Astro 448: Problem Set 5

Due Mar. 7

This is the final problem set and is therefore a bit longer and more involved. The intent is for you to become familiar with the calculation of the power spectrum through a worked example. There is lots of formal baggage here but don't be intimidated. The actual calculation I'm asking you to do is separated off by the horizontal lines.

## 1. Power Spectra, Sources and the Addition of Angular Momenta:

Consider the general description of the temperature field $\Theta$ as a function of both position $\mathbf{x}$ and direction $\mathbf{n}$ at $\mathbf{x}$. It can in general be expanded in a complete set of modes denoted $G_j^m$ as

$$\Theta(\mathbf{x}, \mathbf{n}) = \int \frac{d^3k}{(2\pi)^3} \sum_{\ell m} \Theta_\ell^{(m)}(k) G_\ell^m(\mathbf{x}, \mathbf{k}, \mathbf{n}) \,, \tag{1}$$

where

$$G_\ell^m = (-i)^\ell \sqrt{\frac{4\pi}{2\ell+1}} Y_\ell^m(\hat{\mathbf{n}}) \exp(i\mathbf{k}\cdot\mathbf{x}) \,. \tag{2}$$

You know this is true since plane waves are a complete orthonormal set of modes for a spatial field and spherical harmonics are the same for an angular field.

The angular power spectrum of the field is defined as

$$\begin{aligned} \langle \Theta(\mathbf{x}, \mathbf{n_1}) \Theta(\mathbf{x}, \mathbf{n_2}) \rangle &\equiv \sum_{\ell_1 m_1} \sum_{\ell_2 m_2} Y_{\ell_1}^{m_1}(\mathbf{n_1})^* Y_{\ell_2}^{m_2}(\mathbf{n_2}) C_{\ell_1} \delta_{\ell_1 \ell_2} \delta_{m_1 m_2} \\ &\equiv \sum_\ell \frac{2\ell+1}{4\pi} C_\ell P_\ell(\mathbf{n_1}\cdot\mathbf{n_2}) \end{aligned} \tag{3}$$

under the assumption of translational and rotational invariance of the two point function. $C_\ell$ is the famous angular power spectrum.

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- Show that

$$C_\ell = 4\pi \int \frac{d^3k}{(2\pi)^3} \sum_m \frac{\left\langle \Theta_\ell^{(m)*} \Theta_\ell^{(m)} \right\rangle}{(2\ell+1)^2} \,. \tag{4}$$

The extra sum over $m$ anticipates the following calculation where we temporarily break rotational invariance by considering a single $\mathbf{k}$ with a particular direction that picks out special values of $m$. Invariance is restored by the integral over $\mathbf{k}$ directions above. Since we are guaranteed $m$-invariance of the power spectrum at the end we instead calculate the $m$-averaged contribution for each mode.

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Let us suppose that the temperature field $\Theta$ is generated by the line-of-sight integral of another positionally and directionally dependent field $S$ evaluated at a $\mathbf{x} = D\hat{\mathbf{n}}$

$$\begin{aligned} \Theta(\mathbf{x}, \mathbf{n}) &= \int dD\, S(\mathbf{x}, \mathbf{n}) \,, \\ &= \int dD \int \frac{d^3k}{(2\pi)^3} \sum_{jm} S_j^{(m)}(D, k) G_j^m \,. \end{aligned} \tag{5}$$

Note that the radial distance $D = (\eta_0 - \eta)$.

