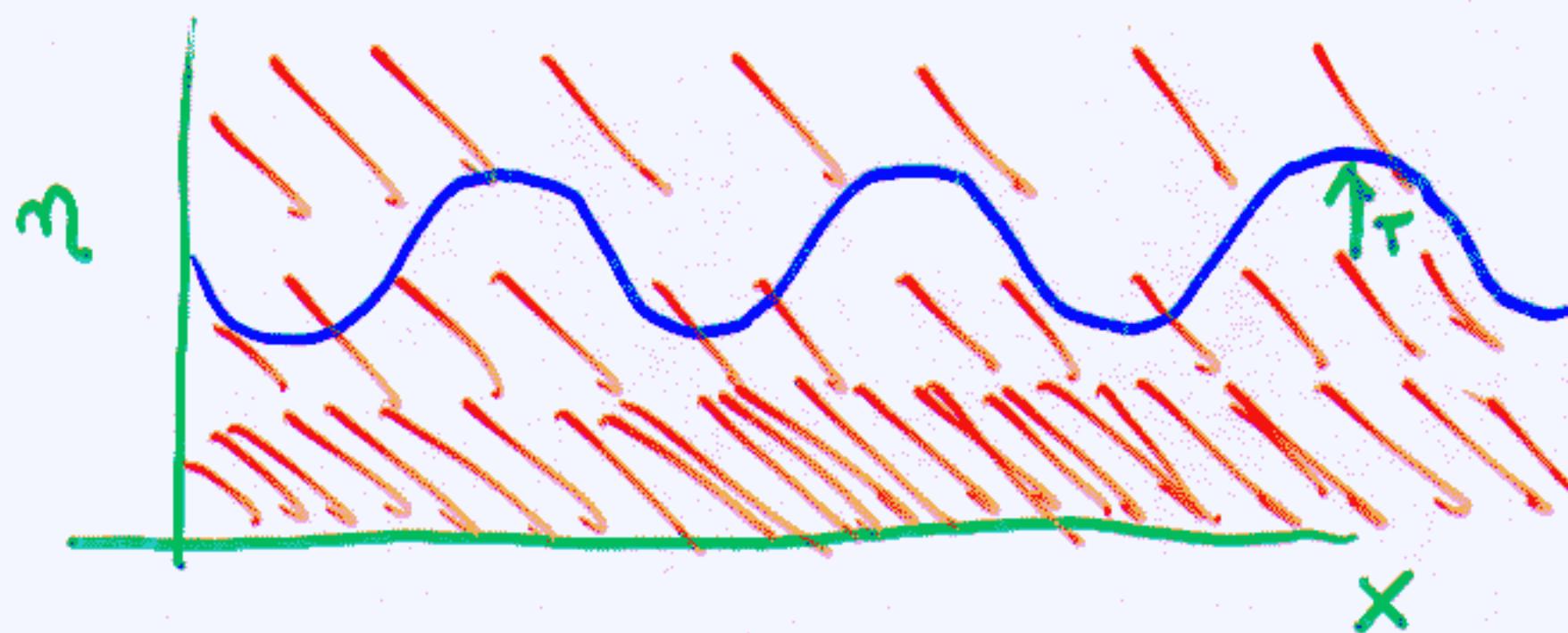


Gauge

- Qualitative

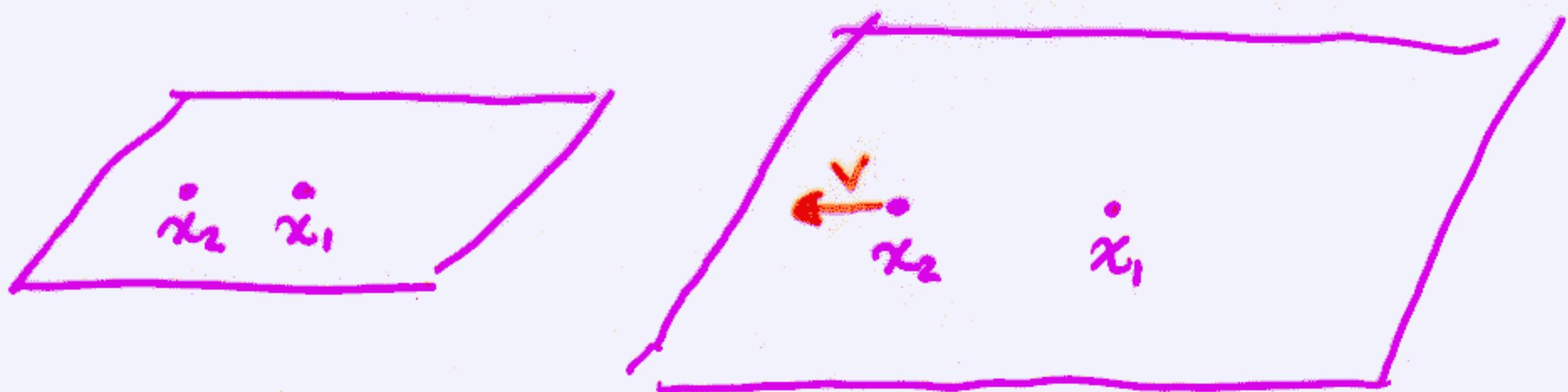
- A redefinition of the temporal coordinate $\xrightarrow{\text{as a fn of position}}$ \Rightarrow redefinition of density perturbation due to expansion?



