

Inflationary Perturbations

- Linear perturbations $\phi = \phi_0 + \phi_1$

$$\delta R_\phi = a^{-2}(\dot{\phi}_0 \dot{\phi}_1 - \dot{\phi}_0^2 A) + V' \phi_1$$

$$\delta P_\phi = \delta R_\phi - 2V' \phi_1$$

$$(R_0 + P_\phi)(V_\phi - B) = a^{-2} k \dot{\phi}_0 \dot{\phi}_1$$

$$\therefore \boxed{\ddot{\phi}_1 + 2\frac{\dot{a}}{a}\dot{\phi}_1 + (\kappa^2 + a^2 V'')\phi_1 = (\dot{A} - 3\dot{H}_L - kB)\dot{\phi}_0 - 2a^2 V' A}$$

- Comoving gauge $V_\phi = B \Rightarrow \phi_1 = 0$

has no scalar field perturbation

- Gauge transformation

$$\tilde{\phi}_1 = \phi_1 - \dot{\phi}_0 T = 0 \text{ in comoving frame}$$

\therefore to get to comoving frame

$$\boxed{T = \frac{\phi_1}{\dot{\phi}_0}}$$

$$\begin{aligned} \tilde{H}_T &= H_T + kL \\ L &= -H_T/\kappa \end{aligned}$$

$$\therefore \tilde{J} = H_L + \frac{H_T}{3} - \frac{\dot{a}}{a} \frac{\phi_1}{\dot{\phi}_0}$$

- Calculate in a frame where $H_L = H_T = 0$
then ϕ_1 is simply related to $\boxed{\tilde{J} = -\frac{\dot{a}}{a} \frac{\phi_1}{\dot{\phi}_0}}$

• Spatially Unperturbed Gauge ($H_L = H_T = 0$)

$$\ddot{\phi}_1 + 2\frac{\dot{a}}{a}\dot{\phi}_1 + (\kappa^2 + a^2 V'')\phi_1 = (\dot{A} + k\dot{B})\dot{\phi}_0 - 2a^2 V'A$$

$$\begin{aligned}\frac{d}{dt} A &= 4\pi G a^2 (\rho_\phi + p_\phi) (v_\phi - B) / \kappa \\ &= 4\pi G \dot{\phi}_0 \phi_1\end{aligned}$$

$$\therefore A = \left(\frac{\dot{a}}{a}\right)^{-1} 4\pi G \dot{\phi}_0 \phi_1$$

$$\frac{d}{dt} kB = 4\pi G a^2 [8\rho_\phi + 3\frac{\dot{a}}{a}(\rho_\phi + p_\phi)(v_\phi - B)]$$

$$\begin{aligned}kB &= 4\pi G a^2 \left(\frac{\dot{a}}{a}\right)^{-1} \left\{ a^{-2} \dot{\phi}_0 \dot{\phi}_1 - a^{-2} \dot{\phi}_0^2 A + V' \phi_1 \right\} \\ &\quad + 12\pi G \dot{\phi}_0 \phi_1 \\ &= 4\pi G \left(\frac{\dot{a}}{a}\right)^{-1} \dot{\phi}_0 \dot{\phi}_1 - 4\pi G \dot{\phi}_0 3\phi_1 + 12\pi G \dot{\phi}_0 \phi_1\end{aligned}$$

$$kB = 4\pi G \left(\frac{\dot{a}}{a}\right)^{-1} \dot{\phi}_0 \dot{\phi}_1$$

$$\begin{aligned}\dot{A} &= \left(\frac{\dot{a}}{a}\right)^2 \frac{d}{dt} \left(\frac{\dot{a}}{a}\right) 4\pi G \dot{\phi}_0 \phi_1 + \left(\frac{\dot{a}}{a}\right)^{-1} 4\pi G (\dot{\phi}_0 \phi_1 + \dot{\phi}_0 \dot{\phi}_1) \\ &\quad \dot{\phi}_0 \approx \dot{\phi}_0 \frac{\dot{a}}{a}\end{aligned}$$

