}{a}\right)^3 \Omega_0 + \left(\frac{a_0}{a}\right)^2 (1 - \Omega_0 - \Omega_\Lambda) + \Omega_\Lambda. \tag{B.25}$$

In the matter or radiation-dominated epoch, $D = a/a_{eq}$ by construction. In a $\Lambda = 0$ universe, D becomes

$$D = \frac{5}{2x_{eq}}\left[1 + \frac{3}{x} + \frac{3(1+x)^{1/2}}{x^{3/2}}\ln[(1+x)^{1/2} - x^{1/2}]\right], \tag{B.26}$$

where $x = (\Omega_0^{-1} - 1)a/a_0$. Fitting formulae for the growth factor, valid for the general case, are occasionally useful [26]:

$$\frac{D_0}{a_0} \simeq \frac{5}{2}\Omega_0\left[\Omega_0^{4/7} - \Omega_\Lambda + \left(1 + \frac{1}{2}\Omega_0\right)\left(1 - \frac{1}{70}\Omega_\Lambda\right)\right]^{-1}, \tag{B.27}$$

$$\frac{d\ln D}{d\ln a} \simeq \left[\frac{\Omega_0(1+z)^3}{\Omega_0(1+z)^3 - (\Omega_0 + \Omega_\Lambda - 1)(1+z)^2 + \Omega_\Lambda}\right]^{4/7}. \tag{B.28}$$

The latter relation is often employed to relate the velocity to the density field.

B.2.5 Optical Depth

For the CMB, the optical depth τ to Compton scattering is a useful lookback time parameterization,

$$\begin{aligned}\tau(a,a_0) &= \int_\eta^{\eta_0} d\eta' x_e n_e \sigma_T a' \\ &= 6.91\times 10^{-2}(1-Y_p/2)x_e\Omega_b h \int_a^{a_0} \frac{da'}{a'}\frac{H_0}{H}\left(\frac{a_0}{a'}\right)^3, \end{aligned} \tag{B.29}$$

for constant ionization fraction. If $a \gg a_{eq}$, this has closed form solution,

$$\begin{aligned}\tau(a,a_0) &= 4.61\times 10^{-2}(1-Y_p/2)x_e\frac{\Omega_b h}{\Omega_0^2} \\ &\times \begin{cases} 2-3\Omega_0+(1+\Omega_0 z)^{1/2}(\Omega_0 z+3\Omega_0-2) & \Omega_\Lambda=0 \\ \Omega_0[1-\Omega_0+\Omega_0(1+z)^3]^{1/2}-\Omega_0. & \Omega_0+\Omega_\Lambda=1 \end{cases}\end{aligned} \tag{B.30}$$

Furthermore, since the optical depth is dominated by early contributions the distinction between open and Λ universes for $\tau \gtrsim 1$ is negligible.

B.3 Critical Scales

B.3.1 Physical Scales

Several physical scales are also of interest. We always use comoving measures when quoting distances. The most critical quantity is the horizon scale η given in the last section and the curvature scale $(-K)^{-1/2} = 2997.9h(1-\Omega_0-\Omega_\Lambda)^{1/2}$Mpc. There are two related quantities of interest, the Hubble scale and the conformal angular diameter distance to the horizon. The Hubble scale is often employed instead of the horizon scale because it is independent of the past evolution of the universe. The wavenumber corresponding to the Hubble scale is

$$k_H = \frac{\dot a}{a} = \begin{cases} (\Omega_r H_0^2)^{1/2}(a_0/a) & \text{RD} \\ (\Omega_0 H_0^2)^{1/2}(a_0/a)^{1/2} & \text{MD} \\ (-K)^{1/2} & \text{CD} \\ (\Omega_\Lambda H_0^2)^{1/2} a/a_0. & \Lambda\text{D} \end{cases} \tag{B.31}$$

Comparison with the relations for η shows that $k_H\eta \sim 1$ during radiation and matter domination but not curvature or Λ domination. Indeed, due to the exponential expansion, the Hubble scale goes to zero as $a/a_0 \to \infty$, reflecting the fact that regions which were once in causal contact can no longer communicate. This is of course how inflation solves the horizon problem. Throughout the main text we have blurred the distinction between the Hubble scale and the horizon scale when discussing the radiation- and matter-dominated epochs.