$$\tilde{\delta\rho} = \delta\rho - T_r \dot{\rho}$$

trivial example: CMB temperature is measured to be higher at higher redshift in molecular lines
if we warp the time coordinate so that at the distance d_{molecule} z_{molecule} is now
we assign a spatially varying temperature to CMB even though it is in fact homogeneous in the "right" coordinates

- A redefinition of the spatial coordinate as a function of time
 \Rightarrow redefinition of velocity field



trivial example: recession velocity

$$d = a x$$

$$v = \dot{d} = \dot{a}x = H d$$

generally under a spatial coordinate shift \dot{L} peculiar velocities

$$\tilde{v} = v + \dot{L}$$

- Analogous changes in metric
 e.g. H_L is a spatially varying change to scale factor

$$\tilde{H}_L = H_L - \frac{\dot{a}}{a} T \quad \text{under spatially varying time redefn.}$$

Gauge freedom / coordinate transformation

Matter & Metric take on different values in different coordinate systems (no such thing as "gauge invariant" perturbation)

General Coordinate Transformation

$$\begin{aligned}\tilde{\eta} &= \eta + T \\ \tilde{x}^i &= x^i + L^i\end{aligned}$$

Simple Case: scalar function

$$\begin{aligned}f(\tilde{\eta}, \tilde{x}^i) &= f(\eta, x^i) \\ &= f(\eta, x^i) - \frac{\partial f}{\partial \eta} T - \frac{\partial f}{\partial x^i} L^i \quad \rightarrow \text{2nd order}\end{aligned}$$

General Case: tensor function

$$\tilde{T}_{\mu\nu}(\tilde{\eta}, \tilde{x}^i) = \frac{\partial x^\alpha}{\partial \tilde{x}^\mu} \frac{\partial x^\beta}{\partial \tilde{x}^\nu} T_{\alpha\beta}(\eta, x^i)$$

Fourier Modes

$$T(\eta, x) = T(\eta, k) Q^{(0)}$$

$$L_i(\eta, x) = \sum_{m=-1}^1 L^{(m)} Q_i^{(m)}$$

- Metric

$$\tilde{A} = A - \dot{T} - \frac{\dot{a}}{a} T$$

$$\tilde{B} = B + \dot{L} + kT$$

$$\tilde{H}_L = H_L - \frac{k}{3} L - \frac{\dot{a}}{a} T$$

$$\tilde{H}_T = H_T + kL$$

Scalar

Vector

$$\tilde{B}^{(\pm 1)} = B^{(\pm 1)} + \dot{L}^{(\pm 1)}$$

$$\tilde{H}_T^{(\pm 1)} = H_T^{(\pm 1)} + kL^{(\pm 1)}$$

- Matter

$$\tilde{\delta\rho} = \delta\rho - \dot{\rho}T$$

$$\tilde{\delta p} = \delta p - \dot{\phi}T$$

$$\tilde{v} = v + \dot{L}$$

(superscript C suppressed)

$$\tilde{v}^{(\pm 1)} = v^{(\pm 1)} + \dot{L}^{(\pm 1)}$$

- Gauge Choice fixes $\underbrace{T, L}_{\text{scalar}}, \underbrace{L^{(+1)}, L^{(-1)}}_{\text{vector}}$ 4 functions

Choose gauge so as to
and/or

- 1) eliminate variables
- 2) Simplify equations

go between gauges by defining $T, L, L^{(+1)}, L^{(-1)}$
 $(\Rightarrow$ restoring general covariance = Bardeen's "gauge invariance")

- "Gauge Modes"

If $T, L, L^{(+1)}, L^{(-1)}$ not completely specified
 Gauge mode solutions (unphysical) appear

• Newtonian Gauge

- Good: intuitive, Newtonian-like gravity matter & metric algebraically related

- Bad: numerically unstable

- Best: solve in another gauge
transform to Newtonian to interpret

Defn: $B = H_T = 0 \Rightarrow$ metric diagonal

$A \equiv \Psi \Rightarrow$ time-time pert; Newtonian Pot.