$$= 4\pi G \dot{\phi}_0 \phi_1 + 4\pi G \dot{\phi}_0 \phi_1 + \left(\frac{\dot{a}}{a}\right)^{-1} 4\pi G \dot{\phi}_0 \dot{\phi}_1$$

$$\dot{A} - kB = 8\pi G \dot{\phi}_0 \phi_1$$

- $(\dot{A} - kB) \ddot{\phi}_0 - 2a^2 V' A$

$$= 8\pi G \dot{\phi}_0^2 \phi_1 - 2(-3\dot{\phi}_0(\frac{\dot{a}}{a}))(\frac{\dot{a}}{a})' 4\pi G \dot{\phi}_0 \phi_1$$

$$8 \cdot 4\pi G \dot{\phi}_0^2 \phi_1$$

$$\begin{aligned} 4\pi G \dot{\phi}_0^2 &= \frac{2}{3} V a^3 \epsilon \\ &= (\frac{\dot{a}}{a})^2 \epsilon \end{aligned}$$

$$= 8\epsilon (\frac{\dot{a}}{a})^2 \phi_1$$

- $-a^2 V' \phi_1 = -8\pi G (e - \delta) a^2 V \phi_1$

$$= -2(\frac{3}{2})(\frac{\dot{a}}{a})^2 (e - \delta)$$

$$= (\frac{\dot{a}}{a})^2 (-3e + 3\delta) \phi_1$$

- Egn of motion

$$\ddot{\phi}_1 + 2\frac{\dot{a}}{a}\dot{\phi}_1 + (K^2 - (5e + 3\delta)(\frac{\dot{a}}{a})^2) \phi_1 = 0$$

- Transform to comoving

$$u = a\phi_1 \quad \phi_1 = u/a$$

$$\dot{\phi}_1 = \dot{u}/a - (u/a)(\frac{\dot{a}}{a})$$

$$\ddot{\phi}_1 = \ddot{u}/a - 2(\dot{u}/a)(\frac{\dot{a}}{a}) + \frac{u}{a}(\frac{\dot{a}}{a})^2 - \frac{u}{a} \frac{d}{dt}(\frac{\dot{a}}{a})$$

$$\dot{\phi}_1 + 2\frac{\dot{a}}{a}\dot{\phi}_1 = \frac{\ddot{u}}{a} + 2\frac{\dot{u}}{a}(\frac{\dot{a}}{a}) + \frac{u}{a}(\frac{\dot{a}}{a})^2 - (\frac{u}{a})(\frac{\dot{a}}{a})^2(1-e) + 2\frac{\dot{a}}{a}(\frac{\dot{u}}{a} - \frac{u}{a}(\frac{\dot{a}}{a}))$$

$$= \frac{\ddot{u}}{a} + \frac{u}{a}(\frac{\dot{a}}{a})^2 e - 2(\frac{\dot{a}}{a})^2(\frac{u}{a})$$

$$\ddot{u} + \left(\kappa^2 - (2+4\epsilon+3\delta) \frac{\dot{a}}{a}^2 \right) u = 0$$

define new time coordinate

$$\tilde{\eta} = \eta - \eta_{\text{end}} \quad (= -\text{time to end of inflation})$$

$$\begin{aligned}\tilde{\eta} &= \int_{a_{\text{end}}}^a \frac{da}{Ha^2} \\ &= -\frac{1}{aH} \Big|_{a_{\text{end}}}^a - \int_{a_{\text{end}}}^a \frac{da}{a} \frac{dH/da}{H^2} \\ &= -\frac{1}{aH} \Big|_{a_{\text{end}}}^a + \epsilon \int_{a_{\text{end}}}^a \frac{da}{Ha^2} \\ &= -\frac{1}{aH} \Big|_{a_{\text{end}}}^a + \epsilon \tilde{\eta}\end{aligned}$$

