Astro 448: Problem Set 2

Due Jan. 31

Classical Scalar Fields

Scalar fields are the basis of inflation and dark energy models. In this problem set we derive the classical equations of motion for a scalar field and its perturbations. Most of the hard work has already been done for you in class by deriving the equations of motion for an arbitrary stress-energy tensor.

The stress-energy tensor of a minimally coupled scalar field φ with a potential $V(\varphi)$ is given by

$$T^{\mu}{}_{\nu} = \nabla^{\mu}\varphi\,\nabla_{\nu}\varphi - \frac{1}{2}(\nabla^{\alpha}\varphi\,\nabla_{\alpha}\varphi + 2V)\delta^{\mu}{}_{\nu}\,. \tag{1}$$

We will expand the scalar field fluctuations about its background value ϕ_0 as $\varphi = \phi_0 + \phi_1$.

1 Homogeneous Case

(1) Using the FRW metric for the background and the general relation for the components of the stress energy tensor, derive the expressions for the energy density of the field $\rho_\phi(\phi_0, \dot{\phi}_0)$ and the pressure $p_\phi(\phi_0, \dot{\phi}_0)$. You will need this below so if you are unsure of the result check it in Scott's book.

(2) If the energy density is dominated by the potential term what is the equation of state $w_\phi = p_\phi/\rho_\phi$? If the energy density is dominated by the kinetic term ($\dot{\phi}$) what is the equation of state?

(3) Show that the continuity equation

$$\dot{\rho}_\phi = -3(\rho_\phi + p_\phi)\frac{\dot{a}}{a}\,, \tag{2}$$

implies the homogeneous scalar field equation

$$\ddot{\phi}_0 + 2\frac{\dot{a}}{a}\dot{\phi}_0 + a^2 V' = 0\,, \tag{3}$$

primes are derivatives with respect to the argument ϕ_0, and overdots are derivatives with respect to conformal time.

2 Fluctuations

The same procedure as in (1) works for the fluctuations. This is a bunch of tedious algebra so I will just give you the answer

$$\begin{aligned}
\delta\rho_\phi &= a^{-2}(\dot{\phi}_0\dot{\phi}_1 - \dot{\phi}_0^2 A) + V'\phi_1\,,\\
\delta p_\phi &= a^{-2}(\dot{\phi}_0\dot{\phi}_1 - \dot{\phi}_0^2 A) - V'\phi_1\,,\\
(\rho_\phi + p_\phi)(v_\phi - B) &= a^{-2}k\dot{\phi}_0\phi_1\,,\\
p_\phi\pi_\phi &= 0\,,
\end{aligned} \tag{4}$$

where A is the time-time metric perturbation and B is the scalar time-space metric perturbation in covariant perturbation theory. Note that there are no vector and tensor modes associated with the fluctuations. They are higher order in ϕ_1. The linearization of the Einstein equations preserves the scalar nature of the scalar field.

(4) Using the gauge transformation properties of the stress energy components derive the gauge transformation properties of ϕ_1. Hint: look at the velocity component. Argue that your result had to be true given that a scalar field is a scalar field!

(5) Show that the adiabatic sound speed

$$c_\phi^2 \equiv \dot{p}_\phi/\dot{\rho}_\phi = 1 + \frac{2V'\dot{\phi}}{3(\rho_\phi + p_\phi)}\left(\frac{\dot{a}}{a}\right)^{-1} \tag{5}$$

This looks like a "bad thing" since c_ϕ^2 is not guaranteed to be positive. An imaginary sound speed means accelerated collapse and a scalar field is supposed to be the most gravitationally stable type of matter possible - hence its utility in the dark energy game.

(6) Show that pressure fluctuation

$$\delta p_\phi = \delta\rho_\phi + 3(\rho_\phi + p_\phi)\frac{v_\phi - B}{k}\frac{\dot{a}}{a}(1 - c_\phi^2)\,. \tag{6}$$

Argue that in the right coordinate system, in this case the comoving frame the relevant sound speed squared

$$\delta p_\phi/\delta\rho_\phi = 1\,. \tag{7}$$

The sound speed relevant for gravitational collapse in this frame is the speed of light. The density fluctuation in the comoving frame really is what you want to think of as the non-relativistic density perturbation - e.g. it obeys the usual Poisson equation when related to the Newtonian gravitational potential. Thus a (slowly rolling) scalar field is gravitationally stable (smooth) inside the horizon.

(7) Use the continuity equation to derive the equation of motion for the scalar field perturbation

$$\ddot{\phi}_1 \quad = \quad -2\frac{\dot{a}}{a}\dot{\phi}_1 - (k^2 + a^2V'')\phi_1 + (\dot{A} - 3\dot{H}_L - kB)\dot{\phi}_0 - 2Aa^2V'\,. \tag{8}$$

Give the expression in the Newtonian and synchronous gauges. What happens to ϕ_1 in the comoving frame? (hint: re-examine the expression for the energy flux).