$H_L \equiv \Phi \Rightarrow$ space-space pert; $\xrightarrow{\text{pert to}} \text{Spatial Curvature}$

Gauge Transformation into Newtonian

$$\tilde{H}_T = 0 \Rightarrow L = -H_T/k$$

$$\tilde{B} = 0 \Rightarrow kT = -B - L \Rightarrow T = -B/k + H_T/k^2$$

Gauge freedom entirely fixed
(no gauge modes)

Einstein Eqs

conclns: unstable

$$(k^2 - 3K)\Phi = 4\pi G a^2 (\delta\rho + 3\frac{\dot{a}}{a}(\rho + p)v/k)$$

$$k^2(\Psi + \Phi) = -8\pi G a^2 p \bar{\Pi} \quad (\Rightarrow \Psi \approx -\Phi)$$

Conservation

$$\left[\frac{d}{dt} + 3\frac{\dot{a}}{a} \right] \delta\rho + 3\frac{\dot{a}}{a} \delta p = -(\rho + p)(kv + 3\dot{\Phi})$$

$$\left[\frac{d}{dt} + 4\frac{\dot{a}}{a} \right] [(\rho + p)v/k] = \delta\rho - \frac{2}{3}(1 - 3\frac{K}{k^2})p \bar{\Pi} + (\rho + p)\Psi$$

• Comoving Gauge

- Good: Causality simple (Bardeen curvature conserved)
- Bad: time-space metric has no Newtonian analogue
- Best: use this gauge to prove general relations and then transform them to Newtonian

Defn: $T^0_i = 0$ energy flux vanishes $\Rightarrow B = V$

$H_T = 0$ no metric shear

$$A \equiv \xi$$

$H_L \equiv \zeta \Rightarrow$ "Bardeen Curvature"

Gauge Transformation into Comoving

$$\tilde{v} - \tilde{B} = 0 \Rightarrow T = (v - B)/k$$

$$\tilde{H}_T = 0 \Rightarrow L = -H_T/k$$

Gauge freedom entirely fixed (no gauge modes)

Einstein Eqns:

$$\begin{aligned} \dot{\zeta} &= \frac{\dot{a}}{a} \xi - K v/k \\ (K^2 - 3K) \left[\dot{\zeta} + \frac{\dot{a}}{a} v/k \right] &= 4\pi G a^2 \delta p \end{aligned}$$

Conservation Egn

$$\left[\frac{d}{dt} + 3\frac{\dot{a}}{a} \right] \delta p + 3\frac{\dot{a}}{a} \delta p = -(q+p)(kv + 3\dot{\zeta})$$

$$(q+p)\dot{\zeta} = -\delta p + \frac{2}{3}(1-3K)p\pi$$

$$\boxed{\dot{\zeta} = \frac{\dot{a}}{a} (\text{stress gradient}) = 0 \text{ as } k \rightarrow 0}$$

Synchronous Gauge

- Good: stable, compatible with numerical codes
- Bad: off-diagonal metric - hard to interpret - no potential
- Best: numerically solve in this gauge interpret in Newtonian

Defn: $A = B = 0 \Rightarrow$ metric part one spatial

$$-H_L - \frac{1}{3} H_T \equiv \eta \quad \left\{ \begin{array}{l} h = 6H_L \text{ also used} \\ H_T \equiv h_T \end{array} \right. \text{ but less stable}$$

Gauge Transformation into Synchronous

$$\tilde{A} = 0 \Rightarrow T = a^{-1} \int da A + c_1 a^{-1}$$

$$\tilde{B} = 0 \Rightarrow L = - \int d\eta (B + kT) + c_2$$

Gauge freedom remains in (c_1, c_2) fix through initial cond

Einstein Eqns

$$-(\kappa^2 - 3\kappa) [\eta + \frac{\dot{a}}{a} h_T / \kappa^2] = 4\pi G a^2 [\delta\rho + 3\frac{\dot{a}}{a} (\rho + p) v/\kappa]$$

$$\ddot{\eta} + \frac{\kappa}{\kappa^2} \dot{h}_T = 4\pi G a^2 (\rho + p) v/\kappa$$