$$\tilde{\eta} = \frac{1}{1-\epsilon} \left(\frac{1}{a_{\text{end}} H_{\text{end}}} - \frac{1}{aH} \right) \approx -\frac{1}{aH} \frac{1}{1-\epsilon} = -\frac{1}{1-\epsilon} \left(\frac{\dot{a}}{a} \right)^{-1}$$

$$\ddot{u} + \left[\kappa^2 - \frac{(2+4\epsilon+3\delta)}{(1-\epsilon)^2 \tilde{\eta}^2} \right] u = 0$$

$$\boxed{\ddot{u} + \left[\kappa^2 - \frac{2+6\epsilon+3\delta}{\tilde{\eta}^2} \right] u = 0}$$

harmonic oscillator to quantize

• Quantized Harmonic Oscillator (Dodonov 5.4.1)

1) Classical Eqn of Motion

$$\ddot{x} + \omega^2 x = 0$$

2) Quantize $x \rightarrow \hat{x}$ operator

$$\hat{x} = V(\omega, t)\hat{a} + V^*(\omega, t)\hat{a}^\dagger$$

$$\ddot{\hat{x}} + \omega^2 \hat{x} = 0$$

where the creation and annihilation operators satisfy

$$[\hat{a}, \hat{a}^\dagger] = 1 \quad \text{and} \quad \hat{a}|0\rangle = 0$$

3) Normalize the solutions by demanding

$$[\hat{x}, \hat{p}] = [\hat{x}, \frac{d\hat{x}}{dt}] = i$$

$$V(\omega, t) = \frac{e^{-i\omega t}}{\sqrt{2\omega}}$$

4) Zero pt fluctuations of ground state

$$\langle 0 | \hat{x}^2 | 0 \rangle = \langle 0 | x^\dagger x | 0 \rangle$$

$$= \underbrace{\langle 0 | (V^* \hat{a}^\dagger + V \hat{a})}_{x} (\underbrace{V \hat{a} + V^* \hat{a}^\dagger}_{x^\dagger}) | 0 \rangle$$

$$= \langle 0 | \hat{a} \hat{a}^\dagger | 0 \rangle | V(\omega, t) |^2$$

$$= \langle 0 | [\hat{a}, \hat{a}^\dagger] + \hat{a}^\dagger \hat{a} | 0 \rangle | V(\omega, t) |^2$$

$$= | V(\omega, t) |^2 = \frac{1}{2\omega}$$

• Quantized Scalar Field

- 1) In the oscillator limit, quantum fluctuations

$$\ddot{u} + k^2 u = 0$$

imply

$$u = \frac{1}{\sqrt{2k}} e^{-ik\eta}$$

- 2) Classical Egn of Motion carries these outside horizon

take the $\epsilon \ll 1$ $\delta \ll 1$ slow roll limit

$$\ddot{u} + (k^2 - \frac{2}{\eta^2}) u = 0$$

$$|k\eta| \ll 1 \quad u \propto 1/\eta$$

full solution

$$u = \frac{1}{(2k^3)^{1/2}} \left[k - \frac{i}{\eta} \right] e^{-ik\eta}$$

so:

$$\lim_{k\eta \rightarrow 0} u = \frac{i}{(2k^3)^{1/2}} \eta$$

$$= \frac{i H a}{(2k^3)^{1/2}}$$

$$u = a\phi_1$$

$$\phi_1 = \frac{i H}{(2k^3)^{1/2}}$$

$$(S = -\frac{i H}{(2k^3)^{1/2}} (\frac{\partial}{\partial}) \frac{1}{\phi_0})$$

3) Transform to comoving frame

Berrein Curvature

$$\zeta = -\frac{iH}{(2K^3)^{1/2}} \left(\frac{\dot{a}}{a}\right) \frac{1}{\dot{\phi}_0}$$

$$\dot{\phi}_0^2 = \frac{2}{3} a^2 \nu \epsilon$$

$$|\zeta|^2 = \frac{H^2}{2K^3} \left(\frac{\dot{a}}{a}\right)^2 \frac{3}{2} \frac{1}{a^2 \nu \epsilon}$$