The distance inferred for an object of known spatial extent by its angular diameter is known as the conformal angular diameter distance. It multiplies the angular part of the

spatial metric. Moreover, in an open universe, it is not equivalent to the distance measured in conformal time. For an observer at the present, it is given by

$$r_\theta(\eta) = (-K)^{-1/2}\sinh[(\eta_0-\eta)(-K)^{1/2}]. \tag{B.32}$$

Note that the argument of sinh is the difference in development angle χ in an open universe. Of particular interest is the angular diameter distance to the horizon $r_\theta(0)$ since many features in the CMB are generated early

$$r_\theta(0) \simeq \begin{cases} 2(\Omega_0 H_0)^{-1} & \Omega_\Lambda = 0 \\ 2(\Omega_0 H_0^2)^{-1/2}(1+\ln\Omega_0^{0.085}). & \Omega_\Lambda+\Omega_0=1 \end{cases} \tag{B.33}$$

In the flat case, $r_\theta(0)=\eta_0$.

A microphysical scale, the mean free path of a photon to Compton scattering, is also of interest for the CMB,

$$\lambda_C = (x_e n_e \sigma_T a/a_0)^{-1} = 4.3404\times 10^4(1-Y_p/2)^{-1}(x_e\Omega_b h^2)^{-1}(1+z)^{-2}\text{Mpc}. \tag{B.34}$$

The diffusion length is roughly the geometric mean of λ_C and the horizon η. More precisely, it is given by equation (A.55) as

$$\lambda_D^2 \sim k_D^{-2} = \frac{1}{6}\int d\eta \frac{1}{\dot\tau}\frac{R^2+4f_2^{-1}(1+R)/5}{(1+R)^2}. \tag{B.35}$$

where

$$f_2 = \begin{cases} 1 & \text{isotropic, unpolarized} \\ 9/10 & \text{unpolarized} \\ 3/4 & \text{polarized} \end{cases} \tag{B.36}$$

where isotropic means that the angular dependence of Compton scattering has been neglected, and the polarization case accounts for feedback from scattering induced polarization. Throughout the main text, we have used $f_2 = 1$ for simplicity. If the diffusion scale is smaller than the sound horizon, acoustic oscillations will be present in the CMB. The sound horizon is given by

$$r_s = \int_0^\eta c_s d\eta' = \frac{2}{3}\frac{1}{k_{eq}}\sqrt{\frac{6}{R_{eq}}}\ln\frac{\sqrt{1+R}+\sqrt{R+R_{eq}}}{1+\sqrt{R_{eq}}}, \tag{B.37}$$

which relates it to the horizon at equality $\eta_{eq} = (4-2\sqrt{2})k_{eq}^{-1}$, where

$$\begin{aligned} k_{eq} &= (2\Omega_0 H_0^2 a_0/a_{eq})^{1/2} \\ &= 9.67\times 10^{-2}\Omega_0 h^2(1-f_\nu)^{1/2}\Theta_{2.7}^{-2}\text{Mpc}^{-1}, \\ \eta_{eq} &= 12.1(\Omega_0 h^2)^{-1}(1-f_\nu)^{1/2}\Theta_{2.7}^2\text{Mpc}, \end{aligned} \tag{B.38}$$

with k_{eq} as the wavenumber that passes the Hubble scale at equality.

B.3.2 Angular Scales

A physical scale at η subtends an angle or equivalently a multipole on the sky ℓ

$$\ell = k r_\theta(\eta) \simeq \theta^{-1}, \qquad \ell \gg 1 \tag{B.39}$$

where the angle-distance relation r_θ is given by equation (B.32). Three angular scales are of interest to the CMB. The sound horizon at last scattering determines the location of the acoustic peaks

$$\begin{aligned}\ell_A &= \pi \frac{r_\theta(\eta_*)}{r_s(\eta_*)}, \\ \ell_p &= \begin{cases} m\ell_A & \text{adiabatic} \\ (m-\frac{1}{2})\ell_A, & \text{isocurvature}\end{cases}\end{aligned} \tag{B.40}$$

where ℓ_p is the location of the pth acoustic peak. If $R_* \ll 1$, ℓ_A takes on a simple form