Conservation Eqns

$$[\frac{d}{d\eta} + 3\frac{\dot{a}}{a}] \delta\rho + 3\frac{\dot{a}}{a} \delta p = -(\rho + p)(kv - 3\dot{\eta} - \dot{h}_T)$$

$$[\frac{d}{d\eta} + 4\frac{\dot{a}}{a}] [(\rho + p)v/\kappa] = \delta\rho - \frac{2}{3}(1 - 3\frac{\kappa}{\kappa^2}) p \Pi$$

• Hybrid Formulation

- Gauge Transformation Properties } build fluctuation in one gauge from another
- Defn of T, L

• Useful Examples

$$\delta\rho|_{\text{comoving}} = \delta\rho|_{\text{arbit.}} - \dot{\rho}(v-B)|_{\text{arbit.}}/\kappa$$

$$= \delta\rho|_{\text{arbit.}} + 3\dot{a}/a(v+p)|_{\text{arbit.}}/\kappa$$

source of Poisson Egn

eg $(\kappa^2 - 3K)\bar{\Phi}|_{\text{newt.}} = 4\pi G a^2 \delta\rho|_{\text{comoving}}$ hybrid

$$\bar{S}|_{\text{comov}} = \bar{\Phi}|_{\text{newt.}} - \frac{c}{a} v|_{\text{newt.}}/\kappa$$

$$\bar{S}|_{\text{comov}} = \bar{\Phi}|_{\text{newt.}} + 2(\bar{\Psi} - \bar{\Phi}')|_{\text{newt.}} \frac{\rho + \rho_s}{\rho'}$$

$$\begin{aligned} \bar{\Psi}' &= \frac{d}{dt} \bar{\Psi} \\ \rho_s &= -\frac{3}{8\pi G a^2} K \end{aligned}$$

$$\bar{S}' - \bar{S}|_{\text{comov}} = (\bar{\Psi} - \bar{\Phi}')|_{\text{newt.}} \frac{e_s}{\rho'}$$

\therefore Matter in Newtonian metric as labeled

and recall

$$\xi = -\text{stress fluctuation}$$

$$= -\frac{\delta p}{\rho+p} + \frac{2}{3}\left(1 - \frac{3K}{\kappa^2}\right) \frac{P}{\rho+p} \Pi$$

| comov

- Consequences : Stress free

$$\zeta' = 0 \quad \text{if stress gradients negligible}$$

and no smooth components (curvature is only component smooth by defn)

Others can be approximately smooth due to stress gradients preventing collapse)

$$\Rightarrow \zeta = \text{const} \quad \text{Bardeen curvature conserved}$$

$$\Phi + \bar{\Psi} = S_{\text{II}} \equiv -8\pi G a^3 p \pi / k^2$$

Short hand
for anis. stress Einstein Eqns

$$\begin{aligned} \Rightarrow \bar{\Psi} &= \left(1 - \frac{\sqrt{p}}{a} \int \frac{da}{p} \right) \zeta + \underbrace{\frac{\sqrt{p}}{a} \int \frac{da}{p} S_{\text{II}}}_{\text{anisotropic stress}} + \underbrace{C \frac{\sqrt{p}}{a}}_{\text{decaying mode}} \\ &= \left(1 - \frac{2}{5+3w} \right) \zeta \end{aligned}$$

$$\bar{\Psi} = \frac{3+3w}{5+3w} \zeta$$

$$\begin{aligned} \omega = 1/3 &\Rightarrow \bar{\Psi} = \frac{2}{3} \zeta \\ \omega = 0 &\Rightarrow \bar{\Psi} = \frac{3}{5} \zeta \\ \omega \rightarrow -1 &\Rightarrow \bar{\Psi} \rightarrow 0 \end{aligned}$$

$$\bar{\Psi} = 4\pi G a^3 \delta p$$

$$\delta p \propto a^{-2}$$

$$\frac{\delta p}{p} \propto a^{-2+3(1+w)}$$

$$\propto a^{1+3w}$$

$$\Rightarrow \frac{\delta p}{p}$$

grows at a rate
to keep potential
const (as $\zeta \rightarrow 0$)

even for $-1/3 > w > -1$ "dark energy"
if $w \neq -1$ dark energy cannot be perf. smooth

- Why is $\bar{\Phi}$ constant?