Using the expansion of a plane wave in spherical coordinates $(D, \mathbf{n})$

$$\exp(i\mathbf{k}D\cdot\mathbf{n}) = \sum_\ell (-i)^\ell \sqrt{4\pi(2\ell+1)} j_\ell(kD) Y_\ell^0(\mathbf{n}) \,, \tag{6}$$

where we have now chosen the coordinate system so that $\hat{\mathbf{e}}_3 \parallel \mathbf{k}$, we can rewrite the normal mode by adding the local and plane wave angular dependences using the Clebsch-Gordan relation

$$Y_{\ell_1}^{m_1} Y_{\ell_2}^{m_2} = \frac{\sqrt{(2\ell_1+1)(2\ell_2+1)}}{4\pi} \sum_{\ell,m} \langle \ell_1, \ell_2; m_1, m_2 | \ell_1, \ell_2; \ell, m \rangle \langle \ell_1, \ell_2; 0, 0 | \ell_1, \ell_2; \ell, 0 \rangle \sqrt{\frac{4\pi}{2\ell+1}} Y_\ell^m \,. \tag{7}$$

The general form of the result is

$$G_j^m = \sum_\ell (-i)^\ell \sqrt{4\pi(2\ell+1)}\alpha_{j\ell}^{(m)}\, Y_\ell^m\,. \tag{8}$$

Here the $\alpha_{j\ell}^{(m)}$ are linear combinations of spherical Bessel functions with weights given by Clebsch-Gordan coefficients.

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- Anticipating its use for the Doppler effect, let's assume that the source is some vector field so that $j=1$. Show that

$$\alpha_{1\ell}^{(0)}(x) = j_\ell'(x)\,, \tag{9}$$

where primes are derivatives with respect to the argument and

$$\alpha_{1\ell}^{(\pm 1)}(x) = \sqrt{\frac{\ell(\ell+1)}{2}}\frac{j_\ell(x)}{x}\,. \tag{10}$$

Hint: use Mathematica or a good quantum book to evaluate the Clebsch-Gordan coefficient; $j_\ell'$ and $j_\ell/x$ can be written as linear combinations of $j_{\ell\pm1}$.

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Comparing expressions (1) and (5), we see that

$$\Theta_\ell^{(m)} = (2\ell+1)\int dD \sum_j S_j^{(m)}\alpha_{j\ell}^{(m)}\,. \tag{11}$$

This is the integral method for CMB calculations: once the source is known the observable CMB angular moments follows from a simple geometric projection or radial integral over the source.

## 2. Limber Approximation

If the source field $S_j^{(m)}(D)$ varies with $D$ only on scales much larger than the wavelength $2\pi/k$ then the source can be approximated at the peak of the spherical bessel function $D=\ell/k$ and taken out of the radial integral in (11) . This is the all-sky, directional source, generalization of the well-known Limber approximation.

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- Using the relation

$$\int_0^\infty dx j_\ell(x) = \frac{\sqrt{\pi}}{2}\frac{\Gamma[(\ell+1)/2]}{\Gamma[(\ell+2)/2]}\,, \tag{12}$$

evaluate $\Theta_\ell^{(m)}(k)$ for $m=-1,0,1$ and a vector source $j=1$. Noting that $j_\ell(0)=0$ and $j_\ell(\infty)=0$, argue that $m=0$ vanishes.

- Transforming the variable of integration back to radial distance using $D=\ell/k$, show that

$$C_\ell = \frac{\pi^2}{\ell^3}\int dDD \sum_{m=\pm1}\Delta_S^{2\,(m)}\,, \tag{13}$$

where $\Delta_S^2$ is the logarithmic power spectrum of the 3-D source field (with $j=1$) and we have assumed $\ell \gg 1$. Notice that $m=0$ does not contribute.

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A relation like this which expresses a 2-D statistic on the sky as the radial integral of a 3-D statistic of a source is usually called a Limber approximation in cosmology. A useful rule of thumb is that the logarithmic power spectrum of the 2-D observable $\ell^2 C_\ell/2\pi \sim H_0^{-2}\Delta_S^2/\ell$ since the sources are generally at distances of order the Hubble length $H_0^{-1}$. Note the factor of $\ell$ in the denominator implies that a scale-invariant source will produce a spectrum that is not scale-invariant – in particular falling with $\ell$. Thinking about why that is so and what happened to $m=0$ may help you with understanding the implications for the Doppler effect below.