$$\ell_A = 172\left(\frac{z_*}{10^3}\right)^{1/2}\frac{f_G}{f_R}, \tag{B.41}$$

where f_R is the correction for the expansion during radiation domination

$$\begin{aligned} f_R &= (1+x_R)^{1/2} - x_R^{1/2}, \\ x_R = a_{eq}/a_* &= 2.38\times 10^{-2}(\Omega_0 h^2)^{-1}(1-f_\nu)^{-1}\Theta_{2.7}^4(z_*/10^3), \end{aligned} \tag{B.42}$$

and f_G is the geometrical factor

$$f_G \simeq \begin{cases}\Omega_0^{-1/2} & \Omega_\Lambda = 0 \\ 1+\ln\Omega_0^{0.085}. & \Omega_\Lambda+\Omega_0 = 1\end{cases} \tag{B.43}$$

The diffusion damping scale at last scattering subtends an angle given by

$$\ell_D = k_D(\eta_*) r_\theta(\eta_*), \tag{B.44}$$

where $k_D(\eta_*)$ is the effective damping scale at last scattering accounting for the recombination process. From §6.3.4, to order of magnitude it is

$$\ell_D \sim 10^3(\Omega_b/0.05)^{1/4}\Omega_0^{-1/4} f_R^{-1/2} f_G, \tag{B.45}$$

if $\Omega_b h^2$ is low as required by nucleosynthesis. The scaling is only approximate since the detailed physics of recombination complicates the calculation of k_D (see Appendix A.2.2). The curvature radius at the horizon distance (*i.e.* early times) subtends an angle given by

$$\begin{aligned}\ell_K &\simeq \sqrt{-K}\, r_\theta(0) \\ &\simeq \frac{2\sqrt{1-\Omega_0}}{\Omega_0}.\end{aligned} \tag{B.46}$$

This relation is also not exact since for reasonable Ω_0, the curvature scale subtends a large angle on the sky and the small angle approximation breaks down. Note also that at closer distances as is relevant for the late ISW effect, the curvature scale subtends an even larger angle on the sky than this relation predicts.

B.4 Normalization Conventions

B.4.1 Power Spectra

There are unfortunately a number of normalization conventions used in the literature and indeed several that run through the body of this work. Perhaps the most confusing conventions are associated with open universes. The power in fluctuations is expressed alternately per logarithmic intervals of the Laplacian wavenumber k or the eigenfunction index $\nu = \tilde{k}/\sqrt{-K}$, $\tilde{k} = (k^2+K)^{1/2}$. The relation between the two follows from the identity $kdk = \tilde{k}d\tilde{k}$,

$$\tilde{P}_X(\tilde{k}) = \frac{k}{\tilde{k}} P_X(k), \tag{B.47}$$

where P_X is the power spectrum of fluctuations in X. For example, our power law spectra

$$\begin{aligned}|\Phi(0,k)|^2 &= Bk^{n-4}, \\ |S(0,k)|^2 &= Ck^m,\end{aligned} \tag{B.48}$$

become

$$\begin{aligned}|\tilde{\Phi}(0,\tilde{k})|^2 &= B(1-K/\tilde{k}^2)^{(n-3)/2}\tilde{k}^{n-4}, \\ |\tilde{S}(0,\tilde{k})|^2 &= C(1-K/\tilde{k}^2)^{m/2}\tilde{k}^m.\end{aligned} \tag{B.49}$$

To add to the confusion, adiabatic fluctuations are often expressed in terms of the density power spectrum at present $P(k) = |\Delta_T(\eta_0,k)|^2$. The two conventions are related by the Poisson equation,

$$(k^2-3K)\Phi = \frac{3}{2}\Omega_0 H_0^2(1+a_{eq}/a)\frac{a_0}{a}\Delta_T. \tag{B.50}$$

To account for the growth between the initial conditions and the present, one notes that at large scales ($k \to 0$) the growth function is described by pressureless linear theory. From equations (A.8) and (A.9),

$$\Delta_T(\eta_0,k) = \frac{3}{5}(\Omega_0 H_0^2)^{-1}\left[1+\frac{4}{15}f_\nu\right]\left[1+\frac{2}{5}f_\nu\right]^{-1}(1-3K/k^2)\frac{D}{D_{eq}}\frac{a_{eq}}{a}\Phi(0,k). \tag{B.51}$$