- Consider an initial $\bar{\Phi}_i = -\bar{\Psi}_i$

- Generates a potential flow

$$\dot{v} \sim -k\bar{\Phi}_i$$

$$v \sim -(k\eta)\bar{\Psi}_i$$

- Generates a density perturbation

$$(\delta\rho)^\circ \sim -(\rho+p)kv$$

$$\delta\rho \sim -(k\eta)^2 (\rho+p)\bar{\Psi}_i$$

- Generates a curvature perturbation

$$\begin{aligned} k^2\bar{\Phi} &= 4\pi Ga^2 \delta\rho \\ &= -4\pi Ga^2 (k\eta)^2 (\rho+p)\bar{\Psi}_i \end{aligned}$$

- $\gamma = \int \frac{da}{a^2 H} \frac{1}{H} = \frac{1}{aH}$

$$\eta^2 \approx \frac{1}{a^2 H^2} \approx \frac{3}{8\pi G} \frac{1}{\rho a^2}$$

$\bar{\Phi} \sim -\bar{\Psi}_i$ remains the same

- If stress gradients oppose gravity

$$v < -(k\eta)\bar{\Psi}_i$$

and potential / curvature decays

• Newtonian Density Perturbation

$$\bar{\Psi} \sim \frac{1}{(km\eta)^2} \left. \frac{\delta\rho}{\rho} \right|_{\text{comoving}}$$

$$\bar{\Psi} \gg \left. \frac{\delta\rho}{\rho} \right|_{\text{comoving}} \quad \text{if } (km\eta) \ll 1 \quad (\text{Superhorizon})$$

\therefore comoving density perturbation
and pressure perturbations negligible outside
horizon NOT TRUE FOR NEWTONIAN

$$\bar{\Psi} \gg -\left. \frac{\delta p}{p + \rho} \right|_{\text{comoving}} = \xi$$

Relate by gauge transformations A

$$\bar{\Psi} = \xi - \dot{T} - \frac{\dot{a}}{a} T \quad T = -V/\kappa$$

$$= \frac{\dot{V}}{\kappa} + \frac{\dot{a}}{a} \frac{V}{\kappa} \quad V|_{\text{comov}} = V|_{\text{inert}}$$

$$a\bar{\Psi} = \frac{d}{d\eta} \left[\frac{aV}{\kappa} \right]$$

$$\frac{aV}{\kappa} = \int (a\bar{\Psi}) d\eta = \bar{\Psi} \int dt$$

$$\frac{V}{\kappa} = \frac{\bar{\Psi} t}{a}$$

$$\therefore \delta\eta = \frac{\delta t}{a} \quad \frac{\delta t}{t} = -\bar{\Psi}$$

gauge transformation follows
from $\bar{\Psi}$ as a time-time
perturbation

$$\delta \rho_i|_{\text{NEWT}} = \delta \rho_i|_{\text{comov}} - \dot{\rho}_i T$$

$$= 3(1+w_i) \rho_i T \frac{\dot{a}}{a}$$

$$= -3(1+w_i) \rho_i \left(\frac{\dot{a}}{a} \frac{1}{a} \Psi t \right) \quad \frac{\dot{a}}{a} \frac{t}{a} = \frac{1}{a} \frac{da}{dt} + t$$

$$= -2 \frac{(1+w_i)}{(1+w)} \bar{\Psi} \rho_i$$

$\therefore \delta \rho_i|_{\text{NEWT}} \sim \bar{\Psi}$ for $k\eta \ll 1$

e.g. radiation $\Theta_0 = \frac{\delta \rho_\gamma}{4 \rho_\gamma} \quad 1+w_\gamma = 4/3$

$$\Theta_0 = -\frac{2}{3} \frac{1}{(1+w)} \bar{\Psi}$$

$$\Theta_0 + \bar{\Psi} = \frac{1}{3} \left[\frac{-2 + 3 + 3w}{1+w} \right] \bar{\Psi}$$

$$= \frac{1}{3} \left[\frac{1+3w}{1+w} \right] \bar{\Psi}$$

if $w=0$ (matter dominated)

$$\boxed{\Theta_0 + \bar{\Psi} = \frac{1}{3} \bar{\Psi}}$$

"Sachs-Wolfe Effect"