## 3. Doppler and Vishniac Effects

Now let's try an explicit example. The temperature field produced by the Doppler effect is given by

$$\Theta(\mathbf{x},\mathbf{n}) = \int dD\, g(D)\, \mathbf{n}\cdot\mathbf{v}(\mathbf{x})\,, \tag{14}$$

where the $g(D) = \dot{\tau} e^{-\tau}$ is the visibility function or the probability of last scattering within $dD$ of $D$.

Recall that the Fourier decomposition of a vector field

$$\mathbf{v}(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} \sum_{m=-1}^{1} v^{(m)} \mathbf{Q}^{(m)}(\mathbf{k}, \mathbf{x}) \,, \tag{15}$$

where the normal modes consist of one potential (scalar $m = 0$) and 2 vorticity (vector $m = \pm 1$) components

$$\begin{aligned} \mathbf{Q}^{(0)} &= -i\mathbf{e}_3 \exp(i\mathbf{k} \cdot \mathbf{x}) \\ \mathbf{Q}^{(\pm 1)} &= \mp i \frac{1}{\sqrt{2}} (\mathbf{e_1} \pm i\mathbf{e_2}) \exp(i\mathbf{k} \cdot \mathbf{x}) \,, \end{aligned} \tag{16}$$

(note my $Y_l^m$ conventions differ from Jackson by $(-1)^m$).

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- Show that

$$\mathbf{n} \cdot \mathbf{Q}^{(m)} = G_1^{(m)} \,. \tag{17}$$

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Now we've got the source in exactly the right form to use the Limber approximation. The contribution of the Doppler effect to the power spectrum follows immediately:

$$C_\ell = \frac{\pi^2}{\ell^3} \int dD D \, g^2(D) \sum_{m=\pm 1} \Delta_v^{2\,(m)} \,, \tag{18}$$

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- In linear theory, the velocity field is a potential flow and hence the $m = \pm 1$ vorticity components vanish identically. To leading order in the Limber approximation and perturbation theory, there is no Doppler effect. Geometrically why is this so? How does vorticity escape this effect? Why are these arguments not true for the Doppler contributions at recombination?

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Naively, the Doppler effect during the reionized epoch would be very large: optical depths are at least a few percent and velocity fields are of order $10^{-3}$ yielding $\Theta > 10^{-5}$, potentially masking the primary anisotropies from recombination. Fortunately, this is not so when you consider the fact that the Doppler effect is directionally dependent and does not follow the rule of thumb given above.

At higher order in perturbation theory, the Doppler source can gain an effective vorticity ($m = \pm 1$ component) even when the underlying velocity field is still potential. What happens is that the spatial variations in the visibility function effectively randomizes the direction of $k$ so that it is no longer related to the direction of the velocity field. The power in each $m$-component the becomes a third of the total.

If the variations in the visibility function are due to the linear density fluctuations of the baryons $g(D) \to g(D)(1 + \delta_b)$, this second order Doppler effect is called the Vishniac effect.

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- Show that the power spectrum of the Doppler effect in the limit of small scales where the baryon density fluctuations and velocity field are independent becomes

$$C_\ell = \frac{2\pi^2}{3\ell^3} \int dD D \, g^2(D) v_{\rm rms}^2 \Delta_{\delta_b}^2 \,, \tag{19}$$

  where $v_{\rm rms}^2 = \int d\ln k \Delta_v^{2\,(0)}$ which you should think of as just giving the amplitude of a *coherent* bulk flow on these scales.

Because it is second order, the Vishniac effect is small - of order $\mu$K in the temperature anisotropies. It is potentially detectable on scales below the damping tail where primary anisotropies drop out. Note that any spatial variation in the visibility function produces an analogous effect. A variant of this effect that has received much recent attention is the potential inhomogeneity of the ionization fraction at the onset of reionization.

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You are now ready to calculate CMB anisotropies in the real world!