If the neutrino anisotropic stress is neglected, drop the f_ν factors for consistency. Thus for a normalization convention of $P(k) = Ak^n$ at large scales

$$A = \frac{9}{25}(\Omega_0 H_0^2)^{-2}\left[1+\frac{4}{15}f_\nu\right]^2\left[1+\frac{2}{5}f_\nu\right]^{-2}(1-3K/k^2)^2\left(\frac{D}{D_{eq}}\frac{a_{eq}}{a}\right)^2 B. \tag{B.52}$$

Notice that in an open universe, power law conditions for the potential do not imply power law conditions for the density,

$$\begin{aligned}P(k) &\propto (k^2-3K)^2 k^{n-4}, \\ \tilde{P}(\tilde{k}) &\propto \tilde{k}^{-1}(\tilde{k}^2-K)^{-1}(\tilde{k}^2-4K)^2(\tilde{k}^2-K)^{(n-1)/2}.\end{aligned} \tag{B.53}$$

Experiment	ℓ_0	ℓ_1	ℓ_2	$Q_{\rm flat}(\mu$K)	Ref.
COBE	–	–	18	19.9 ± 1.6	[62]
FIRS	–	–	30	19 ± 5	[57]
Ten.	20	13	30	26 ± 6	[70]
SP94	67	32	110	26 ± 6	[68]
SK	69	42	100	29 ± 6	[119]
Pyth.	73	50	107	37 ± 12	[49]
ARGO	107	53	180	25 ± 6	[43]
IAB	125	60	205	61 ± 27	[131]
MAX-2 (γUMi)	158	78	263	74 ± 31	[2]
MAX-3 (γUMi)	158	78	263	50 ± 11	[67]
MAX-4 (γUMi)	158	78	263	48 ± 11	[44]
MAX-3 (μPeg)	158	78	263	19 ± 8	[117]
MAX-4 (σHer)	158	78	263	39 ± 8	[32]
MAX-4 (ιDra)	158	78	263	39 ± 11	[32]
MSAM2	143	69	234	40 ± 14	[30]
MSAM3	249	152	362	39 ± 12	[30]

Table B.3: Anisotropy Data Points

A compilation of anisotropy measurements from [146]. The experimental window function peaks at ℓ_0 and falls to half power at ℓ_1 and ℓ_2. Points are plotted in Fig. 1.3.

$\tilde P(\tilde k)$ is the form most often quoted in the literature [175, 82, 134].

The power spectrum may also be expressed in terms of the bulk velocity field. At late times, pressure can be neglected and the total continuity equation (5.6) reduces to

$$\begin{aligned} kV_T &= -\dot\Delta_T \\ &= -\frac{\dot a}{a}\frac{d\ln D}{d\ln a}\Delta_T, \end{aligned} \tag{B.54}$$

and in particular

$$kV_T(\eta_0,k) = -H_0 \frac{d\ln D}{d\ln a}\bigg|_{\eta_0} \Delta_T(\eta_0,k), \tag{B.55}$$

or

$$P_V(k) \equiv |V_T(\eta_0,k)|^2 = H_0^2 \left(\frac{d\ln D}{d\ln a}\right)^2\bigg|_{\eta_0} P(k), \tag{B.56}$$

for the velocity power spectrum. Recall from equation (B.27) that $d\ln D/d\ln a \simeq \Omega_0^{0.6}$ in an open universe.

B.4.2 Anisotropies

The anisotropy power spectrum C_ℓ is given by

$$\frac{2\ell+1}{4\pi}C_\ell = \int \frac{dk}{k} T_\ell^2(k) \times \begin{cases} k^3|\Phi(0,k)|^2 & \text{adiabatic} \\ k^3|S(0,k)|^2, & \text{isocurvature} \end{cases} \tag{B.57}$$

where $T_\ell(k)$ is the radiation transfer function from the solution to the Boltzmann equation. Examples are given in §6. The power measured by a given experiment with a window function W_ℓ has an ensemble average value of

$$\left(\frac{\Delta T}{T}\right)^2_{rms} = \frac{1}{4\pi}\sum_\ell (2\ell+1)C_\ell W_\ell. \tag{B.58}$$