• Smooth Components

defn: $\delta\rho_s |_{\text{comov}} \ll \delta\rho_m$ even if $\rho_s > \rho_m$

we obtain this when stress gradients prevent collapse inside (sound) horizon

derived equations with $\rho_K = -\frac{3}{8\pi G a^2} K$

generalize to arbitrary smooth component

$$\zeta = \bar{\Psi} + 2(\bar{\Psi} - \bar{\Psi}') \frac{\rho_m + \rho_s}{\rho_m}$$

$$\zeta' = (\bar{\Psi} - \bar{\Psi}') \frac{\rho_s}{\rho_m}$$

Eliminate ζ' :

$$\bar{\Psi}'' + \left(1 - \frac{\rho_m''}{\rho_m} + \frac{1}{2} \frac{(\rho_m + \rho_s)'}{\rho_m + \rho_s}\right) \bar{\Psi}' + \left(\frac{1}{2} \frac{2\rho_m'' + \rho_s' - \frac{\rho_m''}{\rho_m}}{\rho_m + \rho_s} - \frac{\rho_s''}{\rho_m}\right) \bar{\Psi}$$

$$= 0$$

\Rightarrow decay in $\bar{\Psi}$

interpretation: potential flow $\delta\rho_m \propto \rho_m (kn)^2 \bar{\Psi}$

Poisson

$$\bar{\Psi} = (kn)^{-2} \frac{\delta\rho_m}{(\rho_m + \rho_s)}$$

gradual decay

• Effectively Smooth Component Domination

inside horizon stress gradients prevent collapse

→ decay in potential

$$\Phi = C_1 \bar{a}^{-1} + C_2 \bar{a}^{-1} \int d\ln a \underbrace{\frac{a \rho_m}{R_m + R_s}}_{\text{stress free matter}}$$

i.e. CDM

$$\sim \int d\ln a \frac{a^{3(1+w)/2} \bar{a}^{-2}}{\bar{a}^{-2(1-w)}}$$

\leftarrow decay for $w < 1$

$$\Rightarrow \frac{\delta \rho_m}{\rho_m} \propto \bar{a}^{\frac{1}{2}(1-3w)} \quad (w = 1/3 \Rightarrow lma)$$

• Transfer Function

the transfer function accounts for the scale-dependent growth of structure due to stress gradients

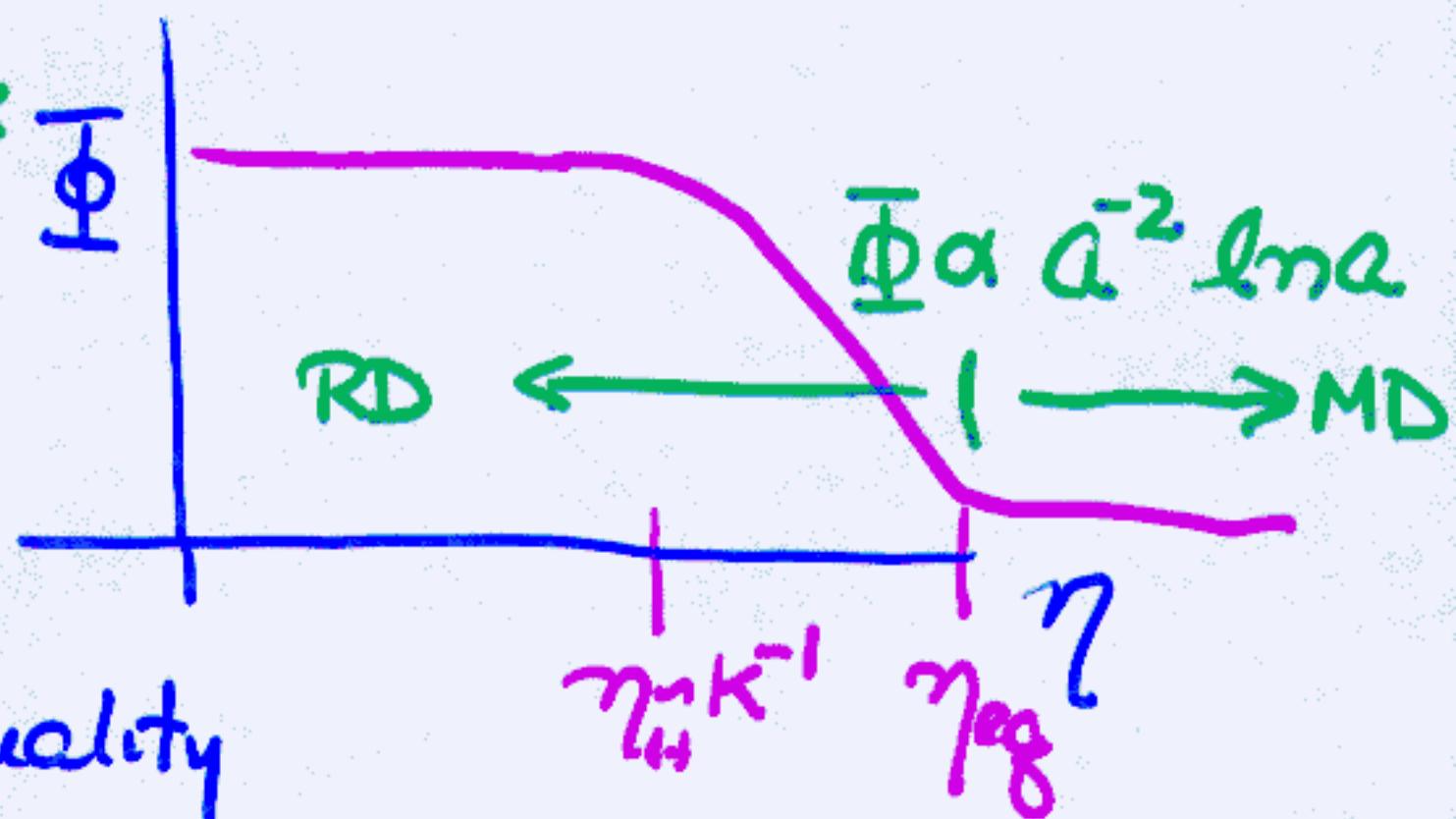
$$T(k) \propto \frac{\Phi(k, a_{\text{now}})}{\Phi(k, a_{\text{initial}})}$$