$$= \frac{H^2}{K^3} \frac{2\pi G}{\epsilon}$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G a^2 \nu}{3}$$

$$G^{-1} h = m_{PL}$$

$$\Delta_S^2 =$$

$$\frac{K^3 |\zeta|^2}{2\pi^2} = \frac{H^2}{m_{PL}^2} \frac{1}{\pi \epsilon}$$

to the extent $H = \text{const}$ $\epsilon = \text{const}$
during inflation
 \rightarrow scale invariant spectrum
of curvature fluctuations

in terms of the Newtonian curvature in the
Radiation Dominated epoch

$$\bar{\Phi} = \frac{3+3W}{5+3W} \zeta$$

$$\frac{K^3 |\bar{\Phi}|^2}{2\pi^2} \Big|_{RD} = \frac{1}{9\pi} \frac{H^2}{m_{PL}^2 \epsilon}$$

$$= \frac{2}{3} \zeta$$

4) Deviations from Scale invariance

$$\frac{d \ln \Delta_S^2}{d \ln K} = n_s - 1$$

$n_s \equiv$ scalar tilt

$$\frac{d \ln \Delta^2 S}{d \ln K} = 2 \frac{d \ln H}{d \ln K} - \frac{d \ln \epsilon}{d \ln K} = n - 1$$

n.b. H, ϵ evaluated at freeze in $K\eta = -1$

$$\begin{aligned} \left. \frac{d \ln H}{d \ln K} \right|_{K\eta=1} &= \left. \frac{1}{H} \frac{dH}{d\eta} \right|_{K\eta=1} \left. \frac{d\eta}{dK} \right|_{K\eta=-1} \\ &= \left. \frac{\kappa}{I} (-\alpha H^2 \epsilon) \right|_{K\eta=1} \frac{1}{K^2} \quad \alpha H = -\frac{1}{\eta} \\ &= -\epsilon \end{aligned}$$

$$\begin{aligned} \left. \frac{d \ln \epsilon}{d \ln K} \right|_{K\eta=1} &= \left. \frac{\kappa}{\epsilon} \frac{d\epsilon}{d\eta} \right|_{K\eta=1} \left. \frac{d\eta}{dK} \right|_{K\eta=-1} \\ &= \left. \frac{\kappa}{\epsilon} \frac{2\alpha H \epsilon (\epsilon + \delta)}{K^2} \right. \\ &= 2(\epsilon + \delta) \end{aligned}$$

$$n = 1 - 4\epsilon - 2\delta$$

- Fluctuation Spectrum for ϕ_1 applies to any massless (scalar) field the inflaton is special in that it dominates the energy density & generates curvature

\Rightarrow Isocurvature features & gravity waves

• Gravity-waves

Classical eqns of motion (+, ×)

$$\ddot{h} + 2\frac{\dot{a}}{a}\dot{h} + \kappa^2 h = 0$$

Compare with scalar field in slow roll limit

$$\ddot{\phi}_i + 2\frac{\dot{a}}{a}\dot{\phi}_i + \kappa^2 \phi_i = 0$$

normalization fixed by examining Lagrangian

$$\mathcal{L} = \frac{ah}{16\pi G}$$

fluctuations: $\mathcal{L} = \frac{iH a}{(2\kappa^3)^{1/2}} \Rightarrow h = \sqrt{16\pi G} \frac{iH}{(2\kappa^3)^{1/2}}$

Power spectrum: $\Delta_h^2 = \frac{16\pi G}{2\pi^2} \frac{H^2}{2} = \frac{4 H^2}{\pi m_{Pl}^2}$

note $H^2 \propto V$ so amp gives energy scale

tensor tilt $\frac{d \ln \Delta_h^2}{d \ln k} = n_T = \frac{2 d \ln H}{d \ln k} = -2\epsilon$

$\frac{\Delta_h^2}{\Delta_S^2} = 4\epsilon = -2n_T$ consistency relation