Only if the whole sky is measured at high signal to noise does the variance follow the "cosmic variance" prediction of a χ^2 with $2\ell+1$ degrees of freedom. Real experiments make noisy measurements of a fraction of the sky and therefore require a more detailed statistical treatment. To employ likelihood techniques, we must assume some underlying power spectrum. In order to divorce the measurement from theoretical prejudice, experimental results are usually quoted with a model independent choice. The two most common conventions are the gaussian autocorrelation function $C_{\rm gacf}(\theta) = C_0 \exp(-\theta^2/2\theta_c^2)$ and the "flat" power spectrum motivated by the Sachs-Wolfe tail of adiabatic models (see *e.g.* [174]),

$$\begin{aligned} C_{\ell\rm gacf} &= 2\pi C_0 \theta_c^2 \exp[-\ell(\ell+1)\theta_c^2/2], \\ C_{\ell\rm flat} &= \frac{24\pi}{5}\left(\frac{Q_{\rm flat}}{T_0}\right)^2 [\ell(\ell+1)]^{-1}. \end{aligned} \tag{B.59}$$

The two power estimates are thus related by

$$Q_{\rm flat}^2 \frac{6}{5}\sum_\ell \frac{2\ell+1}{\ell(\ell+1)}W_\ell = C_0\theta_c^2 \frac{1}{2}\sum_\ell (2\ell+1)\exp[-\ell(\ell+1)\theta_c^2/2]W_\ell. \tag{B.60}$$

The current status of measurements is summarized in Tab. B.3 [146].

B.4.3 Large Scale Structure

Large scale structure measurements probe a smaller scale and have yet another set of normalization conventions based on the two point correlation function of astrophysical objects

$$\xi_{ab}(\mathbf{x}) = \langle \delta\rho_a(\mathbf{x}'+\mathbf{x})\delta\rho_b(\mathbf{x}')/\bar\rho_a\bar\rho_b\rangle. \tag{B.61}$$

If all objects are clustered similarly, then all $\xi_{aa} = \xi$ and the two-point correlation function is related to the underlying power spectrum by

$$\begin{aligned} \xi(r) &= \frac{V}{2\pi^2}\int \frac{dk}{k} k^3 P(k) X_\nu^0(\sqrt{-K}r) \\ &\simeq \frac{V}{2\pi^2}\int \frac{dk}{k} k^3 P(k) \frac{\sin(kr)}{kr}, \end{aligned} \tag{B.62}$$

where the approximation assumes that scales of interest are well below the curvature scale. The normalization of the power spectrum is often quoted by the Nth moment of the correlation function $J_N(r) = \int_0^r \xi(x)x^{(N-1)}dx$ which implies

$$J_3(r) = \frac{V}{2\pi^2}\int \frac{dk}{k} P(k)(kr)^2 j_1(kr). \tag{B.63}$$

For reference, $j_1(x) = x^{-2}\sin x - x^{-1}\cos x$. Another normalization convention involves the rms density fluctuation in spheres of constant radii

$$\sigma^2(r) = \frac{V}{2\pi^2}\int \frac{dk}{k} k^3 P(k) \left(\frac{3j_1(kr)}{kr}\right)^2 . \tag{B.64}$$

The observed galaxy distribution implies that

$$J_3(10h^{-1}\mathrm{Mpc}) \simeq 270h^{-3}\mathrm{Mpc}^3 \tag{B.65}$$

$$\sigma_8 \equiv \sigma(8h^{-1}\mathrm{Mpc}) = \begin{cases} 1.1 \pm 0.15 & \text{optical [109]} \\ 0.69 \pm 0.04. & \text{IRAS [55]} \end{cases} \tag{B.66}$$

The discrepancy between estimates of the normalization obtained by different populations of objects implies that they may all be biased tracers of the underlying mass. The simplest model for bias assumes $\xi_{aa} = b_a^2\xi$ with constant b. Peacock & Dodds [122] find that the best fit to the Abell cluster (A), radio galaxy (R), optical galaxy (O), and IRAS galaxy (I) data sets yields $b_A : b_R : b_O : b_I = 4.5 : 1.9 : 1.3 : 1$.