conventionally normalized to

$$\lim_{k \rightarrow 0} T(k) = 1$$

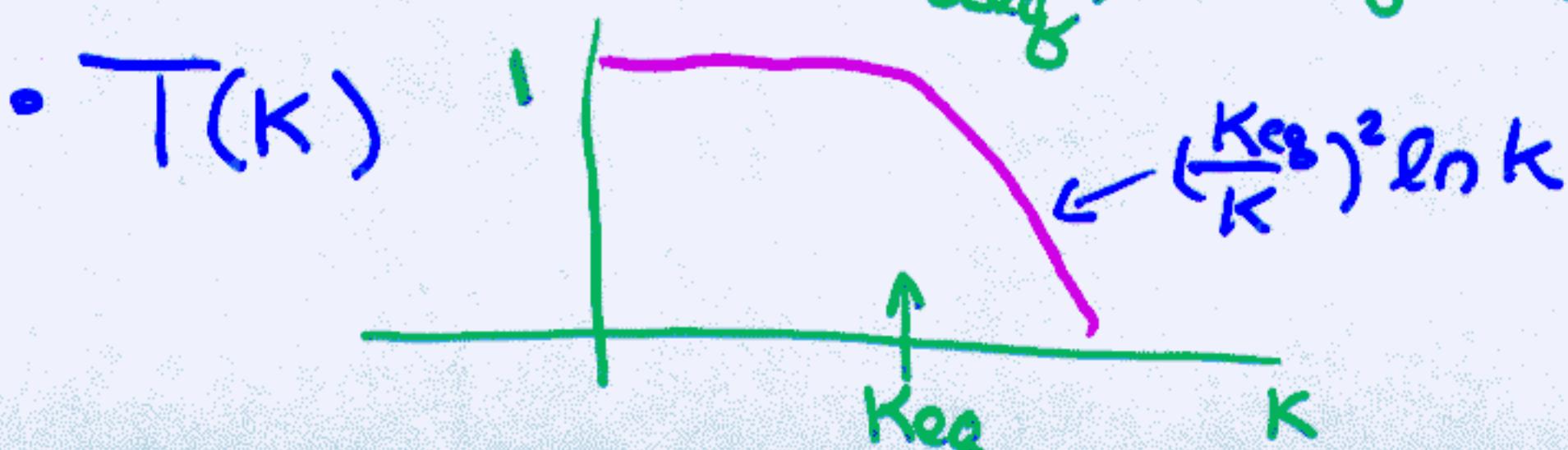
• Radiation Domination:

- decays as $a^{-2} \ln a$ between horizon crossing $\eta_H^{-1} k^{-1}$ and matter-radiation equality η_{eq}



- RD $\eta \alpha a \therefore \left(\frac{a_H}{a_{eq}}\right) = \left(\frac{\eta_H}{\eta_{eq}}\right) \approx \left(\frac{1}{\kappa \eta_{eq}}\right)$

- Suppression in k : $\left(\frac{a_H}{a_{eq}}\right)^2 \ln(a_{eq}/a_H) \propto k^{-2} \ln k$



• Stress Free Vector Modes

$$\left[\frac{d}{d\eta} + 4 \frac{d}{a} \right] [(\rho + p)(\sqrt{\pm 1} - B^{(\pm 1)})/\kappa] = 0$$

$$\Rightarrow \sqrt{\pm 1} - B^{(\pm 1)} \propto a^{-4} (\rho + p)^{-1}$$

decay unless
continuously generated

• Stress Free Tensor Modes

$$\left[\frac{d^2}{d\eta^2} + 2 \frac{d}{a} \frac{d}{d\eta} + (\kappa^2 + 2\kappa) \right] H_T^{(\pm 2)} = 0$$

$$H_T^{(\pm 2)} = C_1 H_1 + C_2 H_2$$

$$H_1 \propto x^{-m} j_m(x)$$

$$x = \sqrt{\kappa^2 - 2\kappa} \eta$$

$$H_2 \propto x^{-m} n_m(x)$$

$$m = \frac{(1-3w)}{(1+3w)}$$

$$w > -\frac{1}{3}$$

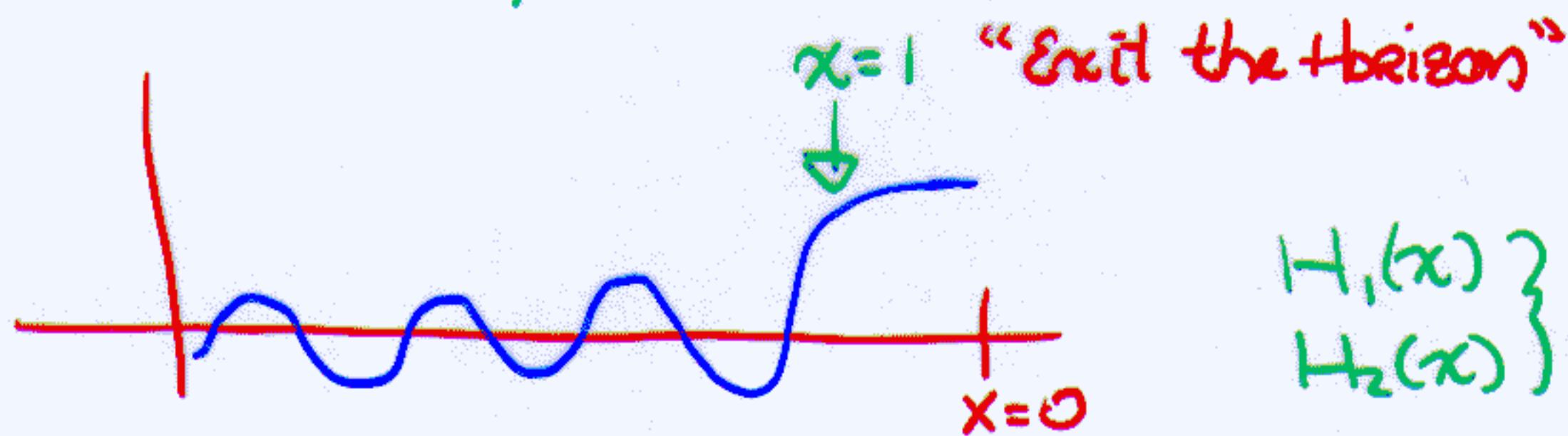
gravity waves
constant about horiz
 $x \ll 1$ and then
oscillate & damp

(important for gravity wave C_e's)

$$\omega < -\frac{1}{3}$$

Reverse: gravity waves oscillate and then freeze in at some value

$$x = -k \int_{\eta}^{\infty} d\eta = -k(\eta_{\infty} - \eta)$$



This is exactly the behavior of scalar field fluctuations: field oscillations freeze in as they "exit the horizon"

Link between scalar field fluctuations & gravitational waves include generation of quantum fluctuations